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BUCKLING AND BARRELLING INSTABILITIES OF NONLINEARLY ELASTIC COLUMNS

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1. Introduction. The question of predicting what happens to an elastic column when it is compressed uniaxially has long been a subject of investigation (see [9, 17] for a bibliography). We address here the compression problem for a circular cylinder of radius h of compressible hyperelastic material which satisfies appropriate ellipticity and growth hypotheses. We study the mixed displacement-traction boundary value problem for the deformation $\mathbf{x}$ of the column in which the compression ratio (ratio of deformed to undeformed length) is prescribed, the surface traction at its ends has no tangential component, and the lateral surface is specified to be stress free.

Results on the compression of a circular cylinder of general compressible material have previously been obtained by Simpson and Spector. They gave a necessary and sufficient condition for the equilibrium equations of elasticity linearized about $\mathbf{x}_\lambda$ to have an axisymmetric or barrelling solution [17], and showed that such a solution does exist for materials with certain specific stored energy functions [17, 19]. Analogous conditions have been obtained by Ogden [16] for the existence of planar solutions to the linearized equations for a rectangular column, and by Davies [9] who considered the two-dimensional case of an elastic rectangle and showed that this would always have a solution for a general class of materials.

As in previous treatments of this and similar problems we first give conditions which ensure that for each compression ratio $\lambda \in (0, 1)$ there is precisely one homogeneous solution $\mathbf{x}_\lambda$ to the nonlinear problem with diagonal gradient. These are stated in Sec. 2 and apply to columns of general constant cross-section. The stability of $\mathbf{x}_\lambda$ (in a sense made precise in Sec. 3) is then investigated, and the Complementing Condition (cf. Simpson and Spector [20]) for the linearized problem is discussed.

In Sec. 5, results of Ball [5] are used to rigorously obtain an equation for λ_{BAR}, the compression ratio λ at which $\mathbf{x}_\lambda$ can first cease to be a weak local minimum of the deformation energy with respect to barrelling perturbations. Section 6 contains a comparison of λ_{BAR} with λ_{BUC}, the compression ratio corresponding to the onset of planar buckling of a column of square cross-section of side $2h$. (Calculation

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of λ_{BUC} is essentially the same as for the two-dimensional problem in [9] and the relevant results are summarized in Sec. 4.) We show that the relative sizes of λ_{BAR} and λ_{BUC} are governed by both h and a material parameter $\overline{g}$. If $\overline{g}$ is positive, then λ_{BUC} is larger than λ_{BAR} whatever the column width, while if $\overline{g}$ is negative, then λ_{BUC} (respectively λ_{BAR}) is the larger for sufficiently thin (respectively thick) columns. This is illustrated by looking at the model studied by Simpson and Spector [19], for which $\overline{g}$ is negative. The physical meaning of these results is discussed in Sec. 7.

Notation. If $B \subset \mathbb{R}^n$ is open and $1 \le p \le \infty$, then we use $L^p(B;\mathbb{R}^m)$ to denote the Banach space of functions $\mathbf{u}\colon B \to \mathbb{R}^m$ for which $\|\mathbf{u}\|_{0,p} < \infty$, where

$$\|\mathbf{u}\|_{0,p} = \begin{cases} \left[\int_B |\mathbf{u}(\mathbf{X})|^p \, d\mathbf{X}\right]^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{\mathbf{X}\in B} |\mathbf{u}(\mathbf{X})| & p = \infty. \end{cases}$$

The corresponding Sobolev space $W^{1,p}(B;\mathbb{R}^m)$ is the Banach space

$$W^{1,p}(B;\mathbb{R}^m) = \{\mathbf{u} \in L^p(B;\mathbb{R}^m) \colon \|\mathbf{u}\|_{1,p} < \infty\},$$

where $\|\mathbf{u}\|_{1,p} = \|\mathbf{u}\|_{0,p} + \|D\mathbf{u}\|_{0,p}$ (the gradient $D\mathbf{u}$ being understood in the sense of distributions). We also denote the space $W^{1,2}(B;\mathbb{R}^m)$ as $H(B;\mathbb{R}^m)$. If k is a nonnegative integer, then $C^k(B;\mathbb{R}^m)$ denotes the space of k times continuously differentiable functions on B with values in $\mathbb{R}^m$.

2. The nonlinear problem. This section is concerned with the nonlinear boundary value problem corresponding to the uniaxial compression of an elastic cylinder of constant cross-section. It has been shown by Simpson and Spector [17] that mild growth conditions on the stored energy function yield the existence of a solution. We also state hypotheses under which this solution is unique in a particular class.

2.1. *Formulation of the problem.* Consider a homogeneous isotropic compressible hyperelastic material with stored energy function $W \in C^2(M_+^{3\times 3};\mathbb{R})$ (here $M_+^{n\times n}$ denotes the set of $n \times n$ matrices with positive determinant), which occupies the domain $\overline{\Omega} \subset \mathbb{R}^3$. We shall assume W to be bounded below by its value at the identity, which without loss of generality we take to be zero. Material points are represented by Cartesian coordinates $\mathbf{X} = (X_1, X_2, X_3) \in \overline{\Omega}$, and we use Greek indices $\alpha, \beta, \gamma, \ldots$ to denote the components of these reference coordinates. With respect to this coordinate system we take Ω to be

$$\Omega = (0,\pi) \times \mathscr{D},$$

where $\mathscr{D}$ is a convex bounded set in $\mathbb{R}^2$ with boundary $\partial\mathscr{D}$, and denote the lateral surface $(0,\pi)\times\partial\mathscr{D}$ by $\mathscr{B}$. By a *deformation* we mean a function $\mathbf{x} \in [W^{1,1}(\Omega;\mathbb{R}^3)]$ which satisfies the local invertibility condition

$$\det(D\mathbf{x}(\mathbf{X})) > 0 \quad \text{for a.e. } \mathbf{X} \in \Omega.$$

The components of $\mathbf{x}$ are assigned Latin indices i, j, k, etc., and

$$[D\mathbf{x}(\mathbf{X})]_{i\alpha} \equiv x_{i,\alpha} \equiv \frac{\partial x_i(\mathbf{X})}{\partial X_\alpha}, \qquad i, \alpha \in \{1, 2, 3\}.$$

The total energy E associated with the deformation $\mathbf{x}$ is given by

$$E(\mathbf{x}) = \int_\Omega W(D\mathbf{x}(\mathbf{X}))\,d\mathbf{X},$$

and the equilibrium equations of nonlinear elasticity when no body forces are acting are the Euler-Lagrange equations for the functional E, i.e.,

$$\frac{\partial}{\partial X_\alpha}\left[\frac{\partial W}{\partial F_{i\alpha}}(D\mathbf{x}(\mathbf{X}))\right] = 0, \qquad i = 1, 2, 3. \tag{2.1.1}$$

We are interested in studying the problem in which the cylinder is subjected to equal and opposite compressive forces at its ends $X_1 = 0, \pi$, which are assumed to be lubricated to eliminate friction. The lateral surface $\mathscr{B}$ is taken to be stress free. The nonlinear problem for a given compression ratio λ is thus to find a deformation $\mathbf{x}$ which satisfies (2.1.1) and the following boundary conditions:

$$x_1(0, X_2, X_3) = 0, \qquad x_1(\pi, X_2, X_3) = \lambda\pi, \tag{2.1.2}$$

$$\frac{\partial W}{\partial F_{21}}(D\mathbf{x}(\mathbf{X})) = \frac{\partial W}{\partial F_{31}}(D\mathbf{x}(\mathbf{X})) = 0 \quad \text{on } X_1 = 0, \pi, \tag{2.1.3}$$

$$\frac{\partial W}{\partial F_{i2}}(D\mathbf{x}(\mathbf{X}))n_2 + \frac{\partial W}{\partial F_{i3}}(D\mathbf{x}(\mathbf{X}))n_3 = 0 \quad \text{on } \mathscr{B}, \; i = 1, 2, 3, \tag{2.1.4}$$

where $\mathbf{n} = (0, n_2, n_3)$ is the outward unit normal to $\mathscr{B}$ at $\mathbf{X}$.

Here (2.1.3) is the condition corresponding to the ends $X_1 = 0, \pi$ being greased, and (2.1.4) states that the lateral surfaces are free of stress. Notice that if $\mathbf{x}$ solves (2.1.1)–(2.1.4), then so will any deformation of the form $Q\mathbf{x} + \mathbf{c}$, where $\mathbf{c}$ is a constant vector with zero first component and Q is a rotation about the X_1 axis. To eliminate this trivial nonuniqueness we impose the additional constraints

$$\int_\Omega x_2(\mathbf{X})\,d\mathbf{X} = \int_\Omega x_3(\mathbf{X})\,d\mathbf{X} = 0 \tag{2.1.5}$$

and

$$\int_\Omega [x_{2,3}(\mathbf{X}) - x_{3,2}(\mathbf{X})]\,d\mathbf{X} = 0 \tag{2.1.6}$$

on $\mathbf{x}$.

2.2. *Existence of a homogeneous solution.* Following Simpson and Spector [17] we shall give conditions on W which are sufficient to guarantee that the system (2.1.1)–(2.1.6) has a solution for each compression ratio $\lambda \in (0, 1]$. Any homogeneous deformation (one for which $D\mathbf{x}(\mathbf{X})$ is independent of $\mathbf{X}$) solves (2.1.1), and in order to show that there will be a homogeneous solution to (2.1.2)–(2.1.6) we use the following well-known result (cf., e.g., Ball [6]).

Proposition 2.2.1. If $W \in C^2(M_+^{3\times 3}; \mathbb{R})$ is isotropic, then there exists a symmetric function $\Phi \in C^2(\mathbb{R}_+^3; \mathbb{R})$ such that

$$W(F) = \Phi(v_1(F), v_2(F), v_3(F)),$$

where $v_i(F)$, $i = 1, 2, 3$, are the singular values of F (the eigenvalues of $\sqrt{F^TF}$).

Moreover, if $F = \operatorname{diag}[v_1, v_2, v_3]$ for $v_i > 0$, $i = 1, 2, 3$, then

$$\frac{\partial W}{\partial F}(F) = \operatorname{diag}[\Phi_{,1}(\mathbf{v}), \Phi_{,2}(\mathbf{v}), \Phi_{,3}(\mathbf{v})],$$

where $\mathbf{v} = (v_1, v_2, v_3)$ and $\Phi_{,i} = \partial\Phi(\mathbf{v})/\partial v_i$.

This implies that any deformation $\mathbf{x}$ of the form $\mathbf{x}(\mathbf{X}) = (\lambda X_1, v_2 X_2, v_3 X_3)$ for positive constants v_2 and v_3 will satisfy (2.1.1)–(2.1.3) and reduce (2.1.4) to the problem of finding $v_2, v_3 > 0$ such that

$$\Phi_{,2}(\lambda, v_2, v_3) = \Phi_{,3}(\lambda, v_2, v_3) = 0. \tag{2.2.1}$$

If Φ is assumed to satisfy the physically reasonable hypothesis

(H1) $\Phi(v_1, v_2, v_3) \to \infty$ if any of the arguments v_i, $i = 1, 2, 3$, become zero or infinite,

then it can be shown (see [17]) that for each $\lambda \in (0, 1)$ there is a constant $v = v(\lambda)$ for which (2.2.1) holds when $v_2 = v_3 = v$. We now impose some extra conditions on Φ which guarantee that this is the only homogeneous solution of (2.1.1)–(2.1.4) with diagonal gradient. Suppose that the following additionally hold:

(H2) The identity is a natural state, i.e.,

$$\Phi_{,i}(1, 1, 1) = 0 \quad \text{for } i = 1, 2, 3;$$

(H3) Φ satisfies the Baker-Ericksen inequalities,

$$\alpha_{ij} > 0 \quad \text{for } i, j \in \{1, 2, 3\}, \ i \neq j,$$

where

$$\alpha_{ij} = \begin{cases} (v_i \Phi_{,i} - v_j \Phi_{,j})/(v_i^2 - v_j^2) & \text{if } v_i \neq v_j, \\ (v\Phi_{,ii} - v\Phi_{,ij} + \Phi_{,i})/2v & \text{if } v_i = v_j = v; \end{cases} \tag{2.2.2}$$

and

(H4) $\Phi_{,22}(\lambda, v, v) + \Phi_{,23}(\lambda, v, v) > 0$ for $\lambda \in (0, 1]$ and $v > 0$.

Hypothesis (H3) implies that any solution (v_2, v_3) of (2.2.1) must satisfy $v_2 = v_3 = v$ for some $v > 0$, and that $\Phi_{,22} - \Phi_{,23} > 0$, where Φ is evaluated at (λ, v, v). For fixed λ define $\phi(v) = \Phi(\lambda, v, v)$; then

$$\phi'(v) = 2\Phi_{,2}(\lambda, v, v)$$

and so solving (2.2.1) is equivalent to finding $v > 0$ for which $\phi'(v) = 0$. As has already been pointed out, the existence of such a v follows directly from (H1). If (H2) and the technical assumption (H4) hold, then the Implicit Function Theorem yields that v is unique and is equal to 1 when $\lambda = 1$. Hence we have the following

THEOREM 2.2.1. If Φ satisfies the constitutive assumptions (H1)–(H4), then there exists a $v \in C^1((0, 1]; \mathbb{R}^+)$ such that for each $\lambda \in (0, 1]$ the deformation $\mathbf{x}_\lambda$ defined by

$$\mathbf{x}_\lambda(\mathbf{X}) = (\lambda X_1, v(\lambda) X_2, v(\lambda) X_3) + \mathbf{c}_\lambda \tag{2.2.3}$$

for some constant $\mathbf{c}_\lambda$ is the unique homogeneous solution of (2.1.1)–(2.1.6) with diagonal deformation gradient. In addition v satisfies $v(1) = 1$.

We shall additionally require Φ to satisfy

(H5) Φ is strongly elliptic at the deformation $\mathbf{x}_\lambda$ defined in (2.2.3) for each $\lambda \in (0, 1]$.

A function $S\colon M_+^{n\times n} \to \mathbb{R}$ is said to be *strongly elliptic* at $F \in M_+^{n\times n}$ if

$$\frac{\partial^2 S(F)}{\partial F_{i\alpha}\partial F_{j\beta}} a_i a_j b_\alpha b_\beta > 0 \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n/\{\mathbf{0}\}.$$

Knowles and Sternberg [13, 14] were the first to give necessary and sufficient conditions for the strong ellipticity of an elastic material in plane strain (corresponding to $n = 2$). One of these conditions has subsequently been simplified by Aubert and Tahraoui [4] (see also [10, 16]). These conditions are obviously necessary for the strong ellipticity of the three-dimensional stored energy function considered here and we state this as the next proposition.

PROPOSITION 2.2.2 [13, 14]. Necessary conditions for the function Φ to be strongly elliptic at $(\lambda, v, v) \in \mathbb{R}_+^3$ are:

(i) $\alpha_{ij} > 0$,
(ii) $\Phi_{,11} > 0$, $\Phi_{,22} > 0$, and
(iii)

$$\left.\begin{aligned} \sqrt{\Phi_{,11}\Phi_{,22}} + \Phi_{,12} + (\Phi_{,1} - \Phi_{,2})/(\lambda - v) > 0 \\ \sqrt{\Phi_{,11}\Phi_{,22}} - \Phi_{,12} + (\Phi_{,1} + \Phi_{,2})/(\lambda + v) > 0 \end{aligned}\right\} \quad \text{for } \lambda \neq v,$$

where the derivatives of Φ are evaluated at (λ, v, v).

We would expect that uniaxial compression of the cylinder would not lead to it contracting radially whilst the lateral surfaces remain stress free. It is thus reasonable to assume also

(H6) The function $v(\lambda)$ defined by $\Phi_{,2}(\lambda, v, v) = 0$ satisfies $v'(\lambda) \leq 0$ for $\lambda \in (0, 1]$.

For convenience we introduce a simplifying notation and then collect together consequences of hypotheses (H4)–(H6) in Theorem 2.2.2. For $\lambda \in (0, 1]$ define

$$a(\lambda) = \Phi_{,11}, \quad b(\lambda) = \Phi_{,12}, \quad c(\lambda) = \Phi_{,22}, \quad d(\lambda) = \Phi_{,23},$$
$$\theta(\lambda) = \Phi_{,1}/(\lambda^2 - v^2), \quad \alpha(\lambda) = \alpha_{12} = \lambda\theta(\lambda), \quad \beta(\lambda) = v(\lambda)\theta(\lambda),$$

where all derivatives of Φ are evaluated at $(\lambda, v(\lambda), v(\lambda))$.

THEOREM 2.2.2. If (H4)–(H6) hold, then the following inequalities are satisfied for each $\lambda \in (0, 1]$:

$$v(\lambda) \geq 1, \quad (\text{with equality only at } \lambda = 1),$$
$$a > 0, \qquad c > 0, \qquad \beta > \alpha > 0,$$
$$c^2 - d^2 > 0, \qquad v'(\lambda) = -b/(c + d) \leq 0,$$
$$\sqrt{ac} + b + (\lambda + v)\theta > 0, \qquad \sqrt{ac} - b + (\lambda - v)\theta > 0.$$

3. The linearized problem.

3.1. *Failure of* $\mathbf{x}_\lambda$ *to be a weak local energy minimizer.* In the previous section we gave hypotheses under which the nonlinear problem (2.1.1)–(2.1.6) has the solution $\mathbf{x}_\lambda$ defined in (2.2.3). We are now interested in finding the first (i.e., largest) value of λ for which $\mathbf{x}_\lambda$ ceases to be a weak local minimizer of the energy.

DEFINITION. The deformation $\mathbf{x}_\lambda$ is said to be a weak local minimizer of E if

$$E(\mathbf{x}_\lambda) \le E(\mathbf{x}_\lambda + \mathbf{u})$$

for all $\mathbf{u} \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ with sufficiently small norm satisfying

$$u_1 = 0 \quad \text{on } X_1 = 0, \pi, \tag{3.1.1}$$

$$\int_\Omega u_2\, d\mathbf{X} = \int_\Omega u_3\, d\mathbf{X} = 0, \tag{3.1.2}$$

$$\int_\Omega (u_{2,3} - u_{3,2})\, d\mathbf{X} = 0, \tag{3.1.3}$$

and

$$\det(D(\mathbf{x}_\lambda + \mathbf{u})) > 0 \quad \text{a.e. in } \Omega .$$

It is shown in Proposition 3.1.2 below that under some technical assumptions $\mathbf{x}_\lambda$ will minimize E in this sense if the second variation $D^2E(\mathbf{x}_\lambda)$ is strictly positive on the space

$$H \equiv \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^3) \colon \mathbf{u} \text{ satisfies } (3.1.1)\text{–}(3.1.3)\}.$$

For $\mathbf{u} \in H$ we define $J_\lambda[\mathbf{u}] = D^2E(\mathbf{x}_\lambda)(\mathbf{u}, \mathbf{u})$. Then

$$J_\lambda[\mathbf{u}] = \int_\Omega \mathscr{A}_{ij}^{\alpha\beta}(\lambda) u_{i,\alpha} u_{j,\beta}\, d\mathbf{X},$$

where the coefficients

$$\mathscr{A}_{ij}^{\alpha\beta}(\lambda) \equiv \frac{\partial^2 W(D\mathbf{x}_\lambda)}{\partial F_{i\alpha} \partial F_{j\beta}}$$

can be computed by using the following result (see, e.g., Ball [6]).

PROPOSITION 3.1.1. If $G \in M_+^{3\times 3}$ and $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}_+^3$, then

$$D_F^2 W(\operatorname{diag}(\mathbf{v}))(G, G) = \sum_{i,j=1}^{3} \Phi_{,ij}(\mathbf{v}) G_{ii} G_{jj} + \sum_{\substack{i,j=1 \\ i \ne j}}^{3} [\alpha_{ij} (G_{ij})^2 + \beta_{ij} G_{ij} G_{ji}],$$

where α_{ij} is defined by (2.2.2) and

$$\beta_{ij} = \begin{cases} (v_j \Phi_{,i}(\mathbf{v}) - v_i \Phi_{,j}(\mathbf{v}))/(v_i^2 - v_j^2) & \text{if } v_i \ne v_j, \\ (v_i \Phi_{,ii}(\mathbf{v}) - v \Phi_{,ij}(\mathbf{v}) - \Phi_{,i}(\mathbf{v}))/2v & \text{if } v_i = v_j = v. \end{cases}$$

Applying this result to our particular case (in which $v_2 = v_3 = v(\lambda) \ne \lambda = v_1$) gives

$$A_{ij}^{\alpha\beta}(\lambda) G_{i\alpha} G_{j\beta} = \mathbf{G}^T M \mathbf{G},$$

where $\mathbf{G}^T = (G_{11}, G_{22}, G_{33}, G_{12}, G_{21}, G_{13}, G_{31}, G_{23}, G_{32})$ and M is the symmetric 9×9 matrix with entries $M_{11} = a$, $M_{12} = M_{13} = b$, $M_{22} = M_{33} = c$, $M_{23} = d$, $M_{44} = M_{55} = M_{66} = M_{77} = \alpha$, $M_{45} = M_{67} = \beta$, $M_{88} = M_{89} = M_{99} = \frac{1}{2}(c-d)$, $M_{ij} = 0$ otherwise if $i < j$.

Hence $J_\lambda[\mathbf{u}] = 0$ if and only if $\mathbf{u} \in H$ is a weak solution of the system of linearized equations,

$$\left.\begin{aligned} au_{1,11} + \alpha(u_{1,22} + u_{1,33}) + (\beta + b)(u_{2,12} + u_{3,13}) = 0 \\ \alpha u_{2,11} + cu_{2,22} + \tfrac{1}{2}(c-d)u_{2,33} + (\beta + b)u_{1,12} + \tfrac{1}{2}(c+d)u_{3,23} = 0 \\ \alpha u_{3,11} + \tfrac{1}{2}(c-d)u_{3,22} + cu_{3,33} + (\beta + b)u_{1,13} + \tfrac{1}{2}(c+d)u_{2,23} = 0 \end{aligned}\right\}, \quad \mathbf{X} \in \Omega, \tag{3.1.4}$$

and boundary conditions,

$$\left.\begin{aligned} \alpha u_{2,1} + \beta u_{1,2} = 0 \\ \alpha u_{3,1} + \beta u_{1,3} = 0 \end{aligned}\right\} \quad \text{on } X_1 = 0, \pi, \tag{3.1.5}$$

$$\left.\begin{aligned} (\alpha u_{1,2} + \beta u_{2,1})n_2 + (\alpha u_{1,3} + \beta u_{3,1})n_3 = 0 \\ (bu_{1,1} + cu_{2,2} + du_{3,3})n_2 + \tfrac{1}{2}(c-d)(u_{3,2} + u_{2,3})n_3 = 0 \\ \tfrac{1}{2}(c-d)(u_{2,3} + u_{3,2})n_2 + (bu_{1,1} + du_{2,2} + cu_{3,3})n_3 = 0 \end{aligned}\right\} \quad \text{on } \mathscr{B}, \tag{3.1.6}$$

when $\mathbf{n} = (0, n_2, n_3)$ is the outward unit normal to $\mathscr{B}$.

It was first shown by van Hove [12] that $\mathbf{x}_\lambda$ is a weak local minimizer of E if J_λ is uniformly positive on H (i.e, if there exists $C > 0$ such that $J_\lambda[\mathbf{u}] \geq C\|\mathbf{u}\|_{1,2}^2 \ \forall\, \mathbf{u} \in H$). It is also well known that if the boundary $\partial\Omega$ of Ω is smooth, then (because W is assumed to be strongly elliptic at the deformation $\mathbf{x}_\lambda$) strict positivity and uniform positivity of J_λ are equivalent, provided that the Complementing Condition (cf. Simpson and Spector [20], and see 3.2 below) holds for the boundary value problem (3.1.4)–(3.1.6). Because of the particular structure of equation (3.1.4) and boundary conditions (3.1.5) this is also true for the region Ω considered here, for which $\partial\Omega$ is not smooth. The proof of this is essentially the same as for the two-dimensional case studied in [9] and is omitted. We thus have the following.

PROPOSITION 3.1.2. The solution $\mathbf{x}_\lambda$ is a weak local minimizer of E in H if the Complementing Condition holds for the boundary value problem (3.1.4)–(3.1.6) and $J_\lambda[\mathbf{u}] > 0$ for all $\mathbf{u} \in H/\{\mathbf{0}\}$. Conversely, if there exists $\mathbf{u} \in H$ with $J_\lambda[\mathbf{u}] < 0$, then $\mathbf{x}_\lambda$ is not a weak local minimizer of E.

3.2. *The Complementing Condition.* We first give a general definition of the Complementing Condition and then discuss it for the particular boundary value problem (3.1.4)–(3.1.6). A good, comprehensive description of this condition and its consequences in nonlinear elasticity is given in the 1987 paper of Simpson and Spector [20].

Let $V \subset \mathbb{R}^n$ and consider the problem

$$\left.\begin{aligned} a\mathbf{u} = \mathbf{0} \quad & \text{in } V \\ b\mathbf{u} = \mathbf{0} \quad & \text{on } \partial V \end{aligned}\right\}, \tag{3.2.1}$$

$$\left.\begin{aligned}(a\mathbf{u})_i &= a_{ij}^{\alpha\beta} u_{j,\alpha\beta} + a_{ij}^{\alpha} u_{j,\alpha} + a_{ij} u_j \\ (b\mathbf{u})_i &= a_{ij}^{\alpha\beta} u_{j,\beta} N_\alpha + b_{ij}^{\alpha} u_j N_\alpha\end{aligned}\right\}, \qquad i = 1, \ldots, n,$$

where the coefficients are allowed to depend on $\mathbf{X} \in \mathbb{R}^n$, and $\mathbf{N}$ is the outer unit normal to ∂V. The Complementing Condition is given in terms of an auxiliary problem defined on the half-space $H = \{\mathbf{y} \in \mathbb{R}^n : y_n > 0\}$ of $\mathbb{R}^n$ bounded by the plane $\partial H = \{\mathbf{y} \in \mathbb{R}^n : y_n = 0\}$. Fix $\mathbf{X}_0 \in \partial V$ and let $A_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(\mathbf{X}_0)$. Then the associated problem is to find a function $\mathbf{v}: H \to \mathbb{R}^n$ satisfying

$$\left.\begin{aligned}A_{ij}^{\alpha\beta} v_{j,\alpha\beta} &= 0 \text{ in } H \\ A_{ij}^{n\beta} v_{j,\beta} &= 0 \text{ on } \partial H\end{aligned}\right\}, \qquad i = 1, \ldots, n. \tag{3.2.2}$$

Definition. The boundary value problem (3.2.1) is said to satisfy the Complementing Condition at $\mathbf{X}_0 \in \partial V$ if for $\mathbf{k} \in \mathbb{R}^n/\{\mathbf{0}\}$, $k_n = 0$, the only solution $\mathbf{v}$ of (3.2.2) of the form

$$\mathbf{v}(\mathbf{y}) = \exp(i\mathbf{k} \cdot \mathbf{y})\mathbf{w}(y_n)$$

is $\mathbf{w} \equiv \mathbf{0}$, where $\mathbf{w} \in C^\infty([0, \infty); \mathbb{R}^n)$ with $\sup\{|\mathbf{w}(t)| : t \in \mathbb{R}\} < \infty$.

If all the coefficients of $\mathbf{a}$ are smooth and the system is elliptic, then the usual arguments (see, e.g., [2, 3, 11]) guarantee interior regularity of weak solutions of (3.2.1). Regularity at the boundary is provided by the Complementing Condition if ∂V is sufficiently smooth. We shall use the particular result given below in Sec. 5.1.

Proposition 3.2.1. Suppose ∂V is smooth and that the Complementing Condition in satisfied at each point of ∂V. If the coefficients $a_{ij}^{\alpha\beta}$, a_{ij}^{α}, a_{ij}, and b_{ij}^{α} are constant, then any weak solution of (3.2.1) will be a classical solution.

We would like to be able to show that the Complementing Condition holds for the boundary value problem (3.1.4)–(3.1.6) (at least for values of λ which are sufficiently far away from zero), and hence compute the first value of λ for which there is a nonzero $\mathbf{u} \in H$ with $J_\lambda[\mathbf{u}] = 0$. Unfortunately it is not feasible to show that this is the case for a general material for any $\mathscr{D}$. Even if it were, the usual method of writing any weak solution $\mathbf{u}$ of (3.1.4)–(3.1.6) as a Fourier series expansion in the axial variable would result in a system of partial differential equations and hence it would not be possible to rigorously find the *first* value of λ for which these have a nontrivial solution. This contrasts with the two-dimensional problem discussed in [9] for which the values of λ where the Complementing Condition holds can be determined. Also, the resulting system after the Fourier series decomposition is one of ordinary differential equations for which solutions can be computed explicitly.

What we shall do here is to look for values of λ which correspond to J_λ ceasing to be uniformly positive with respect to specific types of perturbations when $\mathscr{D}$ is chosen to be either circular or square. In Sec. 5 we suppose $\mathscr{D}$ to be a circle and consider the case for which $\mathbf{u}$ is restricted to be an axisymmetric (barrelling) perturbation. The following section contains a summary of results from [9] corresponding to plane strain instabilities of a square cylinder.

4. Buckling of a square column. We take Ω to be the square column of side $2h$ for which $\mathscr{D} = (-h, h) \times (-h, h)$, and look for the largest value of λ at which $J_\lambda[\mathbf{u}]$

ceases to be uniformly positive when $\mathbf{u} \in H$ is of the form

$$u_i(\mathbf{X}) = v_i(X_1, X_2), \quad i = 1, 2, \qquad u_3(\mathbf{X}) = 0. \tag{4.1}$$

The physical interpretation given to this is that the column is constrained from moving in the X_3-direction by means of two perfectly smooth parallel plates along the planes $X_3 = \pm h$. This will result in the slight modification of the boundary value problem (3.1.4)–(3.1.6)—we shall drop the requirement that the normal stress is zero on $\mathscr{B}$ when $X_3 = \pm h$.

Set $\Omega_1 = (0, \pi) \times (-h, h)$, and let

$$H_{\mathrm{BUC}} = \{\mathbf{v} \in H^1(\Omega_1; \mathbb{R}^2) : v_1 = 0 \quad \text{on } X_1 = 0, \pi, \ \textstyle\int_{\Omega_1} v_2(\mathbf{X})\, d\mathbf{X} = 0\}.$$

If $\mathbf{u} \in \mathbb{R}^3$ and $\mathbf{v} \in \mathbb{R}^2$ are related by (4.1) and $\mathbf{u} \in H$, then $\mathbf{v} \in H_{\mathrm{BUC}}$. Also, $J_\lambda[\mathbf{u}] = P_\lambda[\mathbf{v}]$, where

$$P_\lambda[\mathbf{v}] = \int_{\Omega_1} \mathscr{A}_{ij}^{\alpha\beta}(\lambda) v_{i,\alpha} v_{j,\beta}\, dX_1\, dX_2,$$

the summation being carried out for $\alpha, \beta, i, j \in \{1, 2\}$. The condition that J_λ be uniformly positive on the set of functions $\mathbf{u}$ in H given by (4.1) is equivalent to the uniform positivity of P_λ on H_{BUC}. This follows from the strict positivity of P_λ provided the Complementing Condition holds for the corresponding boundary value problem. Consequently if this condition holds, $\mathbf{x}_\lambda$ can first fail to be a weak local minimizer at E with respect to planar perturbations $\mathbf{u}$ given by (4.1) when $P_\lambda[\mathbf{v}]$ is first zero for some nonzero $\mathbf{v} \in H_{\mathrm{BUC}}$. That is, $\mathbf{v}$ is a weak solution to the system

$$\left.\begin{aligned} a v_{1,11} + \alpha v_{1,22} + (b + \beta) v_{2,12} = 0 \\ \alpha v_{2,11} + c v_{2,22} + (b + \beta) v_{1,12} = 0 \end{aligned}\right\} \quad \text{in } \Omega_1, \tag{4.2}$$

and boundary conditions

$$v_{2,1} = 0 \quad \text{on } X_1 = 0, \pi, \tag{4.3}$$

$$\left.\begin{aligned} \alpha v_{1,2} + \beta v_{2,1} = 0 \\ b v_{1,1} + c v_{2,2} = 0 \end{aligned}\right\} \quad \text{on } X_2 = \pm h. \tag{4.4}$$

This is essentially the same problem as that studied in [9], the only differences being that here boundary conditions (4.4) are prescribed on $X_2 = \pm h$ instead of at $X_2 = 0$, $X_2 = h$, and that $\mathbf{\Phi}$ is now a function of three variables. We accordingly list only the relevant results from [9] and omit the proofs. It is first necessary to introduce some new notation.

We denote by λ_{BUC} the largest value λ for which $P_\lambda[\mathbf{v}]$ is zero for some $\mathbf{v} \in H_{\mathrm{BUC}}/\{\mathbf{0}\}$. Define $\Delta(\lambda)$ and $p(\lambda)$ by

$$\Delta = \sqrt{ac} - b - \alpha - \beta, \tag{4.5}$$

$$p = (a/c)^{1/4}, \tag{4.6}$$

and let $p_1(\lambda)$ and $p_2(\lambda)$ be the roots of the quartic

$$\alpha c q^4 + [(b + \beta)^2 - ac - \alpha^2] q^2 + \alpha a = 0, \tag{4.7}$$

given by

$$p_1 = \left\{ \frac{ac + \alpha^2 - (b+\beta)^2 - [P\Delta]^{1/2}}{2\alpha c} \right\}^{1/2},$$
$$p_2 = \left\{ \frac{ac + \alpha^2 + (b+\beta)^2 + [P\Delta]^{1/2}}{2\alpha c} \right\}^{1/2}, \tag{4.8}$$

where

$$P = [\sqrt{ac} + b + \alpha + \beta][\sqrt{ac} - b + \alpha - \beta][\sqrt{ac} + b - \alpha + \beta]$$

is positive because of the strong ellipticity of Φ (cf. Theorem 2.2.2). If $\Delta(\lambda) = 0$, then (4.8) simplifies to give $p_1 = p_2 = p$. We additionally define

$$\psi_1(\lambda) = \alpha(ac - b^2) + \sqrt{ac}(\alpha^2 - \beta^2), \tag{4.9}$$
$$\psi_2(\lambda) = \beta(ac - b^2) + b(\alpha^2 - \beta^2), \tag{4.10}$$

and

$$E(\lambda, t) = \begin{cases} \dfrac{2p(e^{2p_2 t} - e^{2p_1 t})}{(p_2 - p_1)\sqrt{(e^{4p_2 t} - 1)(e^{4p_1 t} - 1)}} & \text{if } \Delta \neq 0, \\ \dfrac{4pt\, e^{2pt}}{e^{4pt} - 1} & \text{if } \Delta = 0. \end{cases} \tag{4.11}$$

PROPOSITION 4.1 [9]. 1. There is a solution $\mathbf{v} \in H_{\lambda_{\mathrm{BUC}}}/\{\mathbf{0}\}$ to (4.2)–(4.4) if and only if λ satisfies

$$\psi_1(\lambda)/\psi_2(\lambda) = \pm E(\lambda, kh) \tag{4.12}$$

for some positive integer k.

2. The Complementing Condition fails for the boundary value problem (4.2)–(4.4) if and only if $\psi_1(\lambda) = 0$.

It is shown in [9] that $\psi_1(1) > 0$ and (4.12) has a solution λ for any kh provided that $\psi_1(\lambda)$ is negative for λ sufficiently close to zero. The extra hypotheses on Φ used to deduce this for the two-dimensional case in [9], that $v'(\lambda) \to -\infty$ as $\lambda \to 0^+$ and that the growth of $\Phi(\lambda, v(\lambda))$ is at most polynomial in $1/\lambda$ as λ tends to zero, are not sufficient when Φ is a function of three variables. In two dimensions the proof relies upon the fact that $d\Phi_{,1}/d\lambda$ is a multiple of $ac - b^2$, which is no longer true here. This can be overcome by assuming for example that a and $d\Phi_{,1}/d\lambda$ are of the same order of magnitude when λ is small (which is to be expected in practice). Instead of assuming specific technical hypotheses on Φ of this type we will just suppose that there is a solution in $(0, 1]$ to

$$\psi_1(\lambda) = 0 \tag{4.13}$$

and denote the largest such solution by $\bar{\lambda}$.

Then

$$\psi_1(\lambda) > 0 \quad \text{for} \quad \lambda \in (\bar{\lambda}, 1] \tag{4.14}$$

and we have the following:

PROPOSITION 4.2 [9]. Assume that there is a solution $\bar{\lambda}$ to (4.13). Then there exists $\lambda_{\mathrm{BUC}} \in (\bar{\lambda}, 1)$ which is the largest solution λ to

$$F_{\mathrm{BUC}}(\lambda, kh) = 0 \tag{4.15}$$

for some $k \in \mathbb{Z}^+$, where

$$F_{\mathrm{BUC}}(\lambda, t) = \psi_1(\lambda)/\psi_2(\lambda) - E(\lambda, t). \tag{4.16}$$

If in addition $\Delta(\lambda) \geq 0$ for all $\lambda \in (\bar{\lambda}, 1]$, then λ_{BUC} satisfies $F_{\mathrm{BUC}}(\lambda_{\mathrm{BUC}}, h) = 0$, i.e., it solves (4.15) when $k = 1$).

It is also shown in [9] that if λ satisfies (4.12), then the corresponding solution $\mathbf{v} \in H_{\mathrm{BUC}}$ of (4.2)–(4.4) is *classical*, and if $\Delta(\lambda) \geq 0$ then $\mathbf{v}$ satisfies

$$v_2(X_1, X_2) = v_2(X_1, -X_2) \quad \text{for } (X_1, X_2) \in \Omega_1$$

and hence it is a true buckling (and not a two-dimensional barrelling) type perturbation.

5. Barrelling of a circular cylinder. Throughout this section we shall assume $\mathscr{D}$ to be the circular disc

$$\mathscr{D} = \{(X_2, X_3): R < h, \text{ where } R(X_2, X_3) \equiv \sqrt{X_2^2 + X_3^2}\}.$$

5.1. *Axisymmetric perturbations.* Here we shall be concerned with finding the largest value of λ for which there is a nonzero solution $\mathbf{u} \in H$ to $J_\lambda[\mathbf{u}] = 0$ of the form

$$\mathbf{u}(X) = \left(u(X_1, R), \frac{X_2}{R} r(X_1, R), \frac{X_3}{R} r(X_1, R)\right), \tag{5.1.1}$$

where $u, r: \mathscr{R} \equiv (0, \pi) \times (0, h) \to \mathbb{R}$.

The first step is to show that $u, r, \in H^1(\mathscr{R}; \mathbb{R})$ and that they are stationary points of S_λ, the axisymmetric form of J_λ. In order to do this we need the technical result of Ball [5] given below.

PROPOSITION 5.1.1 (Ball [5]). Let $B = \{(Y_1, Y_2): Y_1^2 + Y_2^2 \leq 1\}$ and suppose that

$$\mathbf{y}(\mathbf{Y}) = \frac{S(R)}{R}\mathbf{Y}, \qquad R = |\mathbf{Y}|.$$

If $\mathbf{y} \in W^{1,p}(B; \mathbb{R}^2)$ for some $p \in [1, \infty)$, then S is absolutely continuous on $(0, 1)$,

$$\int_0^1 R\left\{[S'(R)]^2 + \left[\frac{S(R)}{R}\right]^2\right\}^{p/2} dR = \frac{1}{2\pi}\int_B |D\mathbf{y}(\mathbf{Y})|^p \, d\mathbf{Y},$$

and the weak derivatives of y are given by

$$D\mathbf{y}(\mathbf{Y}) = \frac{S(R)}{R}\mathbf{1} + \frac{\mathbf{Y} \otimes \mathbf{Y}}{R^3}[RS'(R) - S(R)] \quad \text{a.e. } \mathbf{Y} \in B.$$

We thus have:

LEMMA 5.1.1. Let $\mathbf{u} \in H$ be given by (5.1.1) for $\mathbf{X} \in \Omega$. Then $(u, r) \in H_{\mathrm{BAR}}$, where $H_{\mathrm{BAR}} = \{(w_1, w_2): \mathbf{w} \in H^1(\mathscr{R}; \mathbb{R}^2),\ w_1 = 0 \text{ at } X_1 = 0, \pi, \text{ and } \int_{\mathscr{R}} w_2\, dX_1\, dR = 0\}$.

Upon substitution of the weak derivatives of $\mathbf{u}$ in terms of u and r given by Proposition 5.1.1 we see that $J_\lambda[\mathbf{u}] = S_\lambda[u, r]$, where

$$S_\lambda[u, r] = 2\pi \int_{\mathscr{R}} R \begin{pmatrix} au_{,1}^2 + \alpha(u_{,R}^2 + r_{,1}^2) + 2\beta r_{,1} u_{,R} + 2bu_{,1}(r_{,R} + r/R) \\ + c(r_{,R}^2 + (r/R)^2) + 2drr_{,R}/R \end{pmatrix} dX_1\, dR.$$

The condition that J_λ is uniformly positive on the set of functions $\mathbf{u} \in H$ given by (5.1.1) is equivalent to S_λ being uniformly positive for $(u, r) \in H_{\mathrm{BAR}}$. The coefficients of the boundary value problem associated with $S_\lambda[u, r] = 0$ are not smooth throughout $\mathscr{R}$. So in order to show that the strict positivity of S_λ implies that it is uniformly positive we need interior regularity as well as regularity at the boundary of weak solutions. Regularity at the boundary is given by the Complementing Condition which is discussed in Sec. 5.2, where we show that it fails only at those values of λ satisfying $\psi_1(\lambda) = 0$ (ψ_1 is defined by (4.9)). We now prove interior regularity of solutions to $S_\lambda[u, r] = 0$.

A similar argument to that used in [9] shows that if $S_\lambda[u, r] = 0$ then u (respectively r) is a linear combination of the functions u^k (respectively r^k) given by

$$u^k(X_1, R) = u_k(R) \sin kX_1,$$
$$r^k(X_1, R) = r_k(R) \cos kX_1 \quad \text{a.e. in } \mathscr{R}.$$

Here $u_k, r_k \in H^1((0, h); \mathbb{R})$, $r_k/R \in L^2((0, h); \mathbb{R})$, and u_k, r_k are weak solutions of the ordinary differential system

$$\begin{pmatrix} ak^2 - \alpha D^2 - \alpha R^{-1} D & (b+\beta)k(D + R^{-1}) \\ (b+\beta)kD & c(D^2 + R^{-1}D - R^{-2}) - \alpha k^2 \end{pmatrix} \begin{pmatrix} u_k \\ r_k \end{pmatrix} = \mathbf{0} \quad \text{in } (0, h), \tag{5.1.2$_k$}$$

and boundary conditions

$$\begin{pmatrix} \alpha D & -\beta k \\ bk & cD + dR^{-1} \end{pmatrix} \begin{pmatrix} u_k \\ r_k \end{pmatrix} = \mathbf{0} \quad \text{on } R = h, \tag{5.1.3$_k$}$$

where $D \equiv d/dR$. That any weak solution to this system must be a classical (in fact a smooth) solution can be seen by considering a solution $\mathbf{u} = (v_1 \sin kX_1, v_2 \cos kX_1, v_3 \cos kX_1)$, $\mathbf{v} = \mathbf{v}(X_2, X_3)$ of (3.14)–(3.1.6). If (u_k, r_k) is a weak solution to $(5.1.2)_k$–$(5.1.3)_k$, then $\mathbf{v}$ given by

$$\mathbf{v}(X_2, X_3) = \left(u_k(R), \frac{X_2 r_k(R)}{R}, \frac{X_3 r_k(R)}{R}\right) \quad \text{for a.e. } (X_2, X_3) \in \mathscr{D}$$

weakly solves

$$\left.\begin{aligned} \alpha(v_{1,22} + v_{1,33}) - (b+\beta)k(v_{2,2} + v_{3,3}) - ak^2 v_1 &= 0 \\ cv_{2,22} + \tfrac{1}{2}(c-d)v_{2,33} + \tfrac{1}{2}(c+d)v_{3,23} + (b+\beta)kv_{1,2} - \alpha k^2 v_2 &= 0 \\ cv_{3,33} + \tfrac{1}{2}(c-d)v_{3,22} + \tfrac{1}{2}(c+d)v_{2,23} + (b+\beta)kv_{1,3} - \alpha k^2 v_3 &= 0 \end{aligned}\right\} \quad \text{in } \mathscr{D}, \tag{5.1.4}$$

with boundary conditions

$$\left.\begin{aligned} (\alpha v_{1,2} - \beta k v_2)n_2 + (\alpha v_{1,3} - \beta k v_3)n_3 = 0 \\ (c v_{2,2} + d v_{3,3} + b k v_1)n_2 + \tfrac{1}{2}(c-d)(v_{3,2} + v_{2,3})n_3 = 0 \\ \tfrac{1}{2}(c-d)(v_{2,3} + v_{3,2})n_2 + (c v_{3,3} + d v_{2,2} + b k v_1)n_3 = 0 \end{aligned}\right\} \quad \text{on } \partial\mathscr{D}, \tag{5.1.5}$$

where $n = (n_2, n_3)$ is the outward unit normal to $\partial\mathscr{D}$.

It is a straightforward computation to show that the Complementing Condition holds for (5.1.4)–(5.1.5) provided $c \neq d$, and this is guaranteed by the hypotheses on Φ introduced in Sec. 2. Hence we can apply Proposition 3.2.1 and deduce that $\mathbf{v}$ is smooth, i.e., any weak solution of $S_\lambda[u, r] = 0$ must be a linear combination of functions of the form $u = u_k(R)\sin kX_1$, $r = r_k(R)\cos kX_1$ for some k, where (u_k, r_k) are classical solutions of $(5.1.2)_k$–$(5.1.3)_k$. Then, provided $\psi_1(\lambda) \neq 0$, S_λ will first fail to be uniformly positive in H_{BAR} at the largest value of λ for which there is a nontrivial solution to $(5.1.2)_k$–$(5.1.3)_k$ for some k, and we denote this value of the compression ratio as λ_{BAR}. Equations $(3.4.2)_k$–$(3.4.3)_k$ have been obtained and solved by Simpson and Spector [17] in their investigations of the stability of a uniaxially compressed elastic column to barrelling perturbations. They show that the general solution to $(5.1.2)_k$ is

$$\left.\begin{aligned} u_k(R) &= \gamma_1 A_1 I_0(p_1 kR) + \gamma_2 A_2 I_0(p_2 kR) \\ r_k(R) &= A_1 I_1(p_1 kR) + A_2 I_1(p_2 kR) \end{aligned}\right\} \tag{5.1.6}$$

if $\Delta \neq 0$, where p_1 and p_2 are the roots of the quartic (4.7) defined by (4.8), A_1 and A_2 are arbitrary constants, and $\gamma_i = (\alpha - c p_i^2)/(b+\beta)p_i$ for $i = 1, 2$. If $\Delta = 0$ the solution is

$$\left.\begin{aligned} u_k(R) &= -(A_1 + \gamma A_2/k)I_0(pkR) - A_2 R I_1(pkR) \\ r_k(R) &= pA_1 I_1(pkR) + pA_2 R I_0(pkR) \end{aligned}\right\} \tag{5.1.7}$$

for arbitrary constants A_1 and A_2, where $\gamma = 2pc/(b+\beta)$, and where I_0 and I_1 are the modified Bessel functions of the first kind. Upon substitution of the general solution into the boundary conditions we see that the system $(5.1.2)_k$–$(5.1.3)_k$ will have a nontrivial solution if and only if

$$F_{\mathrm{BAR}}(\lambda, kh) = 0, \tag{5.1.8}$$

where

$$F_{\mathrm{BAR}}(\lambda, t) = \psi_1(\lambda)h(\lambda, t) + g(\lambda, t) \tag{5.1.9}$$

for ψ_1 as defined by (4.9),

$$g(\lambda, t) = \begin{cases} \dfrac{c(p_1+p_2)}{p_2-p_1}\left[(\beta^2-\alpha^2)\left(\dfrac{\nu(p_1 t)}{p_1} - \dfrac{\nu(p_2 t)}{p_2}\right) - \alpha(p_2-p_1)(c-d)\right] & \text{if } \Delta \neq 0, \\ \dfrac{2c}{p}[(\beta^2-\alpha^2)(\nu(pt) - pt\nu'(pt)) - \alpha p^2(c-d)] & \text{if } \Delta = 0, \end{cases} \tag{5.1.10}$$

and

$$h(\lambda, t) = \begin{cases} \dfrac{1}{p_2 - p_1}\left[\dfrac{\nu(p_1 t)}{p_1^2} - \dfrac{\nu(p_2 t)}{p_2^2}\right] & \text{if } \Delta \neq 0, \\ \dfrac{2\nu(pt) - pt\nu'(pt)}{p^3} & \text{if } \Delta = 0, \end{cases} \tag{5.1.11}$$

where

$$\nu(z) = zI_0(z)/I_1(z). \tag{5.1.12}$$

5.2. *Analysis of barrelling equation (5.1.8).* We first consider the Complementing Condition for the boundary value problem corresponding to $S_\lambda[u, r] = 0$. The principal part of the equation and boundary conditions is

$$\left.\begin{aligned} au_{,11} + \alpha u_{,RR} + (b+\beta)r_{,1R} &= 0 \\ ar_{,11} + cr_{,RR} + (b+\beta)u_{,1R} &= 0 \end{aligned}\right\} \quad \text{in } \mathscr{R},$$

$$\left.\begin{aligned} u &= 0 \\ r_{,1} &= 0 \end{aligned}\right\} \quad \text{on } X_1 = 0, \pi,$$

$$\left.\begin{aligned} \alpha u_{,R} + \beta r_{,1} &= 0 \\ bu_{,1} + cr_{,R} &= 0 \end{aligned}\right\} \quad \text{on } R = h.$$

Hence the auxiliary problem defined on a half-space for this system is identical to that considered for (4.2)–(4.4), and so again the Complementing Condition fails only at those values of λ which satisfy (4.13).

We now investigate (5.1.8); we will study solutions for very thin columns, show that $F_{\mathrm{BAR}}(1, t)$ is positive for all $t > 0$, and look at F_{BAR} as λ approaches $\bar{\lambda}$ (assuming that (4.13) has a solution). We will need the results of Simpson and Spector [18] concerning $\nu(z)$ given below.

PROPOSITION 5.2.1 (Simpson and Spector [18]). The function $\nu(z)$ defined by (5.1.12) satisfies

$$\nu(0) = 2, \tag{5.2.1}$$

$$\begin{aligned} z\nu' &= z^2 + 2\nu - \nu^2 > 0, \\ z\nu'' &= 2z + \nu' - 2\nu\nu' > 0 \quad \text{for all } z > 0. \end{aligned} \tag{5.2.2}$$

We additionally need the following:

LEMMA 5.2.1. The function $\nu(z)$ satisfies

$$\nu'''(0) = 0, \quad \nu'''(z) < 0 \quad \text{for } z > 0. \tag{5.2.3}$$

Proof. The function ν is analytic and its power series expansion about 0 is

$$\nu(z) = 2 + z^2/4 - z^4/192 + O(z^6) \tag{5.2.4}$$

(see, for example, [22]). Hence setting $w(z) = \nu'''(z)$, w satisfies

$$w(0) = 0, \qquad w'(0) = -1/8$$

and so $w(z) < 0$ for sufficiently small $z > 0$. Suppose for a contradiction that there exists $z > 0$ with $w(z) = 0$, and let z_0 be the smallest such, then

$$w(z_0) = 0, \qquad w'(z_0) > 0.$$

Differentiating (5.2.2) twice yields $zw' = -6\nu'\nu'' - (2\nu + 1)w$ and hence $w'(z_0) = -6\nu'\nu''/z_0 < 0$ from (5.2.2), a contradiction.

We now use these results to deduce the following properties of the functions $g(\lambda, t)$ and $h(\lambda, t)$ which were defined at the end of Sec. 5.1.

LEMMA 5.2.2. For fixed λ, $g(\lambda, t)$ is a monotone decreasing function of t. Hence for each $t > 0$, $g(\lambda, t) > g_\infty(\lambda)$, where

$$g_\infty(\lambda) = \begin{cases} [-\Phi_{,1}/p^2 - 2\lambda(c-d)](p_1 + p_2)c\theta/2 & \text{if } \Delta > 0, \\ [-\Phi_{,1}/p^2 - 2\lambda(c-d)]pc\theta & \text{if } \Delta = 0. \end{cases} \tag{5.2.5}$$

Proof. We prove the result for λ such that $\Delta(\lambda) > 0$. The proof when $\Delta(\lambda) = 0$ is similar and will be omitted.

From (5.1.11),

$$\frac{dg}{dt}(\lambda, t) = c\frac{(p_1 + p_2)}{p_2 - p_1}(\beta^2 - \alpha^2)(\nu'(p_1 t) - \nu'(p_2(t)) < 0$$

as $\nu''(z) > 0$ (Proposition 5.2.1). Define

$$g_\infty(\lambda) = \lim_{t\to\infty} g(\lambda, t).$$

To show that $g_\infty(\lambda)$ is as given in the statement of the lemma we use the asymptotic approximation

$$\nu(z) = z + \tfrac{1}{2} + 3/8z + O(1/z^2) \tag{5.2.6}$$

for large z (see, e.g., [22]). Upon substitution of this into (5.1.10) we have

$$\begin{aligned} g(\lambda, t) &= c(p_1 + p_2)[(\beta^2 - \alpha^2)/2p^2 - \alpha(c-d)] + O(1/t) \\ &= c\theta(p_1 + p_2)[-\Phi_{,1}/2p^2 - \lambda(c-d)] + O(1/t) \end{aligned}$$

as required.

LEMMA 5.2.3. For fixed λ, $h(\lambda, t)$ is a monotone increasing function of t. Hence for all $t > 0$, $h(\lambda, t) > h_0(\lambda)$, where

$$h_0(\lambda) = \begin{cases} 2(p_1 + p_2)/p^4 & \text{if } \Delta(\lambda) > 0, \\ 4/p^3 & \text{if } \Delta(\lambda) = 0. \end{cases} \tag{5.2.7}$$

In addition,

$$h(\lambda, t) \to \infty \quad \text{as } t \to \infty. \tag{5.2.8}$$

Proof. Again we shall only consider the case $\Delta(\lambda) > 0$. Set $h_0(\lambda) = h(\lambda, 0)$; then (5.2.7) follows upon substitution of (5.2.8) into the definition of h.

To show that h is monotone increasing in t, it is sufficient to prove that the function $\nu'(z)/z$ is monotone decreasing in z, as

$$(p_2 - p_1)\frac{dh}{dt}(\lambda, t) = \frac{\nu'(p_1 t)}{p_1} - \frac{\nu'(p_2 t)}{p_2}.$$

But

$$\frac{d}{dz}\left(\frac{\nu'(z)}{z}\right) = \frac{\nu_1(z)}{z^2},$$

where $\nu_1(z) = z\nu''(z) - \nu'(z)$. Differentiating gives

$$\nu_1'(z) = z\nu'''(z) < 0 \quad \text{for } z > 0$$

by Lemma 5.2.1. Also $\nu'(0) = 0$, and hence $\nu_1(z) < 0$ for $z > 0$ as required.

Finally, substituting (5.2.6) into (5.1.11) yields

$$h(\lambda, t) = t/p^2 + O(1)$$

for large t, which proves (5.2.8).

It follows from (4.14) and Lemmas 5.2.2 and 5.2.3 that if $\lambda > \bar{\lambda}$, then

$$F_{\mathrm{BAR}}(\lambda, t) > \psi_1(\lambda)h_0(\lambda) + g_\infty(\lambda) \tag{5.2.9}$$

for all $t > 0$. Hence if $t > 0$,

$$F_{\mathrm{BAR}}(1, t) > 2\theta b(a-b)|_{\lambda=1} = b(a-b)^2|_{\lambda=1} > 0.$$

We now look at F_{BAR} as $t \to 0$. Expanding h, g about $t = 0$ gives

$$F_{\mathrm{BAR}}(\lambda, t) = c\theta(p_1 + p_2)[(c + d - 2b^2/a)\lambda + \Phi_{,1}t^2/4] + O(t^4),$$

and so as $t \to 0$ any solution λ of (5.1.8) satisfies

$$\lambda = \frac{a|\Phi_{,1}|}{a(c+d) - 2b^2} \cdot \frac{t^2}{4} + O(t^4). \tag{5.2.10}$$

The same hypotheses on Φ which give the existence of λ satisfying (4.13) will also guarantee that (5.2.10) has a solution when t is sufficiently small, i.e., $\lambda_{\mathrm{BAR}} \to 0$ as $h \to 0$. This is because

$$\frac{d}{d\lambda}(\Phi_{,1}) = a - 2b^2/(c+d)$$

and we would expect this to be positive for realistic materials. Let $\underline{\lambda}$ be the smallest solution of (4.13). If $\lambda < \underline{\lambda}$ then $\psi_1(\lambda) < 0$ and hence it follows from Lemmas 5.1.2 and 5.1.3 that $\dot{F}_{\mathrm{BAR}}(\lambda, t) < 0$ where the dot denotes differentiation with respect to t. It thus follows from the Implicit Function Theorem that for each $\lambda < \underline{\lambda}$ there is $t > 0$ with $F_{\mathrm{BAR}}(\lambda, t) = 0$. This means that in the (t, λ) plane the curve $F_{\mathrm{BAR}} = 0$ either always lies below the line $\lambda = \underline{\lambda}$ which is an asymptote, or there exists $t > 0$ with $F_{\mathrm{BAR}}(\underline{\lambda}, t) = 0$. We shall show below that if we make the additional assumption that (4.13) has only one solution, then which of these two possibilities can occur depends solely on the sign of g_∞ evaluated at this solution.

THEOREM 5.2.1. Suppose that (4.13) has unique solution $\bar{\lambda}$, and that $\Delta(\lambda) \geq 0$ for $\lambda \in [\bar{\lambda}, 1]$.

(1) If $g_\infty(\bar{\lambda}) \geq 0$, then (5.1.8) has no solution for any $\lambda > \bar{\lambda}$.

(2) If $g_\infty(\bar{\lambda}) < 0$, then there exists $t^* > 0$ such that for each $t > t^*$ there is $\lambda > \bar{\lambda}$ with $F_{\mathrm{BAR}}(\lambda, t) = 0$.

Proof. Part (1) follows directly from (5.2.9). We have

$$F_{\mathrm{BAR}}(\bar{\lambda}, t) > g_\infty(\bar{\lambda}) \geq 0 \quad \text{for all } t > 0,$$

and so the curve $F_{\mathrm{BAR}} = 0$ cannot cut the line $\lambda = \bar{\lambda}$ in the (t, λ)-plane for any $t > 0$. Hence if $t > 0$ is fixed, a solution λ to $F_{\mathrm{BAR}}(\lambda, t) = 0$ must satisfy $\lambda < \bar{\lambda}$.

For part (2) notice that $F_{\mathrm{BAR}}(\bar{\lambda}, t) = g(\bar{\lambda}, t)$. If $g_\infty(\bar{\lambda}) < 0$ then there exists $t^* > 0$ such that $g(\bar{\lambda}, t)$ (and consequently $F_{\mathrm{BAR}}(\bar{\lambda}, t)$) is negative for $t > t^*$. We showed previously that $F_{\mathrm{BAR}}(1, t) > 0$ for all t. Hence for each $t > t^*$ there exists $\lambda \in (\bar{\lambda}, 1)$ with $F_{\mathrm{BAR}}(\lambda, t) = 0$.

This means that there will be no barrelling type instabilities if $g_\infty(\bar{\lambda}) \geq 0$. If $g_\infty(\bar{\lambda}) < 0$ then $(5.1.2)_k$–$(5.1.3)_k$ will have a nontrivial solution if k is large enough, for each k if h is sufficiently big. It is to be expected that only low mode (i.e., $k = 1$ or 2) barrelling will occur in practice, and if this is the case the column will eventually barrel under uniaxial compression if it is sufficiently thick.

6. Comparison of λ_{BUC} and λ_{BAR}.

6.1. *The general case.* We again assume that (4.13) has a unique solution $\bar{\lambda}$, and that $\Delta(\lambda) \geq 0$ for $\lambda \in [\bar{\lambda}, 1]$.

Recall that $\lambda_{\mathrm{BUC}}(h) \in (\bar{\lambda}, 1)$ is the first compression ratio for which the linearized equations (3.1.4)–(3.1.6) for a column of *square* cross-section (of side $2h$) admit a nontrivial planar solution in H. It is the largest solution λ to

$$F_{\mathrm{BUC}}(\lambda, h) = 0. \tag{6.1.1}$$

Similarly $\lambda_{\mathrm{BAR}}(h)$ is the largest compression ratio at which barrelling of a *circular* cylinder of radius h can occur, and is given by

$$\lambda_{\mathrm{BAR}}(h) = \max_{k \in \mathbb{Z}+} \{\lambda^k_{\mathrm{BAR}}(h)\},$$

where for each $k \in \mathbb{Z}^+$, $\lambda^k_{\mathrm{BAR}}(h)$ is the largest solution λ to

$$F_{\mathrm{BAR}}(\lambda, kh) = 0. \tag{6.1.2}$$

It was shown in the previous section that if $g_\infty(\bar{\lambda}) \geq 0$, then $\lambda_{\mathrm{BAR}} < \bar{\lambda}$ and hence

$$\lambda_{\mathrm{BAR}}(h) < \lambda_{\mathrm{BUC}}(h) \quad \text{for all } h \text{ if } g_\infty(\bar{\lambda}) \geq 0.$$

We now look at the relative sizes of these critical compression ratios when $g_\infty(\bar{\lambda}) < 0$. As has been remarked previously the only barrelling solutions of practical interest will be those for which $k = 1$ or 2. We shall therefore compare $\lambda_{\mathrm{BUC}}(h)$ with $\lambda^1_{\mathrm{BAR}}(h)$; however all the arguments are identical for the comparison of $\lambda_{\mathrm{BUC}}(h)$ with $\lambda^k_{\mathrm{BAR}}(h)$ for any k. In what follows we shall use t to represent the column size instead of h to avoid confusion with the function h defined in (5.1.11).

Define $G_{\mathrm{BUC}}(\lambda, t) = \psi_2(\lambda) F_{\mathrm{BUC}}(\lambda, t)$, $G_{\mathrm{BAR}}(\lambda, t) = F_{\mathrm{BAR}}(\lambda, t))/h(\lambda, t)$, and $G(\lambda, t) = G_{\mathrm{BAR}}(\lambda, t) - G_{\mathrm{BUC}}(\lambda, t)$. Then $G = E\psi_2 + g/h$, and hence it follows from (5.2.6) that

$$G(\bar{\lambda}, t) = p^2 g_\infty(\bar{\lambda})/t + o(1/t). \tag{6.1.3}$$

For large t, $\lambda_{\mathrm{BUC}}(t)$ approaches $\overline{\lambda}$, and substituting the appropriate asymptotic expansion for λ_{BUC} into (6.1.1) yields $\lambda_{\mathrm{BUC}}(t) = \overline{\lambda} + o(1/t)$. Together with (6.1.3) this then gives

$$G(\lambda_{\mathrm{BUC}}(t), t) = p^2 g_\infty(\overline{\lambda})/t + o(1/t),$$

and hence if $g_\infty(\overline{\lambda}) < 0$ there exists $t_2 > 0$ such that

$$G(\lambda_{\mathrm{BUC}}(t), t) < 0 \quad \text{for all } t > t_2.$$

But $G(\lambda_{\mathrm{BUC}}(t), t) = G_{\mathrm{BAR}}(\lambda_{\mathrm{BUC}}(t), t)$, and so

$$F_{\mathrm{BAR}}(\lambda_{\mathrm{BUC}}(t), t) < 0 \quad \text{for } t > t_2,$$

as $h(\lambda, t)$ is always positive. This means that if $t > t_2$, then $\lambda_{\mathrm{BAR}}(t) > \lambda_{\mathrm{BUC}}(t)$.

The condition that the behaviour for large t depends on the sign of $g_\infty(\overline{\lambda})$ can be simplified by substituting (4.13) into (5.2.5). This yields $\operatorname{sgn}(g_\infty(\overline{\lambda})) = \operatorname{sgn}(\overline{g})$, where

$$\overline{g} = (a(2d - c) - b^2)|_{\lambda=\overline{\lambda}}. \tag{6.1.4}$$

We summarize these results below.

THEOREM 6.1.1. Suppose that (4.13) has a unique solution $\overline{\lambda}$ and that $\Delta(\lambda) \geq 0$ for $\lambda \in [\overline{\lambda}, 1]$.

(i) If $\overline{g} \geq 0$, then $\lambda_{\mathrm{BAR}}(h) < \overline{\lambda} < \lambda_{\mathrm{BUC}}(h)$ for each $h > 0$.

(ii) If $\overline{g} < 0$, then there exist $t_2 > t_1 > 0$ such that $\lambda^1_{\mathrm{BAR}}(h) < \overline{\lambda} < \lambda_{\mathrm{BUC}}(h)$ if $h < t_1$ and $\overline{\lambda} < \lambda_{\mathrm{BUC}}(h) < \lambda^1_{\mathrm{BAR}}(h)$ if $h > t_2$.

6.2. *A particular example.* Simpson and Spector [19] consider the stability with respect to barrelling perturbations of a circular cylinder composed of a material with stored energy function:

$$\Phi(v_1, v_2, v_3) = [v_1^2 + v_2^2 + v_3^2]/2 + (v_1 v_2 v_3)^{-m}/m. \tag{6.2.1}$$

They show that such a cylinder will become unstable at sufficiently high compressions if the product of the mode number k and thickness h is large enough. We shall use this particular model to illustrate the analysis of the preceding sections.

The trivial solution $\mathbf{x}_\lambda$ for the material (6.2.1) is

$$\mathbf{x}_\lambda(\mathbf{X}) = (\lambda X_1, vX_2, vX_3) + \mathbf{c}_\lambda \quad \text{for } \mathbf{X} \in \Omega,$$

where $v(\lambda)$ is the solution to $\Phi_{,2}(\lambda, v, v) = 0$, i.e., $v = \lambda^{-m/2(m+1)}$ (see [19]).

The parameter Δ is nonnegative for this material and the Complementing Condition fails only at solutions λ of (4.13) which corresponds to

$$F(v^2/\lambda^2) = 0, \tag{6.2.2}$$

where

$$F(\mu) = \mu^3 - \frac{(11m+6)}{m+2}\mu^2 - 5\mu - 1. \tag{6.2.3}$$

Equation (6.2.2) has precisely one positive root $v^2/\lambda^2 > \overline{\mu}$, where $\overline{\mu}$ is the positive turning point of F, i.e.,

$$\overline{\mu} = \frac{11m + 6 + 2\sqrt{34m^2 + 48m + 24}}{3(m+2)}. \tag{6.2.4}$$

Hence the unique solution $\bar{\lambda}$ to (4.13) satisfies

$$\bar{\lambda} < \bar{\mu}^{-(m+1)/(3m+2)}. \tag{6.2.5}$$

To determine what happens as the cylinder is compressed we need to compute the sign of $g_\infty(\bar{\lambda})$, which as we showed in the previous section is equal to $\operatorname{sgn}\{(a(2d - c) - b^2)|_{\lambda=\bar{\lambda}}\}$. For this material

$$(a(2d-c)-b^2)|_{\lambda=\bar{\lambda}} = m - 2 - \mu(m+2),$$

where $F(\mu) = 0$. We also have $(m-2)/(m+2) < \bar{\mu} < \mu$, and hence $g_\infty(\bar{\lambda}) < 0$. It thus follows from the analysis of the previous section that for this material $\lambda_{\mathrm{BAR}}(h) > h_{\mathrm{BUC}}(h)$ if h is sufficiently large.

7. Conclusion. We have shown (subject to the constitutive hypotheses discussed below) that whether a circular column can barrel under uniaxial compression is determined by the sign of the parameter $\bar{g}$ defined in (6.1.4). The assumptions (H1)–(H6) of Sec. 2 and that there exists a solution $\bar{\lambda}$ to (4.13) are physically reasonable. However, it is not clear what the restriction that $\Delta(\lambda)$ be nonnegative for $\lambda \in [\bar{\lambda}, 1]$ means for a material, nor what $\bar{g}$ represents physically. Certainly $\Delta \geq 0$ for many constitutive models; for instance the model studied by Simpson and Spector [19] and given as an example in Sec. 6.2 is realistic for certain values of m.

The Complementing Condition fails for the linearized problem when the compression ratio is $\bar{\lambda}$. As has been pointed out in [21] (see also [9]), failure of the Complementing Condition may correspond to infinitesimal surface wrinkling of the elastic column under compression, and it is unlikely that the mathematical model is valid for compression ratios smaller than $\bar{\lambda}$. The results of Sec. 5 are that the column will not barrel before $\bar{\lambda}$ if $\bar{g}$ is nonnegative, and if $\bar{g} < 0$ then physically realistic low mode barrelling will occur if the column is sufficiently thick.

At first sight this would seem to agree well with the experiments performed by Beatty and Hook [8] and Beatty and Dadras [7]. They investigated the compression of columns of various aspect ratios between greased parallel plates and reported that whilst thin columns buckled under a sufficiently large applied load those of larger aspect ratio started deforming inhomogeneously by bulging in the middle. However, as is pointed out by Simpson and Spector [19], the theoretical values of λ_{BAR} obtained for the material considered in Sec. 6.2 are far smaller than those at which barrelling is first observed in practice. The likely cause of this discrepancy is that despite lubrication of the endplates the observed bulging of the column is due to friction at its ends, and what is seen is not a true barrelling perturbation but simply the original homogeneous deformation $\mathbf{x}_\lambda$ distorted by these friction effects (see [9] and [15]).

The compression ratios λ_{BUC} (planar buckling of a square column) and λ_{BAR} (barrelling of a circular column) are compared in Sec. 6.1. There are technical reasons for not considering buckling and barrelling instabilities of columns of the same geometry here. The perturbations associated with barrelling of a square column will be far more complicated than those of Sec. 5.1. Modelling plane strain instabilities of a circular column is also difficult—such a column cannot be constrained from moving in the X_3-direction by the use of smooth parallel plates. In addition, if perturbations

u of the type (4.1) are considered, then integration of $J_\lambda[\mathbf{u}]$ with respect to X_3 yields an extra factor of $\sqrt{1 - X_2^2}$ in $P_\lambda[\mathbf{v}]$, and hence the system of equations for **v** is different from that studied in [9].

Whilst there is no obvious relationship between instabilities of square and circular columns it is hoped that this comparison may be useful in at least providing a starting point in the study of compressional instabilities of columns of general cross-section.

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