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IDENTIFICATION OF SEMICONDUCTOR CONTACT RESISTIVITY

By

STAVROS BUSENBERG (*Harvey Mudd College, Claremont, California*)

AND

WEIFU FANG (*Claremont Graduate School, Claremont, California*)

1. Introduction. In a VLSI semiconductor component, a contact is the place where the semiconductor is attached to a metal layer, serving as an access to the semiconductor. The main parameter that describes the quality of the contact interface is its resistivity ρ_c. It is well known that ρ_c becomes a dominant design factor as components reduce to submicron size, so it is important to obtain accurate values of ρ_c. Due to the complexity of manufacture and miniaturization, it is impossible to directly measure the contact resistivity and to exactly control the location of the contact window. In order to estimate ρ_c and detect the location of the contact window, current is applied to the component and voltage is measured at some accessible place far away from the contact window. There are extensive experimental and computer simulations on this identification problem (see, e.g., [4] and the references therein).

In [1] this situation is modeled as an inverse problem for an elliptic equation which is formulated as follows: Let $u(x)$, the voltage, be the solution to the elliptic problem

$$\Delta u - p\chi(S)u = 0 \quad \text{in } \Omega \subset R^2, \tag{1}$$

$$\frac{\partial u}{\partial n} = g \geq 0 \quad \text{but } g \not\equiv 0 \text{ on } \partial\Omega, \tag{2}$$

where $S \subset \Omega$ and $p > 0$ are unknown, $\chi(S)$ is the set characteristic function of S, g is the given density of the applied current, S is the location of the contact window, and p is a given positive function of the contact resistivity (e.g., in the following we will assume that $p = R_s/\rho_c$, with R_s the known sheet resistance characterizing the semiconductor layer). We wish to recover p and S from a one-point boundary measurement $u(x_0)$ for some $x_0 \in \partial\Omega$. In [1] the identifiability of the unknown pair $\{p, S\}$ from a one-parameter family $\{p(t), S(t)\}_{t\in(0,1)}$ is studied, and uniqueness, stability, and continuous dependence results for the one-point measurement on the boundary are obtained.

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Second author's current address: Center for Applied Mathematical Sciences, University of Southern California, Los Angeles, CA 90089-1113

In this paper, we consider the identification of p when the contact location S is known. Obviously this is a special case of the one-parameter case studied in [1], so we have the uniqueness, stability, and continuous dependence results, but in this simpler case, we have much more detailed properties, as will be shown in Sec. 2. We also establish some quantitative properties for the measurement on the boundary, and apply these to analyze the formula commonly used in industry to extract the contact resistivity from the measurement. We prove that the formula is good only when the true resistivity is large enough, and it overestimates the true resistivity severely when the true resistivity is small. In Sec. 3, based on the properties we develop in Sec. 2, we establish a numerical iteration scheme for identification of the contact resistivity from a given boundary measurement, and prove the convergence of this scheme. We illustrate the use of this scheme with a numerical example in Sec. 4 by using the elliptic PDE solver ELLPACK (see [7]).

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2. The extracted contact resistivity. In applications, in order to identify the contact resistivity $\rho_c = R_s/p$ in problem (1), (2), the voltage u is measured at some $x_0 \in \partial\Omega$. Then the following formula is used to extract the contact resistivity, that is, the so-called extracted contact resistivity:

$$\rho_{ce} = |S| \frac{u(x_0)}{\frac{1}{R_s}\int_{\partial\Omega} g\, ds}. \tag{3}$$

This formula is based on a simple argument using Ohm's law (see, e.g., [4, 5] for details). Computer simulations of this problem are also given in [5]. In this section we study the quantitative behavior of this extracted contact resistivity ρ_{ce} in terms of the true resistivity ρ_c as given by (3).

Let Ω be a bounded connected domain in R^2 and S a given subdomain in Ω with $C^{1,1}$ boundaries $\partial\Omega$ and ∂S. Points in R^2 are denoted by $x = (x_1, x_2)$. Suppose $g(x) \in C^\alpha(\partial\Omega)$ for some $0 < \alpha < 1$ and $g \geq 0$ but $g \not\equiv 0$.

It is well known that the $H^1(\Omega)$ solution to (1), (2) is in $C^{1,\beta}(\Omega)$ for some $0 < \beta < 1$ (see, e.g., [2]). Denote this solution by $u(x;p)$. Our identification problem is: recover the positive constant p from the one-point boundary measurement $u(x_0)$, given the geometric setting and the applied current g on the boundary.

Since this is a special case of the one-parameter monotone family studied in [1], we have from there the uniqueness, stability, and continuous dependence of this identification problem. In fact, for this special case we have much better properties and an easier proof.

THEOREM 1. The solution $u(x;p)$ to (1), (2) is C^∞ in p, and the kth order derivative with respect to p, denoted by $u^{(k)}(x;p)$, is the solution to

$$\begin{aligned} \Delta u^{(k)} - p\chi(S)u^{(k)} &= k\chi(S)u^{(k-1)} \quad \text{in } \Omega, \\ \frac{\partial u^{(k)}}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4$_k$}$$

where $k = 1, 2, \ldots$ and $u^{(0)} = u$. Moreover, we have

$$(-1)^k u^{(k)}(x;p) \geq 0 \quad \text{in } \overline{\Omega} \qquad \text{and} \qquad (-1)^k u^{(k)}(x;p) > 0 \quad \text{in } \overline{\Omega}\backslash S. \tag{5_k}$$

Proof. For a fixed $p > 0$, the solution $u = u(x;p) \in H^1(\Omega)$ satisfies

$$\iint_\Omega \{\nabla u \cdot \nabla \phi + p\chi(S)u\phi\}\,dx = \int_{\partial\Omega} g\phi\,ds \quad \text{for all } \phi \in H^1(\Omega). \tag{6}$$

For small $h \neq 0$ $(|h| < p/2)$ we set

$$v_h(x;p) = \frac{u(x;p+h) - u(x;p)}{h}.$$

Then v_h is in $H^1(\Omega)$ and satisfies

$$\iint_\Omega \{\nabla v_h \cdot \nabla\phi + (p+h)\chi(S)v_h\phi\}\,dx = -\iint_S u\phi\,dx \quad \text{for all } \phi \in H^1(\Omega).$$

By setting $\phi = v_h$ we have

$$\iint_\Omega |\nabla v_h|^2\,dx + (p+h)\iint_S v_h^2\,dx = -\iint_S uv_h\,dx,$$

which implies that

$$\frac{p}{2}\iint_S v_h^2\,dx \leq \|u\|_{L^2(S)}\|v_h\|_{L^2(S)},$$

so

$$\|v_h\|_{L^2(S)} \leq \frac{2}{p}\|u\|_{L^2(S)}. \tag{7}$$

Notice that the weak form of $(4)_k$ is

$$\iint_\Omega \{\nabla u^{(k)} \cdot \nabla\phi + p\chi(S)u^{(k)}\phi\}\,dx = -k\iint_S u^{(k-1)}\phi\,dx \quad \text{for all } \phi \in H^1(\Omega),$$

$k = 1, 2, \ldots$. So for $v_h - u^{(1)}$ we have

$$\iint_\Omega \{\nabla(v_h - u^{(1)}) \cdot \nabla\phi + p\chi(S)(v_h - u^{(1)})\phi\}\,dx = -h\iint_S v_h\phi\,dx \tag{8}$$

for all $\phi \in H^1(\Omega)$. Therefore, setting $\phi = v_h - u^{(1)}$ yields

$$\iint_\Omega |\nabla(v_h - u^{(1)})|^2\,dx + p\iint_S (v_h - u^{(1)})^2\,dx = -h\iint_S v_h(v_h - u^{(1)})\,dx, \tag{9}$$

hence

$$p\iint_S (v_h - u^{(1)})^2\,dx \leq |h| \cdot \|v_h\|_{L^2(S)} \cdot \|v_h - u^{(1)}\|_{L^2(S)},$$

i.e., by (7),

$$\|v_h - u^{(1)}\|_{L^2(S)} \leq \frac{|h|}{p}\|v_h\|_{L^2(S)} \leq \frac{2|h|}{p^2}\|u\|_{L^2(S)}. \tag{10}$$

By the Poincaré inequality, we have

$$C_1\|\phi\|^2_{H^1(\Omega)} \leq \|\nabla\phi\|^2_{L^2(\Omega)} + \frac{1}{|S|}\left\{\iint_S \phi\,dx\right\}^2, \tag{11}$$

hence,

$$C_1\|\phi\|^2_{H^1(\Omega)} \le \|\nabla\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(S)}$$

for all $\phi \in H^1(\Omega)$, where C_1 is independent of ϕ. For a proof of the Poincaré inequality as used here, see [9] for the case $S = \Omega$. Notice that the proof given there is also valid for our case $S \subset \Omega$ under the assumptions we have on S and Ω. So (9) yields

$$C_1 \cdot \min\{p, 1\} \cdot \|v_h - u^{(1)}\|^2_{H^1(\Omega)} \le |h| \cdot \|v_h\|_{L^2(S)} \cdot \|v_h - u^{(1)}\|_{L^2(S)},$$

and then by (7), we have

$$\|v_h - u^{(1)}\|_{H^1(\Omega)} \le C_p|h| \cdot \|u\|_{L^2(S)}, \tag{12}$$

where C_p is a constant independent of h.

It is clear that $v_h - u^{(1)}$ satisfies

$$\Delta(v_h - u^{(1)}) = \chi(S)((p+h)v_h - pu^{(1)}) \quad \text{in } \Omega,$$
$$\frac{\partial(v_h - u^{(1)})}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Let $G_0(x, \xi)$ be a Green's function for the Laplace equation in Ω with Neumann boundary condition (it is also called the Neumann function, see, e.g., [8]). Then, since $\partial\Omega \cap \overline{S} = \varnothing$, for any $x \in \overline{\Omega}$ we have

$$v_h(x) - u^{(1)}(x) = \iint_S G_0(x, \xi)\{p[v_h(\xi) - u^{(1)}(\xi)] + hv_h(\xi)\}\, d\xi + \overline{v}_h - \overline{u}^{(1)} \tag{13}$$

(see, e.g., [8, Chap. 9]), where $\overline{\phi}$ denotes the average of ϕ over Ω:

$$\overline{\phi} = \frac{1}{|\Omega|}\iint_\Omega \phi(x)\, dx.$$

Clearly $|\overline{\phi}| \le \|\phi\|_{L^2(\Omega)}/\sqrt{|\Omega|}$. Hence, from (12),

$$|\overline{v}_h - \overline{u}^{(1)}| \le C_p|h| \cdot \|u\|_{L^2(S)}. \tag{14}$$

Noting that the singularity of G_0 at $x = \xi$ is square integrable, for $x \in \overline{\Omega}$, from (7), (10), and (14), we have

$$\begin{aligned}|v_h(x) - u^{(1)}(x)| &\le \|G_0(x, \cdot)\|_{L^2(S)}(p\|v_h - u^{(1)}\|_{L^2(S)} + |h| \cdot \|v_h\|_{L^2(S)}) + |\overline{v}_h - \overline{u}^{(1)}| \\ &\le |h|\left(\frac{4}{p}\|G_0(x, \cdot)\|_{L^2(S)} + C_p\right)\|u\|_{L^2(S)} \to 0 \quad \text{as } h \to 0.\end{aligned}$$

Therefore, $u^{(1)}(x; p)$ is the derivative of $u(x; p)$ with respect to p for each $x \in \overline{\Omega}$. The same argument applies to show that $u^{(k)}(x; p)$ satisfying $(4)_k$ is the kth order derivative of $u(x; p)$ with respect to p, $k = 2, 3, \dots$.

To prove $(5)_k$, we consider the equation

$$\Delta w - p\chi(S)w = \chi(S)F \quad \text{in } \Omega$$

with homogeneous Neumann condition on $\partial\Omega$, where F is continuous in Ω. It is known that $w \in C^1(\overline{\Omega})$. By applying the maximum principle for elliptic equations (see, e.g., [6]), we can obtain the following conclusion:

$$F \le 0 \text{ in } S \quad \text{implies} \quad w \ge 0 \text{ in } \overline{\Omega} \text{ and } w > 0 \text{ in } \overline{\Omega}\backslash S. \tag{15}$$

From Lemma 2.1 in [1], we have $u \ge 0$ in $\overline{\Omega}$. Therefore, we can establish $(5)_k$ by applying a simple induction on n and (15). Thus the proof is completed.

REMARK. The Green's function G_0 we chose in the proof is unique up to an arbitrary constant, but expression (13) does not depend on the choice of the constant. We can see this by setting $\phi \equiv 1$ in (8).

Next, we study the quantitative behavior of the extracted contact resistivity given by (3). For simplicity, we set $R_S = 1$, i.e., $\rho_c = 1/p$. Then the extracted contact resistivity ρ_{ce} is a function of the true ρ_c given by

$$\rho_{ce} = f(\rho_c) \equiv \frac{|S|}{\int_{\partial\Omega} g\, ds} u\left(x_0\, ; \frac{1}{\rho_c}\right), \tag{16}$$

where $u(x\, ; p)$ is the solution to (1), (2) and x_0 is a point at $\partial\Omega$ where the measurement is made. For the so-called Kelvin resistor (a special choice of the density distribution g and the measurement location x_0), some asymptotic properties of ρ_{ce} are observed in [5] by using computer simulations. In general, we have the following properties for $f(\cdot)$ which agree with the observations made in [5]. Let

$$A_0 = \frac{\int_{\partial\Omega} g\, ds}{|S|} > 0.$$

THEOREM 2. The function $f(\cdot)$ given by (16) has the following properties:

(i) $f(\cdot)$ is in $C^\infty(0, \infty)$ and is strictly increasing in $(0, \infty)$.

(ii) As $\rho_c \to \infty$,

$$\frac{f(\rho_c)}{\rho_c} = 1 + O\left(\frac{1}{\rho_c}\right).$$

(iii)

$$\lim_{\rho_c \to 0+} f(\rho_c) = f_0 = \frac{|S|}{\int_{\partial\Omega} g\, ds} u_0(x_0) > 0,$$

where $u_0(x)$ is the solution to

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega\backslash\overline{S}, \\ u_0 = 0 & \text{on } \partial S, \\ \partial u_0/\partial n = g & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Proof. (i) follows from Theorem 1 directly.

(ii) Let $v(x\, ; p) = u(x\, ; p) - A_0/p$. Then we have

$$v\left(x_0\, ; \frac{1}{\rho_c}\right) = A_0(f(\rho_c) - \rho_c) \tag{18}$$

and $v(x\, ; p)$ satisfies

$$\begin{aligned} \Delta v - p\chi(S)v = A_0\chi(S) \quad & \text{in } \Omega, \\ \partial v/\partial n = g \quad & \text{on } \partial\Omega. \end{aligned} \tag{19}$$

Notice that setting $\phi = 1$ in (6) leads to

$$p \iint_S u(x; p)\, dx = \int_{\partial\Omega} g\, ds.$$

Therefore

$$\iint_S v(x; p)\, dx = 0, \tag{20}$$

and the Poincaré inequality (11) applied to v gives

$$C\|v\|_{L^2(\Omega)} \le \|\nabla v\|_{L^2(\Omega)},$$

or, equivalently,

$$C\|v\|_{H^1(\Omega)} \le \|\nabla v\|_{L^2(\Omega)}, \tag{21}$$

where $C > 0$ depends only on S and Ω. In the weak form of (19), that is,

$$\iint_\Omega \{\nabla v \cdot \nabla\phi + p\chi(S) v\phi\}\, dx = \int_{\partial\Omega} g\phi\, ds - A_0 \iint_S \phi\, dx \quad \text{for all } \phi \in H^1(\Omega),$$

we set $\phi = v$ and use (20) to obtain

$$\|\nabla v\|^2_{L^2(\Omega)} = \iint_\Omega |\nabla v|^2\, dx \le \int_{\partial\Omega} gv\, ds \le C\|v\|_{H^1(\Omega)},$$

where the trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ is used in the second inequality, and $C > 0$ is a constant independent of p. Hence, by (21) we have

$$\|v\|_{L^2(\Omega)} \le \|v\|_{H^1(\Omega)} \le C. \tag{22}$$

From (19), for $x \in \Omega$,

$$v(x; p) = -\int_{\partial\Omega} G_0(x, \xi) g(\xi)\, d\xi + A_0 \iint_S G_0(x, \xi)\, d\xi + p \iint_S G_0(x; \xi) v(\xi; p)\, d\xi + \overline{v}, \tag{23}$$

where G_0 is the Green's function defined in the proof of Theorem 1 (see, e.g., [8, Chap. 9]). By the continuity of v up to $\partial\Omega$ and the integrable singularity of $G_0(x, \xi)$ at $x = \xi$, (23) is also true for $x \in \partial\Omega$. Therefore, from (22) it easily follows that

$$|v(x_0; p)| \le C_1 + C_2 p \quad \text{for any } p > 0,$$

with C_1, C_2 independent of p. That is, $v(x_0; p) = O(1)$ as $p \to 0$. So we have (ii) by (18).

(iii) Setting $\phi = u$ in (6) yields

$$\iint_\Omega \{|\nabla u|^2 + p\chi(S) u^2\}\, dx = \int_{\partial\Omega} gu\, ds \le C_2\|u\|_{H^1(\Omega)},$$

where the trace theorem is used. Hence, by (11),

$$C_1\|u\|^2_{H^1(\Omega)} + (p-1)\|u\|^2_{L^2(S)} \le C_2\|u\|_{H^1(\Omega)} \tag{24}$$

for $p > 2$. That is,

$$C_1\left(\|u\|_{H^1(\Omega)} - \frac{C_2}{2C_1}\right)^2 + (p-1)\|u\|^2_{L^2(S)} \le \frac{C_2^2}{4C_1}.$$

Hence

$$\|u\|^2_{L^2(S)} \le \frac{C_2^2}{4C_1(p-1)} \le \frac{C}{p}. \tag{25}$$

Also from (24), we have

$$\|u\|_{H^1(S)} \le \frac{C_2}{C_1}. \tag{26}$$

For a fixed $\varepsilon \in (0, 1/2)$, $H^{1-\varepsilon}(S)$ is defined as the interpolation space between $H^0(S) = L^2(S)$ and $H^1(S)$ (see e.g., [3, Chap. 1, Sec. 9]) and there holds the interpolation inequality

$$\|w\|_{H^{1-\varepsilon}(S)} \le C\|w\|^{\varepsilon}_{H^0(S)} \cdot \|w\|^{1-\varepsilon}_{H^1(S)} \tag{27}$$

for any $w \in H^1(S)$, where C depends only on S and ε. Combining (25), (26), and (27), we have

$$\|u\|_{H^{1-\varepsilon}(S)} \le \frac{C}{p^{\varepsilon/2}}.$$

Then the trace embedding in $H^{1-\varepsilon}(S)$ gives

$$H^{1-\varepsilon}(S) \hookrightarrow H^{1/2-\varepsilon}(\partial S) \hookrightarrow H^0(\partial S) = L^2(\partial S)$$

since $0 < \varepsilon < \frac{1}{2}$ (see, e.g., [3]). Hence

$$\|u\|_{L^2(\partial S)} \le \frac{C}{p^{\varepsilon/2}}. \tag{28}$$

Notice that $u - u_0$ (u_0 given by (17)) satisfies

$$\begin{cases} \Delta(u-u_0) = 0 & \text{in } \Omega\backslash\overline{S}, \\ u - u_0 = u & \text{on } \partial S, \\ \partial(u-u_0)/\partial n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $G(x, \xi)$ be the Green's function of the Laplace equation in $\Omega\backslash\overline{S}$ with Neumann boundary condition on $\partial\Omega$ and Dirichlet on ∂S. Then for $x \in \Omega$,

$$u(x; p) - u_0(x) = \int_{\partial S} \frac{\partial G}{\partial n}(x, \xi) u(\xi; p)\, d\xi. \tag{29}$$

Since $\operatorname{dist}(\partial\Omega, \partial S) > 0$, $\partial G/\partial n(x, \cdot)$ is regular on ∂S for each $x \in \partial\Omega$. Also G is independent of p. Therefore, (29) is true for $x \in \partial\Omega$, and by (28)

$$|u(x; p) - u_0(x)| \le C\|u\|_{L^2(\partial S)} \le \frac{C}{p^{\varepsilon/2}},$$

i.e., for each $x \in \partial\Omega$,

$$u(x; p) \to u_0(x) \quad \text{as } p \to \infty.$$

Applying the maximum principle to u_0 easily yields $u_0 > 0$ on $\partial\Omega$. Thus (iii) is proved.

From (ii) of this theorem, we can see that this extracted resistivity is a good estimate when the true resistivity ρ_c is large. However, from (iii) it is seen that for small ρ_c the extracted resistivity overestimates ρ_c severely, since the limiting value f_0 of ρ_{ce} as $\rho_c \to 0$ in (iii) is a positive constant. Also (iii) provides a method to calculate f_0. For physical interpretations of these properties, see [5].

3. Identification of the contact resistivity. In this section, we construct a numerical iteration scheme for identification of the contact resistivity ρ_c, or, equivalently, the constant p, from the one-point boundary measurement. Assume the geometric settings of the problem (1), (2) are given, i.e., Ω, the contact window S, the point $x_0 \in \partial\Omega$ where the measurement is made, and the applied current density g are given. Then we wish to estimate the constant p^* from the measurement u^*.

From Theorem 1, $u(x_0; p)$ is strict decreasing in p and is convex. So for each measurement $u^* > u_0(x_0)$ ($u_0(x)$ is given by (17)) there is a unique $p^* > 0$ such that $u^* = u(x_0; p^*)$, where $u(x; p)$ is the solution to (1), (2). To ensure unique identifiability, in the following we assume that $u^* > u_0(x_0)$.

By Newton's method of finding a zero of a function, we construct the following iteration scheme:

$$p_{j+1} = p_j - \frac{u(x_0; p_j) - u^*}{u^{(1)}(x_0; p_j)}, \tag{30$_j$}$$

where $u(x; p_j)$ is the solution to (1), (2) with $p = p_j$, and $u^{(1)}(x; p_j)$ is the derivative of $u(x; p)$ with respect to p given by $(4)_1$ with $p = p_j$.

Besides the quadratic convergence property of the general Newton's method, for this specific problem, we have

THEOREM 3. If the initial guess $p_0 > 0$ is such that $u(x_0; p_0) > u^*$, then the sequence $\{p_j\}_0^\infty$ given by (30) is strictly increasing and convergent from below to p^*, the unique value such that $u^* = u(x_0; p^*)$.

Proof. First we prove the strict monotonicity. For $j = 0$,

$$p_1 = p_0 - \frac{u(x_0; p_0) - u^*}{u^{(1)}(x_0; p_0)} > p_0$$

since $u^{(1)}$ is negative and by the assumption $u(x_0; p_0) > u^*$. For $j \geq 1$, from $(30)_{j-1}$,

$$u^* = u(x_0; p_{j-1}) + (p_j - p_{j-1})u^{(1)}(x_0; p_{j-1}),$$

therefore, for $(30)_j$,

$$\begin{aligned}(p_{j+1} - p_j)u^{(1)}(x_0; p_j) &= u^* - u(x_0; p_j) \\ &= -\{u(x_0; p_j) - u(x_0; p_{j-1})\} + (p_j - p_{j-1})u^{(1)}(x_0; p_{j-1}) \\ &= -\frac{(p_j - p_{j-1})^2}{2} u^{(2)}(\xi_j; p_{j-1})\end{aligned}$$

(ξ_j is between p_j and p_{j-1}). By Theorem 1, we have $u^{(1)}(x_0; p) < 0$ and $u^{(2)}(x_0; p) > 0$, hence, from above, $p_{j+1} > p_j$ for $j = 1, 2, \ldots$. Thus the sequence $\{p_j\}_0^\infty$ is strictly increasing.

Therefore $p_j \uparrow \tilde{p}$ for some $\tilde{p} > 0$ as $j \to \infty$. Letting j tend to ∞ in (30), we have $u(x_0; \tilde{p}) = u^*$. Hence $\tilde{p} = p^*$ by the strict monotonicity of u in p. Thus the proof is completed.

Next we consider an appropriate choice of the initial guess p_0. We call p_0 an eligible initial value if $u(x_0; p_0) \geq u^*$, or, equivalently, $p_0 \leq p^*$. So if p_0 is eligible, by Theorem 3 the Newton iteration scheme (30) starting at this p_0 is monotonely increasing and converges to p^*. On the other hand, if p_0 is not eligible, i.e., $u(x_0; p_0) < u^*$, then $p_0 > p^*$, and clearly the next iteration p_1 by (30) may become nonpositive, so the iteration cannot proceed. In this case, instead of using (30) to obtain the next iteration, we set $p_1 = p_0/4$. We can continue this procedure until we come up with a p_i such that $u(x_0; p_i) > u^*$, then we turn to the Newton scheme (30) for the rest of the iterations, using this p_i as the initial value. At each iteration step, two elliptic problems, one for u and one for $u^{(1)}$, need to be solved to obtain the next iteration value.

We remark that we can also use the bisection method to identify p^*. For this method we need to search for the interval on which we start the bisection procedure. Similar ideas as in the search for the eligible initial value p_0 above can be used to find this interval. At each iteration step, only one elliptic problem (for u itself) needs to be solved.

Comparing these two schemes, we notice that we compensate the speed of convergence in the bisection method for the ease of solving only one elliptic problem.

Finally, we remark that in the above we assume that there is no noise in the measurement. In the case that there is noise in the measurement but the experiment is repeated for a number of times, we should take the mean value for these data first and then use the above scheme to find the p corresponding to this mean value as the estimate for the true p.

4. A numerical example. In this section we consider an example where Ω and S are two nonconcentric discs. Let

$$\Omega = \text{the unit disc}, \qquad S = \text{a disc centered at } (0, 0.2) \text{ with radius } \frac{1}{2}.$$

The applied current density is given by

$$g(x) = \begin{cases} |x_2 + \frac{1}{2}| & \text{when } x_2 \leq -\frac{1}{2}, \\ 0 & \text{when } x_2 > -\frac{1}{2}, \end{cases}$$

and the measurement is made at $x_0 = (1, 0)$.

For any given u^*, we pick an arbitrary initial p_0. Then we solve (1), (2) with this p_0 and test if the p_0 is eligible. If not, we reduce to a quarter of p_0 and test again, until we get an eligible initial value. Once we get an eligible p_0, we use the Newton iteration scheme (30) to obtain the next value, until the present value is close enough to the previous one, or the calculated u value is close enough to the measured u^*. At each iteration step, we use the elliptic problem solver called ELLPACK (see [7]) to solve for $u(x; p_j)$ and $u^{(1)}(x; p_j)$. Notice that in the problem for $u^{(1)}(x; p_j)$, that is, $(4)_1$, we need the values of $u(x; p_j)$. The result is given in Table 1. For comparison we use the bisection method shown in Table 2. Notice that the same problem takes Newton's method eight iteration steps (i.e., solve 16 elliptic problems) to converge while the bisection method takes 21 steps (solve 21 elliptic problems).

TABLE 1. Newton's method with $p_0 = 2.0$ and $u^* = 1.0$. The estimated $p^* = 0.971166$.

j	p_j	$u(x_0; p_j)$
0	2.000000	0.527925
1	0.500000	1.863598
2	0.734298	1.295691
3	0.904528	1.067542
4	0.962841	1.007927
5	0.970565	1.000568
6	0.971127	1.000037
7	0.971164	1.000002
8	0.971166	1.000000

TABLE 2. Bisection method with $p_0 = 2.0$ and $u^* = 1.0$. The estimated $p^* = 0.971166$.

j	p_j	$u(x_0; p_j)$	j	p_j	$u(x_0; p_j)$
0	2.000000	0.527925	11	0.971680	0.999516
1	1.000000	0.973563	12	0.971191	0.999976
2	0.500000	1.863598	13	0.970947	1.000207
3	0.750000	1.270314	14	0.971069	1.000092
4	0.875000	1.100757	15	0.971130	1.000034
5	0.937500	1.032924	16	0.971161	1.000005
6	0.968750	1.002287	17	0.971176	0.999991
7	0.984375	0.987697	18	0.971169	0.999998
8	0.976563	0.994934	19	0.971165	1.000002
9	0.972656	0.998596	20	0.971167	0.999999
10	0.970703	1.000437	21	0.971166	1.000000

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