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ON THE QUENCHING RATE ESTIMATE

By

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§1. Introduction. In this paper, we consider the problem

$$\begin{aligned} &u_t - u_{xx} = (1-u)^{-\beta}, \qquad |x| < l,\ t > 0, \\ &u(\pm l, t) = 0, \qquad t > 0, \\ &u(x, 0) = u_0(x), \qquad |x| \le l, \end{aligned} \tag{1.1}$$

where $\beta > 0$, $l > 0$, $0 \le u_0 < 1$ is smooth. The solution u of (1.1) is said to be quenching if u reaches 1 in finite time T. Note that in this case u_t blows up at the same time T. This phenomenon has been studied by many authors (see, for example, the references cited in [8] and [11]). In particular, for any $\beta > 0$ there exists a positive constant $l_* = l_*(\beta)$ such that u quenches for any u_0 if $l > l_*$. Hereafter we shall assume that u quenches and that u_0 satisfies

$$u_0'' + (1-u_0)^{-\beta} \ge 0. \tag{1.2}$$

Let $T \in (0, \infty)$ be the quenching time for u. We say that a is a quenching point for u if there exists a sequence $\{(x_n, t_n)\}$ such that $x_n \to a$, $t_n \uparrow T$, and $u(x_n, t_n) \to 1$ as $n \to \infty$. The set of all such points (for the same T) is called the quenching set.

In [8] we first proved that the quenching set consists of finite points which remain a positive distance from $x = \pm l$. Then we studied the quenching rate of the solution near any quenching point. Let $\gamma = (\beta + 1)^{-1}$ and $k = \gamma^{-\gamma}$. We obtained the following quenching rate estimate.

Theorem A. If a is a quenching point, then

$$\lim_{t \uparrow T} (1 - u(x, t))(T - t)^{-\gamma} = k \tag{1.3}$$

uniformly for $|x - a| \le C\sqrt{T - t}$ for any positive constant C.

But, there we only proved this theorem for $\beta \ge 3$. In [11], Levine conjectured that this theorem should hold for all $\beta > 0$. Recently, Fila and Hulshof [3] improved this result to any $\beta \ge 1$ using a convexity argument of [5]. The purpose of this paper is to complete this quenching rate estimate for any $\beta > 0$.

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Recall the similarity variables

$$y = (x-a)/\sqrt{T-t}, \qquad s = -\ln(T-t),$$
$$w(y, s) = (1-u(x, t))(T-t)^{-\gamma}.$$

Then w satisfies the equation

$$w_s = w_{yy} - \tfrac{1}{2} y w_y + \gamma w - w^{-\beta} \tag{1.4}$$

in the set $W = \{(y, s) : |y \exp(-s/2) + a| < l, s > -\ln T\}$. The key step for the proof of Theorem A is to show that the only positive global solution to

$$w'' - \tfrac{1}{2} y w' + \gamma w - w^{-\beta} = 0, \qquad y \in \mathbf{R}, \tag{1.5}$$

under some growth condition at $|y| = \infty$, is the constant solution $w \equiv k$.

We note that the quenching rate estimate for the corresponding higher-dimensional radial case for $\beta > 1$ was obtained by the author in [9]. Recently, Fila, Hulshof, and Quittner [4] have completed this result for all $\beta > 0$.

This paper is organized as follows. In Sec. 2, we give the asymptotic behavior of solutions w of (1.5) at $|y| = \infty$ by using the method of [9](see also [2]). Then we apply this result to obtain the quenching rate estimate (1.3) for $0 < \beta < 1$ in Sec. 3.

§**2. Asymptotic behaviors.** In this section, we let $w = w(y)$ be any positive global solution of (1.5). The main result of this section is as follows.

THEOREM 2.1. If $w(y)$ is not identically equal to k, then $w(y)$ behaves either as $|y|^{2\gamma}$ or as $|y|^{-(1+2\gamma)} \exp(y^2/4)$ at $|y| = \infty$.

Let

$$f(w) = \gamma w - w^{-\beta} \quad \text{and} \quad F(w) = \int_k^w f(s)\,ds, \qquad w > 0.$$

Note that $f(w) > 0$ if $w > k$, < 0 if $w < k$; and $F(w) > 0$ for $w \neq k$. Rewrite (1.5) as

$$[w'^2/2 + F(w)]'(y) = y w'^2(y)/2. \tag{2.1}$$

Since the right-hand side of (2.1) is nonnegative for $y \geq 0$, the limit

$$\lim_{y \to \infty} [w'^2/2 + F(w)](y) = l_+$$

exists and is nonnegative. We claim that $l_+ > 0$ if $w \not\equiv k$. Indeed, if $l_+ = 0$, then $w'(y) \to 0$ and $w(y) \to k$ as $y \to \infty$. By integrating (2.1) from y to ∞, we obtain that $w'(y) \equiv 0$ and $w \equiv k$ for $y \geq 0$. This contradiction leads to the conclusion $l_+ > 0$. Let $v = w'$. Then $v' = (y/2)v - f(w)$.

LEMMA 2.2. Let α be a positive constant. Then the region

$$A_\alpha \equiv \{(w, v) : w \geq k, v \geq \alpha w\}$$

is a positively invariant region, i.e., there exists $y_0 = y_0(\alpha)$ such that if $y_1 \geq y_0$ and $(w(y_1), v(y_1)) \in A_\alpha$ then $(w(y), v(y)) \in A_\alpha$, $\forall y \geq y_1$.

Proof. Take $y_0 = 2(\alpha + \gamma/\alpha)$. Since the vector field $(w'(y), v'(y))$ is always pointing inward to the region A_α for $y \geq y_0$, the lemma follows. □

LEMMA 2.3. Suppose that $w(y) \not\equiv k$. Then w cannot assume the value k at infinitely many points as $y \to \infty$.

Proof. Suppose that there is a sequence $y_m \to \infty$ as $m \to \infty$ such that $w(y_m) = k$, $\forall m$. Without loss of generality we may assume that $w'(y_m) > 0$ for all m. Recall that $l_+ > 0$. Hence

$$w'(y_m) \to \sqrt{2l_+} \quad \text{as } m \to \infty.$$

Take any number $\alpha \in (0, \sqrt{2l_+})$ and consider the positively invariant set $A = A_{\alpha/k}$. Let $y_0 = y_0(\alpha/k)$ be the constant obtained in Lemma 2.2. Then there is an m_0 sufficiently large such that $w'(y_{m_0}) \geq \alpha$ and $y_{m_0} \geq y_0$. Since $(w(y_{m_0}), v(y_{m_0})) \in A$, from Lemma 2.2 it follows that $(w(y), v(y)) \in A$ for all $y \geq y_{m_0}$. Thus $v(y) > 0$, $\forall y \geq y_{m_0}$, which is a contradiction and the lemma is proved. □

Let $w(y)$ be a nonconstant positive global solution of (1.5). From Lemma 2.3 it follows that either $w(y) > k$, $\forall y \geq \bar{y}$, or $w(y) < k$, $\forall y \geq \bar{y}$, for some $\bar{y} > 0$. From the differential equation (1.5), we observe that any critical point y of w is a local maximum point if $w(y) > k$, and is a local minimum point if $w(y) < k$. Moreover, by the local uniqueness of solutions of ordinary differential equations, there cannot exist a point y with $w(y) = k$ and $w'(y) = 0$ except when $w \equiv k$. Therefore, there is $y_0 \geq \bar{y}$ such that either

$$w'(y) > 0, \quad \forall y \geq y_0, \quad \text{or} \quad w'(y) < 0, \quad \forall y \geq y_0.$$

Let

$$L = \lim_{y \to \infty} w(y).$$

We claim that $L > k$, if $w'(y) > 0$ for all $y \geq y_0$. Indeed, if $L \leq k$, then from the equation

$$w'' = yw'/2 - f(w)$$

we obtain that $w''(y) > 0$ for $y \geq y_0$. Thus

$$w(y) = w(y_0) + \int_{y_0}^{y} w'(\xi)d\xi \geq w'(y_0)(y - y_0) \to \infty$$

as $y \to \infty$, a contradiction. Hence $L > k$, if $w' > 0$, $\forall y \geq y_0$. Similarly, $L < k$, if $w'(y) < 0$ for all $y \geq y_0$.

LEMMA 2.4. If $w \not\equiv k$, then w must be strictly increasing to $+\infty$ as $y \to \infty$.

Proof. Suppose that $L < \infty$. Since the integral

$$\int_{y_0}^{\infty} w'(y)dy = L - w(y_0)$$

is finite, there is a sequence $y_m \to \infty$ such that $w'(y_m) \to 0$ as $m \to \infty$. Rewrite (1.5) as

$$w''/y - w'/2 = -f(w)/y$$

and integrate it from y_0 to y_m. Using an integration by parts for the first term, the integral on the left-hand side remains bounded as $m \to \infty$. But,

$$\left| \int_{y_0}^{y_m} \frac{f(w(y))}{y} dy \right| \to \infty$$

as $m \to \infty$. This contradiction leads to the conclusion of the lemma. □

Proof of Theorem 2.1. From Lemma 2.4 we can easily obtain the asymptotic behavior of $w(y)$ at $y = \infty$ by the method used in [9]. The motivation is from Remark 2 in [2] and the proof is based on using L'Hôpital's rule (cf. [2] and [9]). For the reader's convenience, we outline the proof here.

First, using Lemma 2.2 and 2.4, we can show that the limit

$$\alpha \equiv \lim_{y \to \infty} \frac{w'(y)}{w(y)}$$

exists and that α is either 0 or ∞. Suppose that $\alpha = 0$. Then we have

$$\lim_{y \to \infty} \frac{y w'(y)}{w(y)} = 2\gamma,$$

by applying L'Hôpital's rule and using the formula

$$\frac{w'(y)}{w(y)} = \exp(y^2/4) \int_y^\infty [\gamma + a(s)] \exp(-s^2/4) ds,$$

where $a(s) \to 0$ as $s \to \infty$. Thus, from

$$\lim_{y \to \infty} y^\delta \left[\frac{y w'(y)}{w(y)} - 2\gamma \right] = 0$$

for $\delta \in (0, 2)$, we conclude that for any $\delta \in (0, 2)$ there exists a positive constant C_δ such that

$$w(y) = C_\delta y^{2\gamma} [1 + o(y^{-\delta})]$$

as $y \to \infty$. The case for $\alpha = \infty$ is similar.

The asymptotic behavior of $w(y)$ at $y = -\infty$ follows by a similar argument and the theorem is proved. □

In the sequel, we shall call a nonconstant positive global solution of (1.5), which behaves as $|y|^{2\gamma}$ both at $y = \infty$ and at $y = -\infty$, as a *slow orbit*.

REMARK 2.5. For any $\beta > 0$, using the Sturm comparison theorem (cf. [10]), we can show that every nonconstant positive solution of (1.5) must be strictly convex for all y sufficiently large and/or for all $-y$ sufficiently large. Hence there is no slow orbit for $\beta \geq 1$.

The proof of Remark 2.5 is quite similar to that of [10, Theorem 2]. Here we compare the function w'' with the function $w^{-\beta} - w''$ and compare the function $w^{-\beta} - w''$ with w'. Then, for a nonconstant positive solution $w(y)$ of (1.5), if $y_0 \geq 0$ (which we may assume without loss of generality) is such that $w'(y_0) =$ and $w'(y) > 0$ for all $y > y_0$, we have $w''(y) > 0$ for all $y \geq y_0$.

For $0 < \beta < 1$, let

$$\phi(y) = K|y|^{2\gamma}, \qquad K = [2\gamma(2\gamma - 1)]^{-\gamma}.$$

Note that K is positive, since $\gamma > \frac{1}{2}$ in this case. Also, $\phi(y)$ is a continuously differentiable function satisfying (1.5) for all $y \neq 0$. Similar to [9, Theorem 3.4], we have the following result.

THEOREM 2.6. Any slow orbit $w(y)$ must intersect the function $\phi(y)$ at least twice in $y > 0$ and/or in $y < 0$.

Proof. Since the proof is quite similar to that of [9, Theorem 3.4], we only sketch the proof. To begin with, we take the minimum point y_0 of w. Without loss of generality we may assume that $y_0 \geq 0$. First, let

$$g(y) = \gamma w(y) - \tfrac{1}{2} y w'(y), \quad h(y) = w''(y), \quad \text{and} \quad V(y) = g(y)h'(y) - g'(y)h(y).$$

Then

$$V(y) = \rho^{-1}(y)\{\rho(y_0)V(y_0) + \int_{y_0}^{y} \rho(t)\beta(\beta+1)w^{-(\beta+2)}(t)[w'(t)]^2 g(t)\,dt\}$$

where $\rho(y) = \exp(-y^2/4)$, and

$$h(y) = \frac{h(y_0)}{g(y_0)} g(y) + g(y)\int_{y_0}^{y} \frac{V(t)}{g^2(t)}\,dt.$$

Recall that $g(y) \to 0$ and $g'(y) \to 0$ as $y \to \infty$ for any slow orbit $w(y)$. By the choice of y_0, we have $V(y_0) \geq 0$ and $g(y_0) > 0$. If $g(y) > 0$ for all $y \geq y_0$, then we will have $V(y) \to \infty$ as $y \to \infty$. Hence $h(y) \to \infty$ as $y \to \infty$, a contradiction. Therefore, $g(y)$ must have a zero in $y > y_0$.

Next, let

$$U(y) = w(y)\phi'(y) - w'(y)\phi(y).$$

Then $U(0) = 0$ and U satisfies

$$\begin{aligned} &U(y) = 2Ky^{2\gamma-1} g(y), \\ &U' - \frac{y}{2}U = w\phi[\phi^{-(\beta+1)} - w^{-(\beta+1)}], \qquad y \neq 0. \end{aligned}$$

If $w(y_0) < \phi(y_0)$, then clearly w intersects ϕ at least once in $(0, y_0)$. From here and proceeding as in the proof of Lemma 3.6 of [9], we obtain that w intersects ϕ at least twice in $y > 0$. Hence the theorem follows. □

§**3. The quenching rate.** We assume that $\beta \in (0, 1)$ and hence $2\gamma > 1$. Let a be a quenching point for u. Without loss of generality we may assume that $a = 0$. First, we recall that $w(y, s) \geq 1/B$ in W for some positive constant B. Applying the maximum principle (cf. [6]) to the function

$$J(x, t) = \frac{1}{2}u_x^2 - \frac{C}{1-\beta}(1-u)^{1-\beta}$$

for some constant $C \geq 1$, we obtain that

$$|u_x| \leq \sqrt{\frac{C}{1-\beta}}(1-u)^{(1-\beta)/2} \quad \text{in } (-l, l) \times (0, T). \tag{3.1}$$

From (3.1) it follows that

$$w(y, s) \leq C(|y|^{2\gamma} + 1) \quad \text{in } W. \tag{3.2}$$

Now, applying the energy method of [7] (for details see [8]), we can show that $w(y, s)$ tends to a positive global solution of (1.5) as $s \to \infty$. By (3.2), this limit function must be a slow orbit, if it is not identically equal to the constant k.

We claim that this limit function intersects $\phi(y)$ at most once both in $y > 0$ and in $y < 0$. First, consider the case $y > 0$. If $w(y, s) > \phi(y)$ in $W \cap \{y > 0\}$, then we are done. Otherwise, we choose $s_0 < \infty$ such that $w(y_0, s_0) \leq \phi(y_0)$ for some $y_0 > 0$ with $(y_0, s_0) \in W$. Then proceeding as in the proofs of [1, Sec. 3] there is a $\delta > 0$ such that $w(y, s)$ intersects $\phi(y)$ exactly once in $(0, \delta \exp(s/2))$ for all $s > s_0$. The case for $y < 0$ is similar. Therefore, the assertion follows. By Theorem 2.6, this limit function must be identically equal to the constant k. Hence Theorem A is proved for $0 < \beta < 1$.

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References

[1] J. Bebernes and D. Eberly, *A description of self-similar blow-up for dimensions* $n \geq 3$, Ann. Inst. Henri Poincaré **5**, 1–21 (1988)

[2] H. Brezis, L. A. Peletier, and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal. **95**, 185–209 (1986)

[3] M. Fila and J. Hulshof, *A note on the quenching rate*, Proc. Amer. Math. Soc., to appear

[4] M. Fila, J. Hulshof, and P. Quittner, *The quenching problem on N-dimensional ball*, preprint

[5] A. Friedman, J. Friedman, and B. McLeod, *Concavity of solutions of nonlinear ordinary differential equations*, J. Math. Anal. Appl. **131**, 486–500 (1988)

[6] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34**, 425–447 (1985)

[7] Y. Giga and R. V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38**, 297–319 (1985)

[8] J. Guo, *On the quenching behavior of the solution of a semilinear parabolic equation*, J. Math. Anal. Appl. **151**, 58–79 (1990)

[9] J. Guo, *On the semilinear elliptic equation* $\Delta w - \frac{1}{2} y \cdot \nabla w + \lambda w - w^{-\beta} = 0$ *in* $\mathbf{R}^n$, IMA Preprint Series # 531 (June 1989), Minnesota

[10] H. G. Kaper and M. K. Kwong, *Concavity and monotonicity properties of solutions of Emden-Fowler equations*, Differential Integral Equations **1**, 327–340 (1988)

[11] H. A. Levine, *Advances in quenching*, Proceedings of the International Conference on Reaction-Diffusion Equations and Their Equilibrium States, Gregynog, Wales, August 1989