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# VOLTAGE-CURRENT CHARACTERISTICS OF MULTI-DIMENSIONAL SEMICONDUCTOR DEVICES

By

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**Abstract.** The steady state drift-diffusion model for the flow of electrons and holes in semiconductors is simplified by perturbation techniques. The simplifications amount to assuming zero space charge and low injection. The limiting problems are solved and explicit formulas for the voltage-current characteristics of bipolar devices can be obtained. As examples, the $pn$-diode, the bipolar transistor and the thyristor are discussed. While the classical results of a one-dimensional analysis are confirmed in the case of the diode, some important effects of the higher dimensionality appear for the bipolar transistor.

**1. Introduction.** The classical drift-diffusion model for the steady flow of negatively charged electrons (density $n(\mathbf{x})$) and positively charged holes (density $p(\mathbf{x})$) in a semiconductor consists of the continuity equations

$$\operatorname{div} \mathbf{J}_n = -\operatorname{div} \mathbf{J}_p = R, \tag{1.1a}$$

the current relations

$$\delta^4 \mathbf{J}_n = \mu_n(\nabla n - n\nabla V), \qquad \delta^4 \mathbf{J}_p = -\mu_p(\nabla p + p\nabla V), \tag{1.1b}$$

and the Poisson equation

$$\lambda^2 \Delta V = n - p - C \tag{1.1c}$$

for the electrostatic potential $V(\mathbf{x})$. The mobilities $\mu_n, \mu_p > 0$ and the doping profile $C$ are assumed to be given functions of position $\mathbf{x} \in \Omega$, where the bounded domain $\Omega \subset R^k$, $k = 1, 2$, or $3$, represents the semiconductor part of the device. For the recombination-generation rate $R$ we use a mass action law of the form

$$R = Q(n, p, \mathbf{x})(np/\delta^4 - 1), \qquad Q \geq 0,$$

which includes the standard models for band-to-band processes and recombination via traps in the forbidden band. Since the discussion of this work is restricted to low

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injection situations, our model certainly describes the relevant physical phenomena [7].

The equations are in dimensionless form. The reference quantity for the particle densities $n, p, C$ is the maximal doping concentration $C_{\max}$, i.e., $\max_\Omega |C| = 1$. The potential has been scaled by the thermal voltage $U_T = kT/q$ where $k, T$, and $q$ denote the Boltzmann constant, the lattice temperature and the elementary charge, respectively. The reference length $L$ is the diameter of the device and, thus, $\operatorname{diam}(\Omega) = 1$. The scaled minimal Debye length $\lambda$ and intrinsic number $\delta^2$ are dimensionless parameters and given by

$$\lambda = \frac{1}{L}\sqrt{\frac{\varepsilon U_T}{q C_{\max}}}, \qquad \delta^2 = \frac{n_i}{C_{\max}}$$

where $\varepsilon$ is the permittivity and $n_i$ the intrinsic number of the semiconductor. The nondimensionalization differs from that commonly used (see [4, 7]) by the scaling of the current densities $\mathbf{J}_n$, $\mathbf{J}_p$ where the introduction of the factor $\delta^4$ seems somewhat artificial. However, this choice is justified by the analysis below.

Regions where the doping profile $C$ is positive, are called $n$-regions because the positively charged impurity ions attract electrons. On the other hand, in $p$-regions the doping profile is negative. The $(k-1)$-dimensional boundaries between $n$- and $p$-regions are called $pn$-junctions. Here we assume abrupt junctions, i.e., the doping profile has jumps across these junctions and is bounded away from zero within the $n$- and $p$-regions.

The boundary $\partial\Omega$ of the device is the disjoint union of Ohmic contacts $C_1, \dots, C_m$ and artificial or insulating boundary segments $\partial\Omega_n$. At Ohmic contacts we assume zero space charge and thermal equilibrium:

$$n - p - C = 0, \qquad np = \delta^4, \quad \text{at } C_1, \dots, C_m$$

which translates to Dirichlet boundary conditions for the charge carrier densities:

$$n = \frac{1}{2}\left(C + \sqrt{C^2 + 4\delta^4}\right), \qquad p = \frac{1}{2}(-C + \sqrt{C^2 + 4\delta^4}), \quad \text{at } C_1, \dots, C_m. \tag{1.2a}$$

For a voltage controlled device, the values of the potential along the contacts are also prescribed:

$$V = V_{bi} - U_j, \quad \text{at } C_j,\ j = 1, \dots, m \tag{1.2b}$$

where $U_i - U_j$ is the external voltage between the contacts $C_i$ and $C_j$, and the built-in potential is given by

$$V_{bi} = \ln \frac{C + \sqrt{C^2 + 4\delta^4}}{2\delta^2}.$$

Along the artificial and insulating boundary segments, we assume that the normal components of the electric field and the electron and hole current densities vanish. This amounts to homogeneous Neumann conditions for $V, n, p$:

$$\frac{\partial V}{\partial \nu} = \frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0, \quad \text{at } \partial\Omega_N \tag{1.2c}$$

where $\nu$ denotes the unit outward normal.

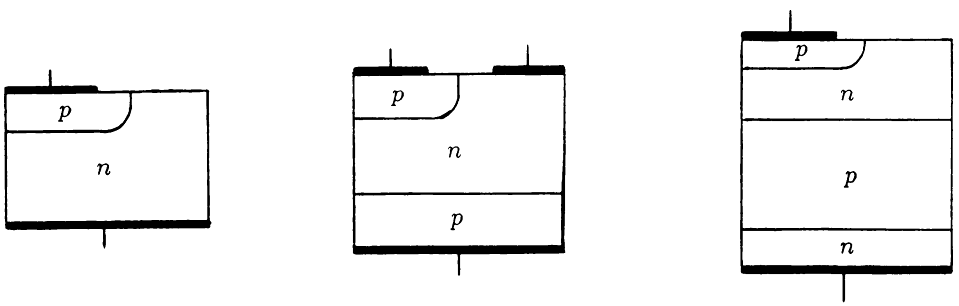
After substitution of the current relations into the continuity equations, (1.1), (1.2) constitutes an elliptic boundary value problem for $V$, $n$, $p$. The important output quantities are the currents through the Ohmic contacts. The current $I_j$ leaving the device through the contact $C_j$ is given by

$$I_j = \int_{C_j} (\mathbf{J}_n + \mathbf{J}_p) \cdot \nu ds.$$

The dependence of the currents on the $m-1$ contact voltages $U_j - U_1$, $j = 2, \ldots, m$ is called the voltage-current characteristic of the device. Obviously, the choice of $U_1$ does not influence the result. Note that we only need to compute $m-1$ currents, since the total current density $\mathbf{J}_n + \mathbf{J}_p$ is divergence free, implying $I_1 + \cdots + I_m = 0$.

The current-voltage characteristic is determined by the number and location of the $n$- and $p$-regions as well as of the Ohmic contacts. We consider devices meeting the following requirements: There is a finite number of open connected $n$-regions whose union is denoted by $\Omega_+$. In the same way, the number of $p$-regions is finite and their union is denoted by $\Omega_-$. Each $n$- or $p$-region has at most one contact and each contact is adjacent to only one $n$- or $p$-region. The union of the $pn$-junctions is denoted by $\Gamma = \overline{\Omega}_+ \cap \overline{\Omega}_-$. Note that these assumptions do not rule out so-called floating regions without any contacts. In Fig. 1 two-dimensional cross sections of three typical devices are depicted. The $pn$-diode consists of one $n$- and one $p$-region, each with a contact. The bipolar transistor has three differently doped regions with contacts. Finally, the thyristor is a $pnpn$-structure. Here we consider the so-called Shockley diode where the two middle layers are floating regions. In Secs. 5, 6, and 7 of this work the voltage-current characteristics of these devices are discussed.

The dimensionless parameters $\lambda$ and $\delta^2$ are small compared to 1 in practical applications. Therefore we shall try to simplify problem (1.1), (1.2) by letting these parameters tend to zero. The limit $\lambda \to 0$ is carried out in Sec. 2. Essentially, it amounts to replacing the left hand side of the Poisson equation (1.1$c$) by zero which




a) b) c)

Fig. 1. Cross sections of a) $pn$-diode, b) bipolar transistor, and c) thyristor.

explains the name zero space charge approximation for the simplified problem. However, since the doping profile has jump discontinuities, the limiting carrier densities also have jumps. The nature of these jumps can be analyzed by introducing slow variables which are continuous in the limit. The construction of formal asymptotic approximations — including layer corrections at the *pn*-junctions — has received a considerable amount of attention in the literature [2, 3, 4, 5, 6, 7, 8 ,10, 19]. In the one-dimensional case, these approximations can be justified by general results for singularly perturbed boundary value problems [14]. For higher dimensions, however, the situation is more involved. A justification for situations close to thermal equilibrium, i.e., small enough applied voltages, can be found in [3]. In [2], a convergence result for a simplified problem with constant mobilities and without recombination-generation is given. The treatment of Sec. 2 is in the spirit of this work. We use a well-known existence result [4] for (1.1), (1.2) and additional a priori estimates for a justification of the limiting procedure. The result is a boundary value problem for a system of two nonlinear elliptic equations.

Although the number of equations has been reduced, the zero space charge problem is essentially not much easier to solve than (1.1), (1.2). Therefore, a further simplification is introduced in Sec. 3 by letting $\delta^2$ tend to zero. Keeping the applied voltages fixed as the built-in potential tends to infinity (for $\delta^2 \to 0$) can be interpreted as a low injection condition. Since the analysis of Sec. 2 provides an existence result for the zero space charge problem, it is only necessary to obtain a few more estimates for justifying the limit. In [7] it has been shown that for the thermal equilibrium problem (consisting only of a nonlinear Poisson equation) the limits $\lambda \to 0$ and $\delta^2 \to 0$ commute. The limit $\delta^2 \to 0$ with $\lambda$ kept positive leads to a free boundary problem [13], which has been analyzed in [12]. Although the author conjectures that the limits commute also in the general case, no proof seems to be available. The problem with $\delta^2 \to 0$, $\lambda$ fixed, has particular importance in VLSI applications, since for very small devices $\lambda$ can be considerably large whereas $\delta^2 << 1$ is always a safe assumption.

It turns out that in the low injection limit the problem is simplified considerably. Its solution is discussed in Sec. 4. The voltage-current characteristic can be given in terms of the solution of a set of algebraic equations containing parameters which can be interpreted as conductivities of the *n*- and *p*-regions in certain reference situations. These conductivities are computed from the solutions of a number of linear elliptic boundary value problems. The set of algebraic equations is nontrivial only if floating regions occur. We can show that the solution is unique if the device has at most one floating region. Thus, we have uniqueness for the *pn*-diode and the bipolar transistor, but in general the question of uniqueness remains open. In Sec. 7 it is shown that the solution is also unique for the Shockley diode. This seems to contradict results [17] that a thyristor has multiple steady states in certain biasing situations. However, in the low injection limit we can only expect to obtain the so-called blocking branch of the characteristic.

This shows that the simplified model cannot describe physical effects caused by large electrical currents (high injection). Another limitation originates from the zero space charge assumption. We neglect the $pn$-junction layers where the space charge density takes appreciable values. This is justified, because the width of these space charge regions is $O(\lambda)$ in our scaling. It is well known, however, that the width grows with the potential jump across the junction. Effects involving large applied biases and, therefore, widening depletion regions can be accounted for by an asymptotic analysis of a rescaled problem [1, 9, 11, 12].

The zero space charge and low injection assumptions have already been used for one-dimensional model problems in the early physical literature on semiconductor devices [15]. In particular, the famous Shockley equation for the voltage-current characteristic of a $pn$-diode and the qualitative behavior of bipolar transistors are derived in this way (see also [18]). Thus, the present work can be viewed as an extension of these results. In Sec. 5, our theory is applied to a multi-dimensional model for a $pn$-diode. The Shockley equation is confirmed, and it is shown how the leakage current can be computed. Section 6 deals with the bipolar transistor. Here the common-emitter current gain is an important parameter. We demonstrate that it strongly depends on the geometry of the base region. This multi-dimensional effect is not captured by the classical one-dimensional analysis. Finally, the Shockley diode is considered in Sec. 7. Steinrück [17] showed that not every $pnpn$-device functions as a thyristor. We extend some of his results to the multi-dimensional case. In particular, it is shown that the voltage-current characteristic changes qualitatively, when a certain parameter passes through a critical value.

**2. The zero space charge approximation.** The analysis of (1.1), (1.2) is greatly facilitated by the introduction of the so-called Slotboom variables [16] $u$ and $v$ instead of the carrier densities:

$$n = \delta^2 e^V u, \qquad p = \delta^2 e^{-V} v.$$

This symmetrizing transformation for the continuity equations changes (1.1), (1.2) to the differential equations

$$\begin{aligned} \operatorname{div}(\mu_n \delta^2 e^V \nabla u) &= \delta^4 R, \\ \operatorname{div}(\mu_p \delta^2 e^{-V} \nabla v) &= \delta^4 R, \\ \lambda^2 \Delta V &= \delta^2 e^V u - \delta^2 e^{-V} v - C, \end{aligned} \tag{2.1}$$

subject to the boundary conditions

$$\begin{gathered} u = e^{U_j}, \quad v = e^{-U_j}, \quad V = V_{bi} - U_j, \quad \text{at } C_j,\ j = 1, \dots, m \\ \frac{\partial V}{\partial \nu} = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad \text{at } \partial\Omega_N. \end{gathered} \tag{2.2}$$

Now the recombination-generation rate is given by

$$R = Q(\delta^2 e^V u, \delta^2 e^{-V} v, \mathbf{x})(uv - 1).$$

Note that the differential operators in the continuity equations are formally self-adjoint. Also we expect the derivatives of the Slotboom variables to be bounded uniformly with respect to $\lambda$. This makes them slow variables in the language of singular perturbation theory. Before we can state an existence result for (2.1), (2.2), a few regularity assumptions for the data are needed: The domain $\Omega$ is Lipschitz and the $(k-1)$-dimensional Lebesgue measure of the union of the contacts is positive. The Dirichlet boundary data for $V$, $u$, $v$ can be extended to $\Omega$ as functions in $H^1(\Omega)$. The doping profile and the mobilities are in $L^\infty(\Omega)$ with the mobilities bounded away from zero. The nonnegative reaction rate $Q$ in the recombination-generation term is a smooth function of the carrier densities as well as a bounded function of position.

A proof of the following theorem can be found in [4].

THEOREM 2.1. Under the above assumptions the problem (2.1), (2.2) has a solution $(V, u, v) \in (H^1(\Omega) \cap L^\infty(\Omega))^3$ which satisfies

$$e^{-U_+} \le u, v \le e^{U_+}, \quad \text{with } U_+ = \max_j |U_j|,$$

$$\|V\|_\infty \le c,$$

where $c$ is independent of $\lambda$ and $\|\cdot\|_p$ denotes the $L^p(\Omega)$-norm

For computing the currents through the contact, the current densities

$$\delta^4 \mathbf{J}_n = \mu_n \delta^2 e^V \nabla u, \qquad \delta^4 \mathbf{J}_p = -\mu_p \delta^2 e^{-V} \nabla v$$

have to be evaluated along the contacts. This is not necessarily possible since the existence result only guarantees current densities in $L^2(\Omega)$. Therefore the currents have to be defined in a different way. Let $\varphi_j$ be in $H^1(\Omega)$ with $\varphi_j = \delta_{ij}$, at $C_i$. Then we obviously have

$$I_j = \int_{\partial\Omega} \varphi_j (\mathbf{J}_n + \mathbf{J}_p) \cdot \nu \, ds = \int_\Omega (\mathbf{J}_n + \mathbf{J}_p) \cdot \nabla \varphi_j \, d\mathbf{x} \tag{2.3}$$

by integration by parts. The term on the right-hand side is certainly well defined and can serve as a definition for the current. It is easy to see that it is independent of the particular choice for $\varphi_j$ if the total current density is in the weak sense divergence free.

It is important in our context that the $L^\infty$ estimates in Theorem 2.1 are uniform with respect to $\lambda$. The following lemma contains estimates for the derivatives of the solution.

LEMMA 2.1. Solutions of (2.1), (2.2) satisfy

$$\lambda \|\nabla V\|_2 \le c, \qquad \|\nabla u\|_2 \le c, \ \|\nabla v\|_2 \le c$$

with $c$ independent of $\lambda$.

*Proof.* Let $V_D$ be an $H^1(\Omega)$-extension of the Dirichlet data for the potential. Then, multiplication of the Poisson equation by $V - V_D$ and integration by parts immediately imply the first estimate. The other two estimates are proved analogously. □

The next result contains an interior estimate, where 'interior' means away from $pn$-junctions and from the boundary. We denote by $\Omega_0 = \Omega \backslash \Gamma$ the union of all the $p$- and $n$-regions. The smoothness assumptions

$$\mu_n, \mu_p \in H^1(\Omega_0), \qquad C \in W^{2,1}(\Omega_0)$$

will be used.

LEMMA 2.2. Under the above assumptions solutions of (2.1), (2.2) satisfy

$$\|\nabla V\|_{L^2(\Omega')} \le c,$$
$$\|\delta^2 e^V u - \delta^2 e^{-V} v - C\|_{L^2(\Omega')} \le \lambda c$$

for $\Omega' \subset\subset \Omega_0$ with $c$ independent of $\lambda$.

*Proof*. We denote the space charge density by

$$F(V, \mathbf{x}) = \delta^2 e^V u(\mathbf{x}) - \delta^2 e^{-V} v(\mathbf{x}) - C(\mathbf{x})$$

and introduce a nonnegative test function $\varphi \in C_0^\infty(\Omega_0)$ with $\varphi = 1$ in $\Omega'$. Multiplication of the Poisson equation by $\varphi F$ and integration by parts gives

$$\lambda^2 \int_\Omega \varphi \frac{\partial F}{\partial V} |\nabla V|^2 d\mathbf{x} + \int_\Omega \varphi F^2 d\mathbf{x} = -\lambda^2 \int_\Omega \nabla V \cdot (\varphi \nabla_{\mathbf{x}} F + F \nabla \varphi) d\mathbf{x} \tag{2.4}$$

where

$$\nabla_{\mathbf{x}} F = \frac{\delta^4}{\mu_n} \mathbf{J}_n + \frac{\delta^4}{\mu_p} \mathbf{J}_p - \nabla C$$

holds, and $F \nabla V$ can be written as

$$F\nabla V = \nabla(\delta^2 e^V u + \delta^2 e^{-V} v - CV) - \delta^2 e^V \nabla u - \delta^2 e^{-V} \nabla v + V \nabla C.$$

The $L^\infty$ estimates of Theorem 2.1 imply that $\frac{\partial F}{\partial V} = \delta^2 e^V u + \delta^2 e^{-V} v$ is bounded away from zero uniformly with respect to $\lambda$. The left-hand side of (2.4) is thus bounded from below by

$$\lambda^2 c \|\nabla V\|^2_{L^2(\Omega')} + \|F\|^2_{L^2(\Omega')}$$

with a $\lambda$-independent positive constant $c$. After integration by parts the right-hand side of (2.4) can be written as

$$\lambda^2 \int_\Omega V \left( \delta^4 \mathbf{J}_n \cdot \nabla \frac{\varphi}{\mu_n} + \delta^4 R \frac{\varphi}{\mu_n} + \delta^4 \mathbf{J}_p \cdot \nabla \frac{\varphi}{\mu_p} - \delta^4 R \frac{\varphi}{\mu_p} - \operatorname{div}(\varphi \nabla C) \right) d\mathbf{x}$$
$$+ \lambda^2 \int_\Omega \Delta\varphi (\delta^2 e^V u + \delta^2 e^{-V} v - CV) d\mathbf{x}$$
$$+ \lambda^2 \int_\Omega \nabla \varphi \cdot (\delta^2 e^V \nabla u + \delta^2 e^{-V} \nabla v - V \nabla C) d\mathbf{x}$$

which can be bounded by $\lambda^2 c$ ($c$ independent of $\lambda$) because of the estimates in Theorem 2.1 and Lemma 2.1 and the assumption on the mobilities and the doping profile. The estimates of the lemma follow immediately.

We are now in the position to prove the main result of this section.

THEOREM 2.2. For every sequence $\lambda_k \to 0$ there exists a subsequence (again denoted by $\lambda_k$) such that the corresponding solutions $(V_k, u_k, v_k)$ of (2.1), (2.2) and the currents $I_{jk}$, $j = 1, \ldots, m$ given by (2.3), satisfy

$$V_k \to V_0 \quad \text{in } L^2(\Omega),$$
$$u_k \to u_0 \quad \text{weakly in } H^1(\Omega),$$
$$v_k \to v_0 \quad \text{weakly in } H^1(\Omega),$$
$$I_{jk} \to I_{j0} \quad \text{for } j = 1, \ldots, m$$

where $(V_0, u_0, v_0)$ is a solution of (2.1), (2.2) with $\lambda$ replaced by zero and the limiting currents can be computed from (2.3).

REMARK. Note that the limiting potential would not satisfy general Dirichlet boundary conditions since it is determined from an algebraic equation (the reduced Poisson equation). However, the assumption of zero space charge at the Ohmic contacts is compatible with the limiting problem, and therefore the limiting solution satisfies the complete set of Dirichlet conditions.

*Proof*. The convergence statements for $u_k$, $v_k$ and $I_{jk}$ follow directly from the a priori estimates. Lemma 2.2 implies convergence for the potential in $L^2(\Omega')$ with $\Omega' \subset\subset \Omega$. It is easy to show that convergence in $L^2(\Omega)$ can be concluded from the uniform $L^\infty(\Omega)$-estimate. It remains to carry out the limit in (2.1), (2.2). The right-hand side of the Poisson equation, considered as a function of $(V, u, v)$, defines a map on a bounded subset of $L^\infty(\Omega)^3$ which is continuous in terms of the $L^2(\Omega)$-norm. Also, by Lemma 2.2 and the $L^\infty(\Omega)$ estimates, this right-hand side tends to zero in $L^2(\Omega)$ as $k \to \infty$. In the limit we obtain

$$0 = \delta^2 e^{V_0} u_0 - \delta^2 e^{-V_0} v_0 - C, \quad \text{a.e. in } \Omega. \tag{2.5}$$

We multiply the electron continuity equation by a test function $\varphi \in C_0^\infty(\Omega \cup \partial\Omega_N)$. Integration by parts gives

$$-\int_\Omega \mu_n \delta^2 e^{V_k} \nabla u_k \cdot \nabla \varphi d\mathbf{x} = \int_\Omega \delta^4 R_k \varphi d\mathbf{x}$$

where $R_k$ is the recombination-generation rate evaluated at $(V_k, u_k, v_k)$. Like the right-hand side of the Poisson equation above, $e^V$ and $R$ are continuous in terms of the $L^2(\Omega)$-norm. The strong convergence of $R_k$ and $e^{V_k}$ and the weak convergence of $\nabla u_k$ allow to go to the limit in the above weak formulation of the electron continuity equation. Analogous arguments can be applied to the hole continuity equation.

The weak convergence in $H^1(\Omega)$ of $u_k$ and $v_k$ implies that their limits satisfy the Dirichlet boundary conditions. Solving (2.5) for $V_0$ at the contacts shows that $V_0$ also takes the values prescribed in (2.2). Finally, the estimates of Lemma 2.1 justify the passage to the limit in (2.3). □

**3. The low injection limit.** After elimination of the potential, the zero space charge problem can be written as

$$\operatorname{div}\left(\mu_n \frac{C + \sqrt{C^2 + 4\delta^4 uv}}{2v} \nabla u\right) = \delta^4 R,$$
$$\operatorname{div}\left(\mu_p \frac{-C + \sqrt{C^2 + 4\delta^4 uv}}{2v} \nabla v\right) = \delta^4 R, \tag{3.1}$$

subject to the boundary conditions

$$u = e^{U_j}, \quad v = e^{-U_j}, \quad \text{at } C_j,$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad \text{at } \partial\Omega_N. \tag{3.2}$$

The results of the previous section imply that (3.1), (3.2) has a solution which satisfies the estimates in Theorem 2.1 and which is close to solutions of (2.1), (2.2) for small values of $\lambda$. In this section we are concerned with the limit $\delta^2 \to 0$. In the zero space charge approximation, the carrier densities are given in terms of the Slotboom variables by

$$n = \frac{1}{2}\left(C + \sqrt{C^2 + 4\delta^4 uv}\right), \qquad p = \frac{1}{2}\left(-C + \sqrt{C^2 + 4\delta^4 uv}\right).$$

Since the a priori estimates for $\|u\|_\infty$ and $\|v\|_\infty$ are uniform with respect to $\delta^2$, it is obvious that

$$n = C + O(\delta^4), \quad p = O(\delta^4), \quad \text{in } n\text{-regions},$$
$$p = -C + O(\delta^4), \quad n = O(\delta^4), \quad \text{in } p\text{-regions}.$$

The statement that the density of the majority carriers (electrons in $n$-regions, holes in $p$-regions) is close to the modulus of the doping profile and the density of the minority carriers is small compared to that, is usually called a low injection condition.

The convergence analysis again relies on a priori estimates. We shall need another regularity assumption on the data: There are continuous extension operators from $H^1(\Omega_+)$ to $H^1(\Omega)$ as well as from $H^1(\Omega_-)$ to $H^1(\Omega)$.

LEMMA 3.1. Every solution of (3.1), (3.2) satisfies

$$\|\nabla u\|_2 \le c, \qquad \|\nabla v\|_2 \le c$$

with $c$ independent from $\delta^2$.

*Proof.* Let $u_D$ denote a $H^1(\Omega)$-extension of the Dirichlet data for $u$. Then, multiplication of the equation for $u$ by $u - u_D$ and integration by parts leads to the estimate

$$\int_\Omega \mu_n \frac{C + \sqrt{C^2 + 4\delta^4 uv}}{2u} |\nabla u|^2 d\mathbf{x} \le c$$

where $c$ is independent from $\delta^2$. The same estimate obviously holds with $\Omega$ replaced by $\Omega_+$. Since, in $\Omega_+$, the diffusion coefficient is bounded away from zero uniformly in $\delta^2$, this implies

$$\|\nabla u\|_{L^2(\Omega_+)} \le c.$$

Now we formulate a boundary value problem determining $u$ in $\Omega_-$. Because of our assumptions there is a uniformly bounded $H^1(\Omega)$-extension of $u|_{\Omega_+}$ which satisfies the Dirichlet conditions for $u$ at contacts adjacent to $\Omega_-$. We take this extension as Dirichlet datum for $u|_{\Omega_-}$ at $\Gamma$ and the contacts adjacent to $\Omega_-$. The continuity equation divided by $\delta^4$ reads

$$\operatorname{div}\left(\mu_n \frac{2v}{-C + \sqrt{C^2 + 4\delta^4 uv}} \nabla u\right) = R.$$

In $\Omega_-$, the diffusion coefficient is uniformly bounded and uniformly bounded away from zero. Thus, it is obvious that $\nabla u$ is uniformly bounded in $L^2(\Omega_-)$. The estimate for $\nabla u$ given in the lemma follows. For $v$ we proceed analogously with the roles of $\Omega_+$ and $\Omega_-$ interchanged. □

For the current densities, now given by

$$\delta^4 \mathbf{J}_n = \mu_n \frac{C + \sqrt{C^2 + 4\delta^4 uv}}{2u} \nabla u , \qquad \delta^4 \mathbf{J}_p = -\mu_p \frac{-C + \sqrt{C^2 + 4\delta^4 uv}}{2v} \nabla v ,$$

Lemma 3.1 implies that the minority carrier current densities are uniformly bounded, i.e.,

$$\|\mathbf{J}_n\|_{L^2(\Omega_-)} \leq c , \qquad \|\mathbf{J}_p\|_{L^2(\Omega_+)} \leq c .$$

An argument for the uniform boundedness of the majority carrier current densities can be given as follows: Let $\Omega_n$ denote an $n$-region with an Ohmic contact, where the boundary condition $u = e^{U_0}$ is prescribed. Then the function $w$, defined by

$$\delta^4 w = u - e^{U_o} ,$$

satisfies in $\Omega_n$ a differential equation with a bounded forcing term. It also satisfies an homogeneous Dirichlet condition at the contact and an homogeneous Neumann condition at $\partial\Omega_n \cap \partial\Omega_N$. At the $pn$-junctions $\partial\Omega_n \cap \Gamma$ adjacent to $\Omega_n$, we prescribe Neumann conditions using the continuity of the normal component of the electron current density. The Neumann data are uniformly bounded because of the uniform boundedness of $\mathbf{J}_n$ in $p$-regions. Thus, $w$, being the solution of a problem with uniformly bounded data, is itself uniformly bounded and so is the electron current density in $\Omega_n$, given by

$$\mathbf{J}_n = \mu_n \frac{C + \sqrt{C^2 + 4\delta^4 uv}}{2u} \nabla w .$$

For a floating $n$-region (without an Ohmic contact) the problem corresponding to that for $w$ is a pure Neumann problem with uniformly bounded Neumann data. A similar argument applies in this case. Obviously, the same reasoning can be used for the hole current density in $p$-regions. Making these arguments precise, however, requires the evaluation of the current densities at $pn$-junctions which is not necessarily possible with the regularity used until now. On the other hand, it turns out that the result is not needed in its full strength for the convergence analysis below. Therefore we settle for less which allows us to get along with the regularity assumptions made so far.

LEMMA 3.2. The majority carrier current densities satisfy the estimates

$$\delta^2 \|\mathbf{J}_n\|_{L^2(\Omega_+)} \leq c , \qquad \delta^2 \|\mathbf{J}_p\|_{L^2(\Omega_-)} \leq c ,$$

with $c$ independent of $\delta^2$.

*Proof.* Let $\varphi \in H^1(\Omega)$ denote a function which is constant in $n$-regions and satisfies the Dirichlet conditions for $u$ at the contacts. The existence of $\varphi$ is guaranteed by the fact that each $n$-region has at most one contact where the Dirichlet datum for $u$ is constant. Multiplication of the electron continuity equation by $u - \varphi$ and integration by parts gives

$$\int_{\Omega_+} \mu_n \frac{C+\sqrt{C^2+4\delta uv}}{2u} |\nabla u|^2 d\mathbf{x}$$
$$= -\int_{\Omega_-} \mu_n \frac{C+\sqrt{C^2+4\delta^4 uv}}{2u} \nabla u \cdot (\nabla u - \nabla\varphi) d\mathbf{x} - \delta^4 \int_\Omega R(u-\varphi) d\mathbf{x}.$$

Note that we have used that $\nabla\varphi$ vanishes in $\Omega_+$. The estimates obtained so far imply that the right-hand side is $O(\delta^4)$. Therefore,

$$\|\nabla u\|_{L^2(\Omega_+)} \le \delta^2 c \tag{3.3a}$$

holds which obviously is equivalent to the estimate for $\mathbf{J}_n$ in the statement of the lemma. The corresponding estimate for $\nabla v$

$$\|\nabla v\|_{L^2(\Omega_-)} \le \delta^2 c \tag{3.3b}$$

is obtained similarly. □

The weakness of this result makes it necessary to change the way currents are computed once more. Because of the assumptions that an $n$- or $p$-region can have at most one contact and considering the divergence theorem, the currents through a contact can be given in terms of the currents through adjacent $pn$-junctions. In the light of our previous results, this is a favourable situation since the currents through $pn$-junctions can be computed by using only minority carrier current densities. In particular, consider a $pn$-junction $\Gamma_0$ separating the $n$-region $\Omega_n$ and the $p$-region $\Omega_p$. Then the current $I_0$ from $\Omega_n$ to $\Omega_p$ is given by

$$I_0 = \int_{\Gamma_0} (\mathbf{J}_n + \mathbf{J}_p) \cdot \nu \, ds \tag{3.4}$$

where $\nu$ is the unit normal vector along $\Gamma_0$ pointing into $\Omega_p$. Similarly to the derivation of (2.3) we use functions $\varphi_n \in H^1(\Omega_n)$ and $\varphi_p \in H^1(\Omega_p)$ which are equal to 1 at $\Gamma_0$ and vanish along other $pn$-junctions or contacts adjacent to $\Omega_n$ and $\Omega_p$. Then the formula for $I_0$ can be rewritten as

$$I_0 = \int_{\Omega_n} (\nabla\varphi_n \cdot \mathbf{J}_p - \varphi_n R) d\mathbf{x} - \int_{\Omega_p} (\nabla\varphi_p \cdot \mathbf{J}_n + \varphi_p R) d\mathbf{x}. \tag{3.5}$$

Note that, differently from (2.3), the recombination-generation rate appears in (3.5) because the individual current densities are not divergence free as opposed to the total current density.

We are now in the position to carry out the limit $\delta^2 \to 0$. The estimates (3.3) imply that

$$\nabla u = 0 \quad \text{in } \Omega_+, \qquad \nabla v = 0 \quad \text{in } \Omega_- \tag{3.6a}$$

holds in the limit. In other words, in each $n$- or $p$-region the Slotboom variable corresponding to the majority carriers is constant. Equations for minority carrier Slotboom variables are derived from (3.1). After dividing the electron continuity equation in $\Omega_-$ as well as the hole continuity equation in $\Omega_+$ by $\delta^4$, the passage to the limit is justified by Lemma 3.1 and arguments analogous to that in the proof of Theorem 2.2. The limiting equations are

$$\operatorname{div}\left(\frac{\mu_n}{|C|}\nabla u\right) = Q(0, |C|)(u - v^{-1}) \quad \text{in } \Omega_-,$$
$$\operatorname{div}\left(\frac{\mu_p}{C}\nabla v\right) = Q(C, 0)(v - u^{-1}) \quad \text{in } \Omega_+. \tag{3.6b}$$

Lemma 3.1 also implies that the limiting solutions satisfy the boundary conditions and that we can go to the limit in (3.5). There we have to substitute

$$\mathbf{J}_n = \frac{\mu_n}{|C|} v\nabla u, \quad R = Q(0, |C|)(uv - 1) \quad \text{in } \Omega_-,$$
$$\mathbf{J}_p = -\frac{\mu_p}{C} u\nabla v, \quad R = Q(C, 0)(uv - 1) \quad \text{in } \Omega_+. \tag{3.7}$$

Summing up our results we have proven the following theorem.

THEOREM 3.1. For every sequence $\delta_k^2 \to 0$ there exists a subsequence (again denoted by $\delta_k^2$) such that the corresponding solutions $(u_k, v_k)$ of (3.1), (3.2) and currents $I_{0k}$, given by (3.5), satisfy

$$u_k \to u_0 \quad \text{weakly in } H^1(\Omega),$$
$$v_k \to v_0 \quad \text{weakly in } H^1(\Omega),$$
$$I_{0k} \to I_{00},$$

where the limits $u_0$, $v_0$ satisfy (3.6) and the boundary conditions (3.2). The limiting current $I_{00}$ is given by (3.5) with the current densities and the recombination term defined by (3.7).

**4. Solution of the low injection problem.** In this section we are concerned with solving the problem (3.6), (3.2) and the computation of currents. Suppose the constant values of $u$ in $n$-regions and of $v$ in $p$-regions are known. Then, $u$ in $\Omega_-$ and $v$ in $\Omega_+$ can be computed by solving boundary value problems for Eq. (3.6*b*). In particular, for a $p$-region $\Omega_p$ let $\Gamma_1, \dots, \Gamma_l$ denote the adjacent $pn$-junctions and (eventually) contact. Then, $u$ takes the constant values $u_1, \dots, u_l$ at these segments of the boundary of $\Omega_p$ because $u$ is constant in $n$-regions and along contacts. The function $u - v^{-1}$ satisfies an homogeneous differential equation and piecewise constant Dirichlet conditions at $\Gamma_1, \dots, \Gamma_l$. Therefore, $u - v^{-1}$ can be written as a linear combination

$$u = v^{-1} + \sum_{j=1}^{l}(u_j - v^{-1})\varphi_j$$

of reference functions $\varphi_j$ satisfying

$$\operatorname{div}\left(\frac{\mu_n}{|C|}\nabla \varphi_j\right) = Q(0, |C|)\varphi_j, \quad \text{in } \Omega_p,$$
$$\varphi_j = \delta_{ij} \quad \text{at } \Gamma_i,\ i = 1, \dots, l \qquad \frac{\partial \varphi_j}{\partial \nu} = 0 \quad \text{at } \partial\Omega_p \cap \partial\Omega_N. \tag{4.1}$$

It is important to notice that the $\varphi_j$'s are independent of the biasing situation, i.e., of the applied voltages. The electron current density in $\Omega_p$ is given by

$$\mathbf{J}_n \frac{\mu_n}{|C|} \sum_{j=1}^{l} (u_j v - 1) \nabla \varphi_j$$

and the electron current through $\Gamma_i$ by

$$\sum_{j=1}^{l} (u_j v - 1) \kappa_{ij} \tag{4.2}$$

where

$$\kappa_{ij} = \int_{\Gamma_i} \frac{\mu_n}{|C|} \nabla \varphi_j \cdot \nu ds$$

is the electron current through $\Gamma_i$ which prevails when the potential at $\Gamma_j$ is raised above equilibrium by a reference value. The values of the $\kappa_{ij}$ $(i, j = 1, \ldots, l)$ provide a complete description of the minority carrier flow through $\Omega_p$. Of course, $v$ and $J_p$ in $n$-regions as well as hole currents can be computed in a similar way. Note that we only have the minority carrier current densities and that we can, therefore, only compute total currents through $pn$-junctions. As already mentioned, however, currents through contacts are immediately given by the divergence theorem.

It remains to determine the constant values of $u$ in the $n$-regions and of $v$ in the $p$-regions. Obviously, for an $n$- or $p$-region with a contact, the value of the majority carrier Slotboom variable is given by the boundary data. Therefore we are done for devices without floating regions. The solution is unique in this case.

Note that uniqueness leads to a stronger version of Theorem 3.1. It implies that the convergence result is not restricted to subsequences.

Now assume we are looking for the majority carrier Slotboom variable in a floating (say $n$-) region $\Omega_n$. An equation for the value of $u$ is provided by the divergence theorem. The total current leaving $\Omega_n$ has to be zero. Since electrons and holes can only leave and enter $\Omega_n$ through $pn$-junctions the current can be given in the form (4.2). Thus, we obtain an algebraic equation for $u$. This equation is linear in terms of $u$. However, the values of the majority carrier Slotboom variables of other $n$- and $p$-regions enter the equation in the form of quadratic terms. Of course, we have such an equation for each floating region. Thus, in general we are dealing with a system of nonlinear algebraic equations. Its solvability is guaranteed by the results of the preceding section but we do not have a uniqueness proof. Nevertheless, uniqueness is obvious for devices with only one floating region because of the above mentioned linearity of the equation. Apart from that, Sec. 7 of this work contains an example of a device with two floating regions where the solution is also unique.

**5. The *pn*-diode.** A $pn$-diode is a device having one $n$-region $\Omega_n$ and one $p$-region $\Omega_p$ with adjacent Ohmic contacts $\Gamma_n$ and $\Gamma_p$, respectively. The $pn$-junction is denoted by $\Gamma$ (see Fig. 2).

The function of a $pn$-diode is that of a valve. Whereas in one direction only a small akage current can pass through the device, significant current flow is possible in the

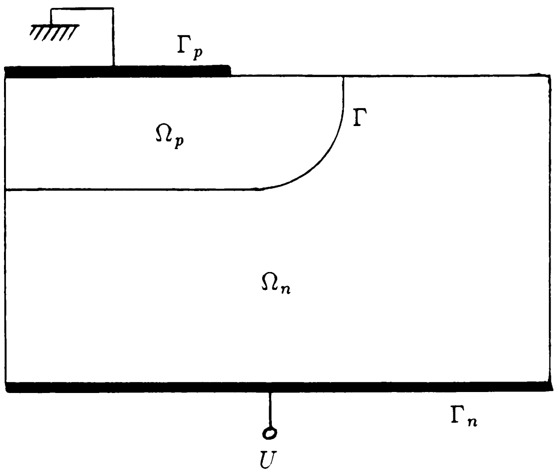



Fig. 2. Cross section of a *pn*-diode.

other direction. By looking at a simple one-dimensional model problem (essentially assuming zero space charge and low injection) Shockley (1949, [15]) computed the approximation

$$I = I_s(e^U - 1) \tag{5.1}$$

for the voltage-current characteristic which is now known as the Shockley equation. In (5.1), $I$ denotes the current through the device and $U$ the contact voltage. The characteristic shows the expected behaviour. Under reverse bias ($U < 0$) the current saturates at the value $I_s$ which has been determined by Shockley as a function of the doping levels in the $n$- and $p$-regions, the mobilities and recombination parameters. For positive voltages (forward bias), on the other hand, the current grows exponentially. An application of the theory of this work will show that the Shockley equation remains valid in the multi-dimensional case with an appropriately chosen value of $I_s$.

With the contact voltage $U$, the Slotboom variables satisfy the boundary conditions

$$u = v = 1, \quad \text{at } \Gamma_p, \qquad u = e^U, \quad v = e^{-U}, \quad \text{at } \Gamma_n.$$

We immediately obtain

$$u = e^U, \quad \text{in } \Omega_n, \qquad v = 1, \quad \text{in } \Omega_p.$$

On the other hand, the procedure described in the previous section leads to the representations

$$u = 1 + (e^U - 1)\varphi_p, \quad \text{in } \Omega_p, \qquad v = e^{-U} + (1 - e^{-U})\varphi_n, \quad \text{in } \Omega_n$$

where $\varphi_n$ and $\varphi_p$ are the solutions of linear elliptic boundary value problems similar to (4.1) with the boundary conditions

$$\varphi_n = \varphi_p = 1, \quad \text{at } \Gamma, \qquad \varphi_n = 0, \quad \text{at } \Gamma_n \quad \varphi_p = 0, \quad \text{at } \Gamma_p.$$

With the formulas for the current densities from the previous section we obtain (5.1) by integration along $\Gamma$. The saturation is current is given by

$$I_s = \int_\Gamma \left( \frac{\mu_n}{|C_p|} \nabla \varphi_p - \frac{\mu_p}{C_n} \nabla \varphi_n \right) \cdot \nu ds$$

where $\nu$ is the unit normal vector along $\Gamma$ pointing into $\Omega_n$, and $C_n$ $(C_p)$ is the doping profile evaluated at the $n$-($p$-)side of the junction. It is easy to see that both terms which sum up to the integrand are positive. If a one-dimensional model with constant mobilities and a piecewise constant doping profile is considered, the differential equations for $\varphi_n$ and $\varphi_p$ are linear ODEs with constant coefficients. In this case, $I_s$ can be computed explicitly recovering Shockley's formulas.

**6. The bipolar transistor.** A bipolar transistor consists of three differently doped regions each having a contact. Among the two possibilities of *npn*- and *pnp*-configurations we choose to consider the latter. The arguments of this section carry over to *npn*-transistors with the obvious changes.

Note that three contacts cannot be incorporated into a one-dimensional model. Therefore we have to assume $k = 2$ or $3$ for the space dimension in this section. Below we shall see that multi-dimensional effects are indeed important for the performance of bipolar transistors.

The outer ($p$-)regions are called emitter $(\Omega_E)$ and collector $(\Omega_C)$, the sandwiched $n$-region is the base $(\Omega_B)$. The corresponding contacts are denoted by $\Gamma_E$, $\Gamma_C$, and $\Gamma_B$, respectively, the emitter junction by $\Gamma_{EB}$ and the collector junction by $\Gamma_{BC}$ (see Fig. 3). Contact voltages are measured with respect to the emitter: $U_{BE}$ is the

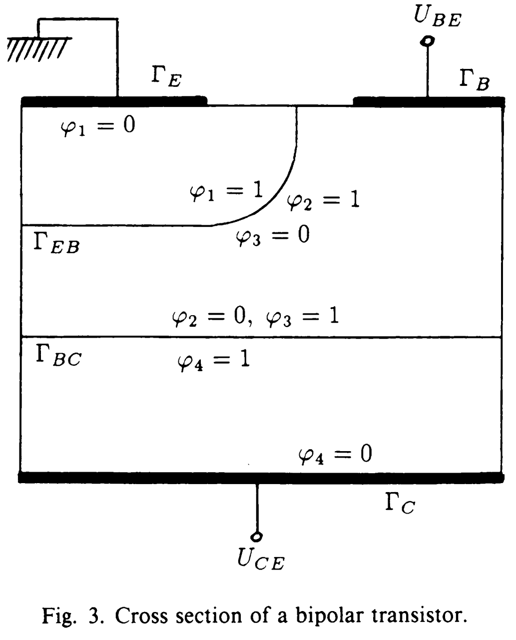



Fig. 3. Cross section of a bipolar transistor.

base-emitter voltage and $U_{CE}$ the collector-emitter voltage. The Slotboom variables then satisfy the boundary conditions

$$u = v = 1, \quad \text{at } \Gamma_E, \qquad u = e^{U_{CE}}, \quad v = e^{-U_{CE}}, \text{ at } \Gamma_C,$$
$$u = e^{U_{BE}}, \quad v = e^{-U_{BE}}, \quad \text{at } \Gamma_B.$$

The arguments of Sec. 4 lead to the representations

$$\begin{aligned} &u = 1 + (e^{U_{BE}} - 1)\varphi_1, \quad v = 1, \quad \text{in } \Omega_E, \\ &u = e^{U_{BE}}, \quad v = e^{-U_{BE}} + (1 - e^{-U_{BE}})\varphi_2 + (e^{-U_{CE}} - e^{-U_{BE}})\varphi_3, \quad \text{in } \Omega_B, \\ &u = e^{U_{CE}} + (e^{U_{BE}} - e^{U_{CE}})\varphi_4, \quad v = e^{-U_{CE}}, \quad \text{in } \Omega_C \end{aligned}$$

where the reference functions $\varphi_1, \ldots, \varphi_4$ solve boundary value problems similar to (4.1) with boundary conditions as indicated in Fig. 3.

The currents $I_E$ entering the device through the emitter, and $I_C$, leaving the device through the collector, can be computed by integrations along the emitter and collector junctions. Then the base current is given by $I_B = I_E - I_C$. The bipolar transistor serves as an amplifier in the following way. A certain collector-emitter voltage is applied and the base current is used for triggering the collector current. Thus, we are interested in the dependence of $I_C$ on $I_B$ and $U_{CE}$. This is achieved by computing $U_{BE}$ from the formula for $I_B$ and substituting the result into the equation for $I_C$. Straightforward algebra gives

$$I_C = (I_B + a_3 + a_4)\frac{a_1 - a_2 e^{-U_{CE}}}{a_3 + a_4 e^{-U_{CE}}} - a_1 + a_2$$

with parameters

$$\begin{aligned} a_1 &= -\int_{\Gamma_{BC}} \frac{\mu_p}{C_B}\nabla\varphi_2 \cdot \nu\, ds, \\ a_2 &= \int_{\Gamma_{BC}} \left(\frac{\mu_p}{C_B}\nabla\varphi_3 - \frac{\mu_n}{|C_C|}\nabla\varphi_4\right) \cdot \nu\, ds, \\ a_3 &= -\int_{\Gamma_{EB}} \frac{\mu_n}{|C_E|}\nabla\varphi_1 \cdot \nu\, ds - \int_{\Gamma_B} \frac{\mu_p}{C_B}\nabla\varphi_2 \cdot \nu\, ds, \\ a_4 &= -\int_{\Gamma_{BC}} \frac{\mu_n}{|C_C|}\nabla\varphi_4 \cdot \nu\, ds - \int_{\Gamma_B} \frac{\mu_p}{C_B}\nabla\varphi_3 \cdot \nu\, ds. \end{aligned}$$

Here $C_B$ denotes the doping profile evaluated at the base-side of the junctions, with similar definitions for $C_E$ and $C_C$.

A measure for the device performance is the so-called common-emitter current gain

$$\beta = \frac{\partial I_C}{\partial I_B} = \frac{a_1 - a_2 e^{-U_{CE}}}{a_3 + a_4 e^{-U_{CE}}}.$$

For significant collector-emitter voltages it can be approximated by $a_1/a_3$ which is large iff both terms summing up to $a_3$ are small compared to $a_1$. Usually the doping in the emitter region is much higher than that in the base region implying that the ratio between $a_1$ and the first term in $a_3$ is large. However, we also require

$$-\int_{\Gamma_B} \frac{\mu_p}{C_B} \nabla\varphi_2 \cdot \nu ds << -\int_{\Gamma_{BC}} \frac{\mu_p}{C_B} \nabla\varphi_2 \cdot \nu ds \tag{6.1}$$

which refers only to the base region. The reference function $\varphi_2$ describes a situation where the potential at the emitter junction is raised. The hole current entering through the emitter junction is split into two parts leaving through the base contact and the collector junction, respectively. The above inequality means that the current through the base contact is much smaller than that through the collector junction, i.e., essentially all the holes injected into the base reach the collector region. Consider a simplified model with constant hole mobility, constant doping in the base region and neglecting recombination-generation effects. Then $\varphi_2$ solves the Laplace equation and the validity of (6.1) only depends on the geometry of the base region.

The classical analysis of bipolar transistors (see e.g. [18]) uses a one-dimensional model. As pointed out above, this means that there is no obvious way of incorporating the base contact into the model. A priori assumptions on the flow in the base region have to be made. It turns out that for the classical model it is assumed that the left-hand side of (6.1) vanishes, i.e., there is no minority carrier current through the base contact. However, situations where in $a_3$ the second term dominates can be easily imagined. Then it is necessary to use the more general theory of the present work.

**7. The thyristor.** A thyristor has four differently doped regions $\Omega_1, \ldots, \Omega_4$. We assume $\Omega_1$ and $\Omega_3$ to be $p$-regions and $\Omega_2$ and $\Omega_4$ to be $n$-regions. A device where only the outer two regions $\Omega_1$ and $\Omega_4$ have contacts ($\Gamma_0$ and $\Gamma_4$, respectively) is called a Shockley diode (see [18]). The $pn$-junctions are denoted by $\Gamma_1, \Gamma_2, \Gamma_3$ (see Fig. 4).

The contact voltage is denoted by $U$, and the Slotboom variables satisfy the boundary conditions

$$u = v = 1, \quad \text{at } \Gamma_0, \qquad u = e^U, \quad v = e^{-U}, \quad \text{at } \Gamma_4.$$

Applying the reasoning of Sec. 4 we obtain the representations

$$\begin{aligned}
&u = 1 + (e^V - 1)\varphi_1, \quad v = 1, \quad \text{in } \Omega_1, \\
&u = e^V, \quad v = e^{-V} + (1 - e^{-V})\varphi_2 + (e^{-W} - e^{-V})\varphi_3, \quad \text{in } \Omega_2, \\
&u = e^W + (e^V - e^W)\varphi_4 + (e^U - e^W)\varphi_5, \quad v = e^{-W}, \quad \text{in } \Omega_3, \\
&u = e^U, \quad v = e^{-U} + (e^{-W} - e^{-U})\varphi_6, \quad \text{in } \Omega_4
\end{aligned}$$

where the reference functions $\varphi_1, \ldots, \varphi_6$ are the solutions of linear boundary value problems similar to (4.1) with boundary conditions as indicated in Fig. 4. The values of the constants $V$ and $W$ determining $u$ in $\Omega_2$ and $v$ in $\Omega_3$, respectively, will

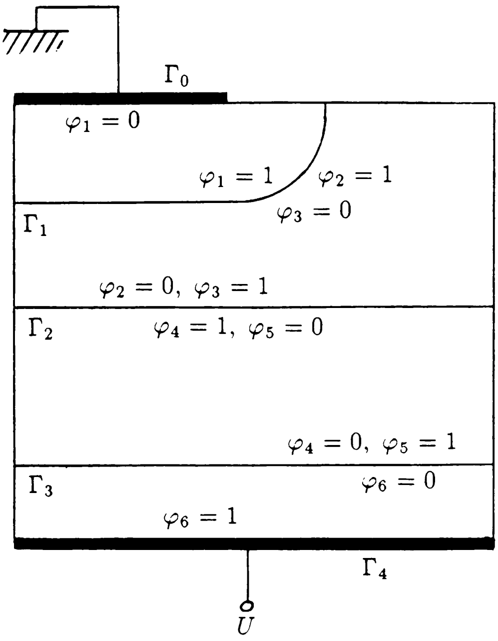



Fig. 4. Cross section of a Shockley diode.

be computed in the following. The currents through the $pn$-junctions are given by

$$\begin{aligned}
I_1 &= \int_{\Gamma_1} \left( \frac{\mu_n}{|C_1|}(e^V - 1)\nabla\varphi_1 - \frac{\mu_p}{C_2}(e^V - 1)\nabla\varphi_2 - \frac{\mu_p}{C_2}(e^{V-W} - 1)\nabla\varphi_3 \right) \cdot \nu\, ds \\
&= (e^V - 1)(\kappa_{11} + \kappa_{12}) - (e^{V-W} - 1)\kappa_{13}, \\
I_2 &= \int_{\Gamma_2} \left( \frac{\mu_n}{|C_3|}(e^{V-W} - 1)\nabla\varphi_4 + \frac{\mu_n}{|C_3|}(e^{U-W} - 1)\nabla\varphi_5 \right. \\
&\qquad \left. - \frac{\mu_p}{C_2}(e^V - 1)\nabla\varphi_2 - \frac{\mu_p}{C_2}(e^{V-W} - 1) \right) \cdot \nu\, ds \\
&= (e^V - 1)\kappa_{22} - (e^{V-W} - 1)(\kappa_{23} + \kappa_{24}) + (e^{U-W} - 1)\kappa_{25}, \\
I_3 &= \int_{\Gamma_3} \left( \frac{\mu_n}{|C_3|}(e^{V-W} - 1)\nabla\varphi_4 + \frac{\mu_n}{|C_3|}(e^{U-W} - 1)\nabla\varphi_5 - \frac{\mu_p}{C_4}(e^{U-W} - 1)\nabla\varphi_6 \right) \cdot \nu\, ds \\
&= -(e^{V-W} - 1)\kappa_{34} + (e^{U-W} - 1)(\kappa_{35} + \kappa_{36})
\end{aligned}$$

where the definition of the positive quantities $\kappa_{ij}$ is obvious from the above formulas and $C_j$ denotes the doping profile evaluated in $\Omega_j$. Note that the $\kappa_{ij}$ are independent of the applied voltage $U$. For a one-dimensional model with constant mobilities and piecewise constant doping profile they can be computed explicitly.

The currents through the junction have to be equal. Their common value is the total current through the device. The equations

$$I_1 = I_2 = I_3$$

can be used for the computation of $V$ and $W$. Elimination of $V$ leads to a quadratic equation of the form

$$ae^{U-2W} + (b + ce^{U})e^{-W} + d = 0 \tag{7.1}$$

fore $e^{-W}$, where the coefficients can be given be in terms of the $\kappa_{ij}$. In particular, we have

$$c = (\kappa_{36} + \kappa_{35} - \kappa_{25})(\kappa_{11} + \kappa_{12} - \kappa_{22}) - \kappa_{25}\kappa_{22} \tag{7.2}$$

which will be used below. It can be shown that $a$ is positive and $d$ is negative. Therefore, (7.1) has a unique solution. Note that the existence but not the uniqueness of the solution has been guaranteed by our previous results. This uniqueness result seems to contradict the common knowledge that thyristors have multiple steady states in certain biasing situations. For the one-dimensional case it has been shown in [17], however, that the voltage-current characteristic consists of two branches called the blocking branch and the conduction branch, where the currents on the conduction branch are $O(\delta^{-4})$ in terms of our scaling. Thus, the present analysis cannot be expected to provide an approximation for the conduction branch. Another result in [17] is that the characteristics of *pnpn*-devices can have two different kinds of qualitative behaviour for positive applied voltages. In one case the characteristic is like that of a diode, i.e., the current grows exponentially with $U$. Such a device would not be called a thyristor. The second possibility is the existence of the two branches mentioned above, with a blocking branch where the current saturates as $U \to \infty$. With the aid of our analysis we can distinguish between the two cases for multi-dimensional problems. Considering (7.1) for $U \to \infty$ leads to

$$\begin{aligned} V &= U + O(1), \quad W = O(1) \quad \text{for } c < 0, \\ V &= O(1), \quad W = U + O(1) \quad \text{for } c > 0. \end{aligned}$$

The above formulas show that the current grows exponentially for negative $c$ whereas it saturates for positive $c$. Thus, only devices with positive $c$ have a chance of behaving like a thyristor should.

When recombination-generation effects are neglected $(Q = 0)$, the formula (7.2) for $c$ is simplified considerably. In this case the conductivities $\kappa_{ij}$ only take the four different values

$$\begin{aligned} a_1 &= \kappa_{11}, \\ a_2 &= \kappa_{12} = \kappa_{22} = \kappa_{13} = \kappa_{23}, \\ a_3 &= \kappa_{24} = \kappa_{34} = \kappa_{25} = \kappa_{35}, \\ a_4 &= \kappa_{36}, \end{aligned}$$

associated with the four regions $\Omega_1, \dots, \Omega_4$, and $c$ is given by

$$c = a_1 a_4 - a_2 a_3.$$

This can be translated to the statement that a *pnpn*-device is a thyristor if the product of the minority carrier conductivities of the outer regions exceeds the product for the inner regions. Specialization to the one-dimensional case shows that this is equivalent to the criterion derived by Steinrück [17].

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