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# LIQUID FILM ON AN UNSTEADY STRETCHING SURFACE

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**Abstract.** A fluid film lies on an accelerating stretching surface. A similarity transform reduces the unsteady Navier-Stokes equations to a nonlinear ordinary differential equation governed by a nondimensional unsteady parameter. Asymptotic and numerical solutions are found. The results represent rare exact similarity solutions of the unsteady Navier-Stokes equations.

**Introduction.** The study of the fluid motion caused by a stretching surface is important in extrusion processes. Steady two-dimensional flow induced by a stretching boundary was first solved by Crane [1]. Other literature on steady stretching boundaries in three dimensions include those of Brady and Acrivos [2] and Wang [3, 4]. All of the above results are also exact solutions of the Navier-Stokes equations.

The present paper studies the behavior of a liquid film on a stretching surface. The problem models fresh paint or protective coating which has been applied to the extrusate. Due to the thinning of the film as it stretches with the boundary, the problem is necessarily unsteady.

In general, the unsteady Navier-Stokes equations can only be solved by tedious numerical integration of the partial differential equations. However, if we restrict the motion to a specified family of time dependence, exact similarity solutions may be obtained. Yang [5] and Birkhoff [6] established the similarity transform

$$\mathbf{q} = \frac{1}{\sqrt{t}}\mathbf{F}\left(\frac{\mathbf{x}}{\sqrt{t}}\right) \tag{1}$$

where $\mathbf{q}(\mathbf{x}, t)$ is the velocity vector. Such a transform would reduce the unsteady Navier-Stokes equations to a set of nonlinear ordinary differential equations. We shall follow this approach.

**Formulation.** Figure 1 shows the situation. The surface at $y = 0$ is being stretched with the velocity

$$U = \frac{bx}{1 - \alpha t}. \tag{2}$$

Here $b$ and $\alpha$ are constants with dimensions [time$^{-1}$]. The stretching is decelerated if $\alpha$ is negative and accelerated if $\alpha$ is positive $(t < \frac{1}{\alpha})$. We assume the end effects and gravity are negligible and the film, of thickness $h(t)$, is uniform and stable.

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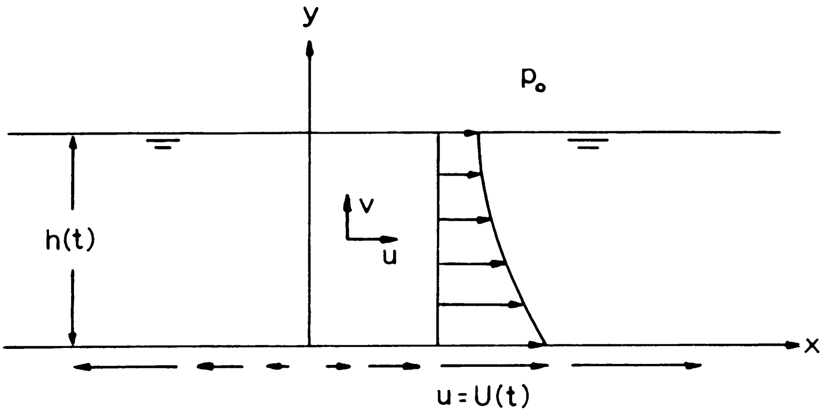


FIG. 1. Film on a stretching surface.

Let $(u, v)$ be the cartesian velocities in the $(x, y)$ directions, respectively. We search for similarity solutions by the transform

$$u = \frac{bx}{1-\alpha t} f'(\eta), \qquad v = \frac{-\sqrt{\nu b}}{\sqrt{1-\alpha t}} f(\eta) \tag{3}$$

where $\eta \equiv \sqrt{b/\nu}\, y/\sqrt{1-\alpha t}$ and $\nu$ is the kinematic viscosity of the fluid. The $x$-momentum equation of the unsteady Navier-Stokes equations reduces to

$$S\left(f' + \frac{\eta}{2} f''\right) + (f')^2 - f f'' = f''' \tag{4}$$

where $S \equiv \alpha/b$ is an important nondimensional parameter signifying the relative importance of unsteadiness to the stretching rate. The $y$-momentum equation gives the pressure $p(y, t)$ which integrates to be

$$p = \frac{\rho \nu b}{(1-\alpha t)} \left[\frac{S}{2}\eta f - \frac{1}{2} f^2 - f'\right] + C(t) \tag{5}$$

where $\rho$ is density. The boundary conditions on the stretching boundary are

$$f'(0) = 1 \tag{6}$$

$$f(0) = 0\,. \tag{7}$$

On the free surface at $y = h(t)$ or $\eta = \beta$, we require $v = dh/dt$ or

$$\frac{S'\beta}{2} = f(\beta)\,. \tag{8}$$

Zero tangential stress condition gives

$$f''(\beta) = 0\,. \tag{9}$$

Also, the normal stress is balanced by ambient pressure $p_0$. Eq. (5) yields

$$C(t) = p_0 - \frac{\rho \nu b}{(1-\alpha t)} \left[f'(\beta) + \frac{S^2 \beta^2}{8}\right]. \tag{10}$$

Thus the basic problem is Eq. (4) and the boundary conditions Eqs. (6–9) with thickness $\beta$ as an unknown.

**Asymptotic solution for thin film.** When the film is thin we expect the tangential velocity $u$ closely follows that of the stretching surface, i.e., $f \approx \eta$. Since $\eta$ is small let

$$\eta = \beta\zeta \tag{11}$$

where $\zeta = O(1)$ and the free surface is at $\zeta = 1$. Set

$$f = \beta\zeta + \beta^3 F_1(\zeta) + \beta^5 F_2(\zeta) + O(\beta^7) \tag{12}$$

$$S = S_0 + \beta^2 S_1 + \beta^4 S_2 + O(\beta^6). \tag{13}$$

The asymptotic sequence was determined *a posteriori*. Eq. (8) gives

$$S_0 = 2, \qquad S_1 = 2F_1(1), \qquad S_2 = 2F_2(1). \tag{14}$$

Upon substitution into Eqs. (4, 6, 7, 9) one obtains the successive equations

$$F_1''' = 3, \qquad F_1'(0) = F_1(0) = F_1''(1) = 0 \tag{15}$$

$$F_2''' = S_1 + 4F_1', \qquad F_2'(0) = F_2(0) = F_2''(1) = 0. \tag{16}$$

The solutions are

$$F_1 = \frac{1}{2}\zeta^3 - \frac{3}{2}\zeta^2 \tag{17}$$

$$F_2 = \frac{1}{10}\zeta^5 - \frac{1}{2}\zeta^4 - \frac{1}{3}\zeta^3 + 3\zeta^2. \tag{18}$$

Thus

$$S = 2 - 2\beta^2 + \frac{68}{15}\beta^4 + O(\beta^6). \tag{19}$$

Inversion gives

$$\beta = \left(1 - \frac{S}{2}\right)^{1/2}\left[1 + \frac{17}{15}\left(1 - \frac{S}{2}\right) + O\left(1 - \frac{S}{2}\right)^2\right]. \tag{20}$$

The third initial condition is

$$f''(0) = \beta F_1''(0) + \beta^3 F_2''(0) + \cdots = -3\beta + 6\beta^3 + O(\beta^5) \tag{21}$$

where $\beta$ is given by Eq. (20).

**Asymptotic solution for thick film.** When $\beta$ approaches infinity and $f$ is bounded, Eq. (8) shows that $S$ approaches zero. In that limit the solution approaches Crane's [1] exact steady solution. We shall perturb from this state as follows

$$f = 1 - e^{-\eta} + Sf_1(\eta) + O(S^2) \tag{22}$$

$$\beta = \frac{\beta_0}{S} + \beta_1 + O(S). \tag{23}$$

Eq. (4) gives

$$f_1''' + (1 - e^{-\eta})f_1''' - 2e^{-\eta}f_1' - e^{-\eta}f_1 = e^{-\eta}\left(1 - \frac{\eta}{2}\right). \tag{24}$$

The boundary conditions are

$$f_1(0) = f_1'(0) = f_1''(\infty) = 0. \tag{25}$$

Eqs. (8, 22, 23) give

$$\beta_0 = 2, \qquad \beta_1 = 2f_1(\infty). \tag{26}$$

After extensive use of variation of parameters, the particular solution for Eq. (24) is $\frac{\eta}{2} - 2$, and the general solution is constructed from the homogeneous solutions found.

$$f_1 = \frac{\eta}{2} - 2 + C_1 e^{-\eta} + C_2(\eta e^{-\eta} + 1) + C_3 Q(r) \tag{27}$$

where

$$Q(r) \equiv 2e^{-r} + (2r - r\ln r + 1)\int_1^r \frac{e^{-r}}{r}\,dr + r\int_1^r \frac{e^{-r}\ln r}{r}\,dr \tag{28}$$

and $r \equiv \exp(-\eta)$. We find

$$Q(0) = 2 + \int_0^1 \frac{1 - e^{-r}}{r}\,dr + \lim_{r\to 0} \ln r. \tag{29}$$

From Eq. (26)

$$\beta_1 = \lim_{\eta\to\infty} \eta - 4 + 2C_2 + 2C_3 Q(0). \tag{30}$$

Since $\beta_1$ is bounded, $C_3$ must be $\frac{1}{2}$ in order for the singularities to cancel. We also find $f_1''(\infty)$ is satisfied. Thus

$$\beta_1 = -2 + 2C_2 + \int_0^1 \frac{1 - e^{-r}}{r}\,dr. \tag{31}$$

Applying the other boundary conditions at $\eta = 0$ or $r = 1$, yields

$$C_1 = \frac{5}{4} - \frac{3}{4e}, \qquad C_2 = \frac{3}{4} - \frac{1}{4e}. \tag{32}$$

The integral in Eq. (31) can be expressed in terms of the exponential function [7] with a numerical value of 0.79660. Thus

$$\beta = \frac{2}{S} + 0.11266 + O(S) \tag{33}$$

$$f''(0) = -1 - \frac{1}{4}\left(1 + \frac{1}{e}\right) S + O(S^2). \tag{34}$$

**Numerical integration.** Numerical integration of Eqs. (4, 6–9) is necessary if $\beta$ is not large or small. Integration of Eq. (4) from zero to $\beta$ yields

$$S\left(\frac{f}{2} + \frac{\eta}{2}f'\right)\Big|_0^\beta + 2\int_0^\beta (f')^2\,d\eta - ff'\Big|_0^\beta = f''\Big|_0^\beta. \tag{35}$$

Using the boundary conditions Eq. (35) simplifies to

$$f''(0) = -\frac{S^2\beta}{4} - 2\int_0^\beta (f')^2\,d\eta. \tag{36}$$

This shows the initial value $f''(0)$ is necessarily negative. Guided by our approximate solutions, for given $S$, we guess $f''(0)$ and integrate Eq. (4, 6, 7) as an initial value problem by the fifth-order Runge-Kutta-Fehlberg algorithm. A step size of

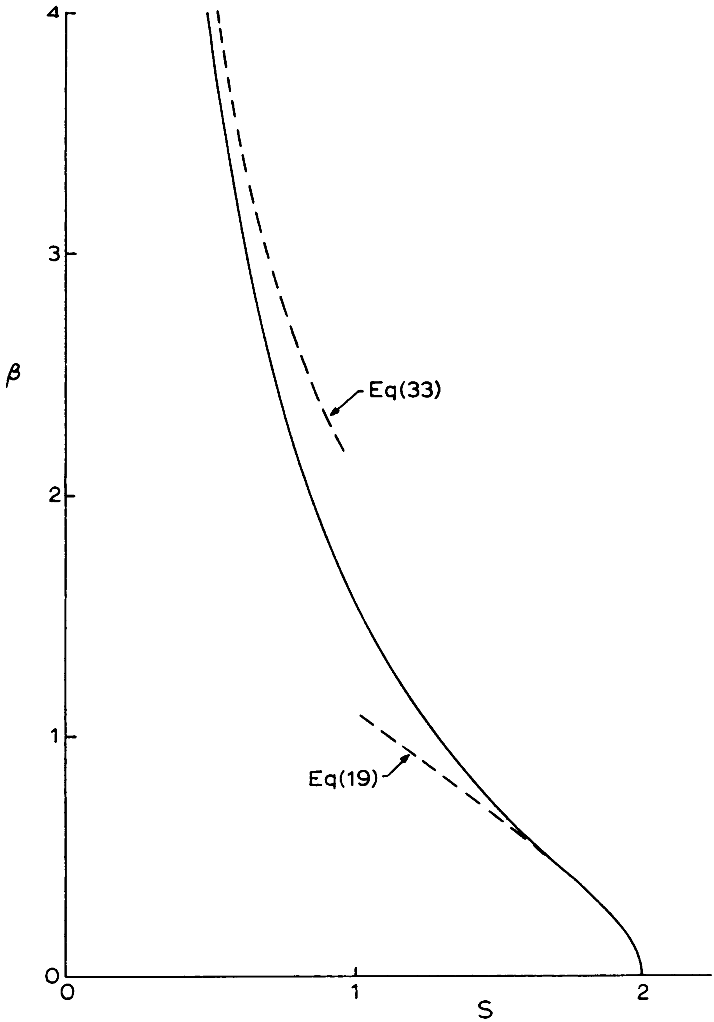


FIG. 2. Unsteady parameter $S$ as function of normalized thickness $\beta$. Dashed lines are approximations.

$\Delta\eta = 0.05$ is found to be sufficient. The integration continues until $f''$ becomes zero where we check whether Eq. (8) is satisfied. If not, the value of $f''(0)$ is adjusted.

Figure 2 shows the computed thickness $\beta$ as a function of $S$. Our approximate formulas compare well with exact numerical integration in their respective ranges of validity. For positive $S$, solutions exist only for $0 \leq S \leq 2$. We see that the faster the stretching rate $S$, the smaller the normalized thickness $\beta$. Actual thickness is a function of time

$$h(t) = \beta\sqrt{\frac{\nu}{b}}\sqrt{1 - \alpha t}. \tag{37}$$

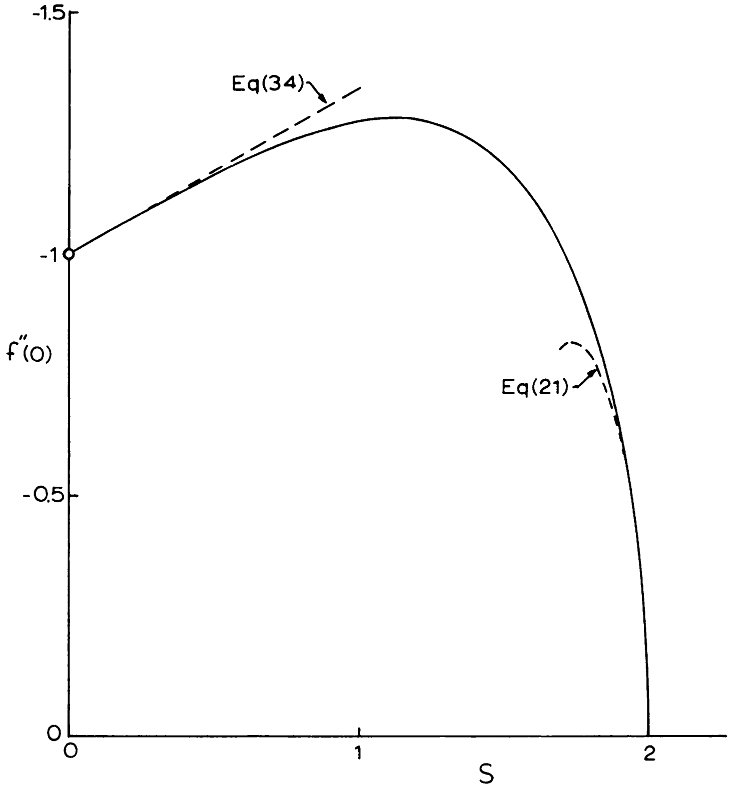


FIG. 3. Initial value $f''(0)$ as function of $S > 0$. Small circle represents Crane's steady solution.

The initial value $f''(0)$ is shown in Fig. 3. The small circle is the Crane solution $f''(0) = -1$, $S = 0$. The value of $|f''(0)|$ reaches a maximum of 1.283 near $S = 1.11$.

The shear stress on the stretching sheet is

$$\tau = \mu \frac{du}{dy}\bigg|_0 = \frac{\rho b\sqrt{b\nu x}}{(1-\alpha t)^{3/2}} f''(0). \tag{38}$$

Figure 4 shows the similarity profiles $f'(\eta)$ which is proportional to the lateral velocity $u$. The $S = 0$ solution is the exact solution of Crane [1], $f' = \exp(-\eta)$. For thin films the lateral velocity is almost uniform.

**The stretching decelerating surface.** In this case $\alpha$ and $S$ are negative. Equation (8) shows $f(\beta)$ must be negative. Then from Eqs. (6,7) one concludes $f'$ is zero somewhere inside the film and thus a flow reversal occurs.

Figure 5 shows the numerical results. Of interest is the steady solution $S = 0$, $f''(0) = -1.1408$, $\beta = 3.6272$. Unfortunately a closed form solution has not been found, and thus a perturbation solution is unavailable. In contrast, Crane's steady solution $(S = 0,\ f''(0) = -1,\ \beta = \infty)$ is in simple closed form.

Figure 6 shows some of the velocity profiles.

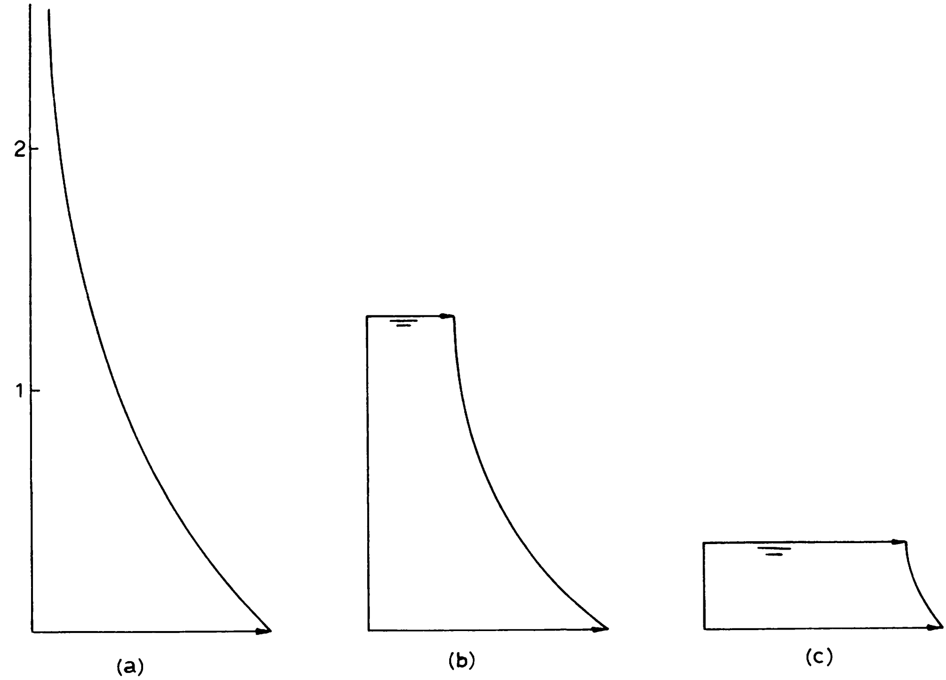


FIG. 4. Lateral normalized velocities. (a) $S = 0$, Crane's solution; (b) $S = 1.11$, maximum shear; (c) $S = 1.8$.

**The contracting decelerating surface.** For a contracting surface, the velocity $u$ is negative. The governing equations are still the same except Eq. (6) is replaced by $f'(0) = -1$. A similar asymptotic perturbation for thin film yields

$$S = -2 - 2\beta^2 - \frac{68}{15}\beta^4 + O(\beta^6) \tag{39}$$

showing $S < -2$. Inversion gives

$$\beta = \left(-\frac{S}{2} - 1\right)^{1/2}\left[1 + \frac{17}{15}\left(1 + \frac{S}{2}\right) + O\left(1 + \frac{S}{2}\right)^2\right]. \tag{40}$$

The initial value is

$$f''(0) = -3\beta - 6\beta^3 + O(\beta^5) \tag{41}$$

The numerical results are shown in Fig. 7. Maximum value of thickness $\beta$, 0.464, occurs at $S = -3.67$. Figure 8 shows the lateral velocity becomes uniform as $s \to -2$, $\beta \to 0$.

**Discussions.** This paper presents a rare exact similarity solution of the unsteady Navier-Stokes equations. The unsteady stretching velocity is prescribed by Eq. (2), which, by varying $b$ or $\alpha$, represents two families of curves. If the actual stretching velocity differs from Eq. (2) or segments of Eq. (2), similarity solutions cannot be obtained and a full numerical integration of the Navier-Stokes equations is necessary.

The nondimensional unsteady parameter $S$ governs the thickness $\beta$ and other fluid dynamic properties. In the steady case, $S = 0$, we recovered not only Crane's

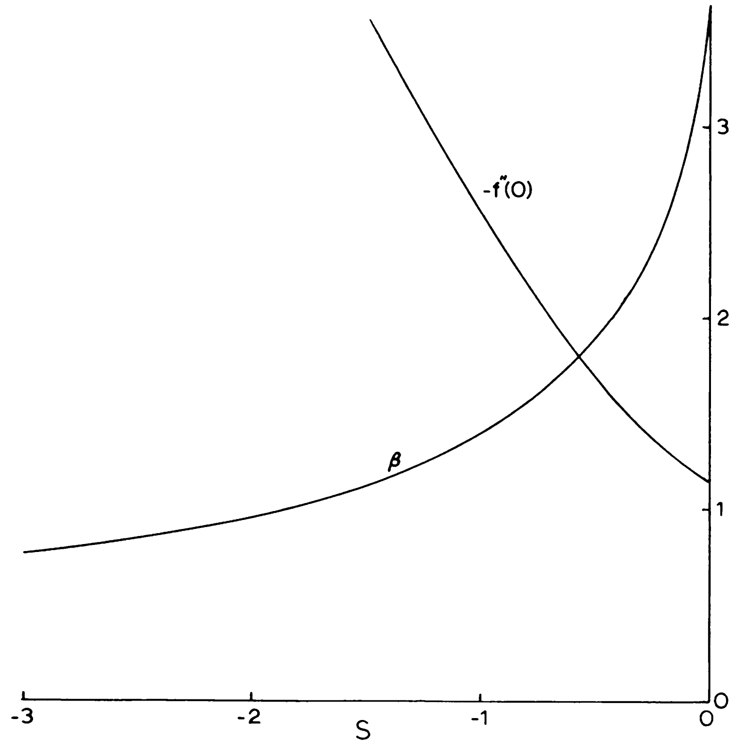


FIG. 5. Decelerated stretching, $S < 0$.

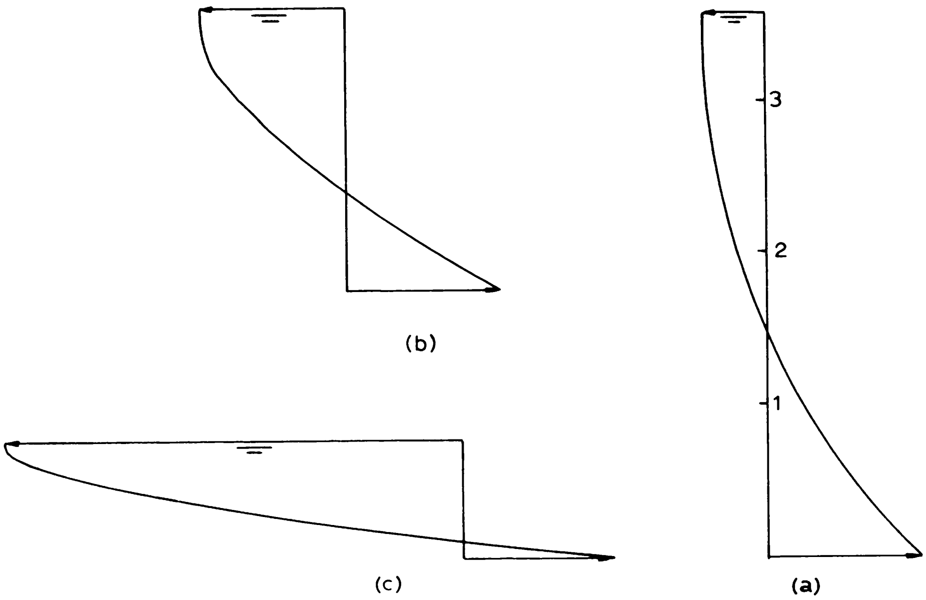


FIG. 6. Lateral velocities for decelerated stretching. (a) $S = 0$, (b) $S = -0.5$, (c) $S = -3$.

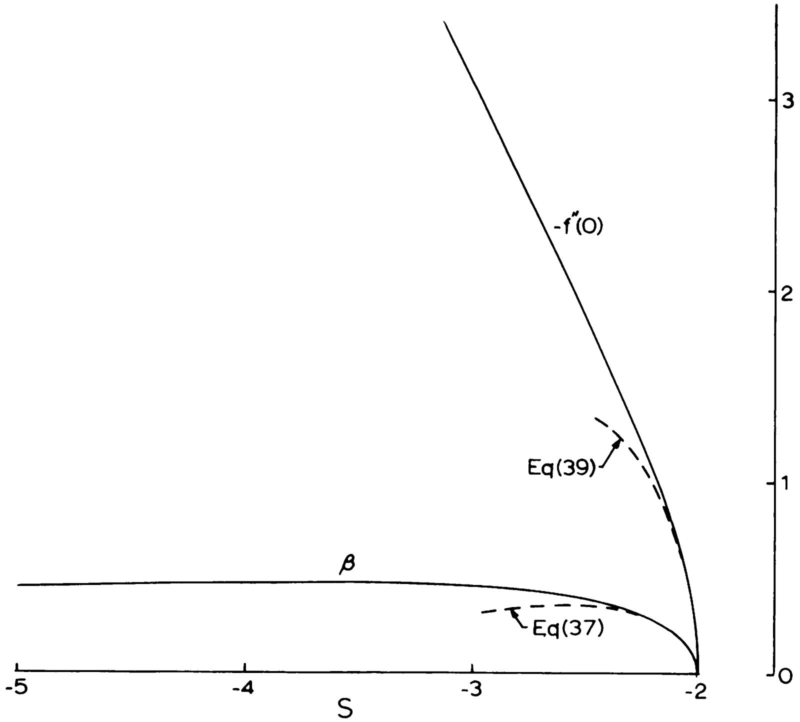


FIG. 7. Decelerated contracting, $S < 0$.

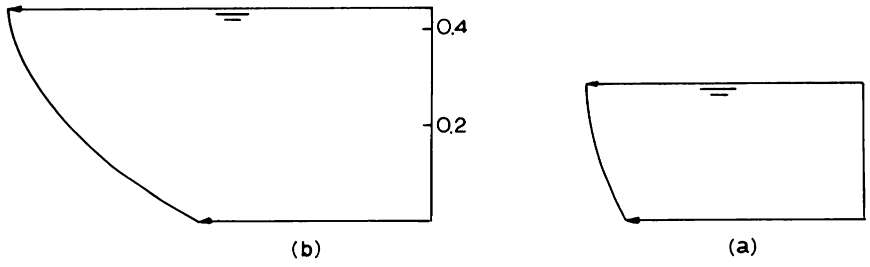


FIG. 8. Lateral velocities for decelerated contracting. (a) $S = -2.2$, (b) $S = -3$.

exact solution but also found another steady solution with finite thickness and reverse flow. No solutions are found for $S > 2$. This only means no *similarity* solutions exist and does not pertain to nonsimilar solutions.

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