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ON CAUSTICS ASSOCIATED WITH HYPERBOLIC SYSTEMS

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Abstract. The Lagrange manifold formalism is adapted to find asymptotic solutions for a class of hyperbolic systems near caustics.

1. Introduction. The asymptotic series technique developed by Keller [1] and his coworkers determines approximate solutions of linear differential equations. In [2, 3] Granoff and Lewis extend their approach to find asymptotic solutions of systems of linear hyperbolic partial differential equations with an emphasis on asymptotically conservative symmetric systems, e.g., wave propagation in plasmas or viscoelastic materials. Near caustics, their approach does not apply. An alternative approach, which determines asymptotic series solutions near caustics associated with the corresponding scalar equations, is the modified Lagrange manifold formalism [4]. Here, we extend this technique to determine asymptotic series solutions at caustics associated with asymptotically conservative symmetric systems of linear hyperbolic differential equations.

2. Background. Because the modified Lagrange manifold formalism parallels the classical approach, for clarity and comparison we present a synopsis of the classical algorithm. Following Granoff and Lewis, we consider hyperbolic systems of the form

$$A_0\frac{\partial \overline{z}}{\partial t}+\sum_{\nu=1}^{n}A_\nu\frac{\partial \overline{z}}{\partial x_\nu}+\lambda B\overline{z}+C\overline{z}=0. \tag{1}$$

In Eq. (1), $\overline{x}=(x_1, x_2, \dots, x_n)$, $\overline{z}=\overline{z}(\overline{x}, t)$ is an m-dimensional column vector, and λ is a large parameter. $A_0, A_1, \dots, A_n, B$, and C are smooth $m\times m$ functions of t and $\overline{x}$, for which A is positive definite, the A_ν are Hermitian, and B is anti-Hermitian, i.e., Eq. (1) represents an asymptotically conservative symmetric system. The algorithm proceeds by assuming a solution of the form

$$\overline{z}\sim\exp\{i\lambda\phi(\overline{x}, t)\}\sum_{j=0}^{\infty}(i\lambda)^{-j}\overline{z}_j(\overline{x}, t), \qquad \lambda\to\infty, \tag{2}$$

where $\phi(\overline{x}, t)$ is referred to as the "phase" and the $\overline{z}_j$ as "amplitudes." Inserting Eq. (2) into Eq. (1) and regrouping by powers of $i\lambda$ one obtains the recursive system

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of equations

$$G\bar{z}_{j+1} = -A_0 \frac{\partial \bar{z}_j}{\partial t} - \sum_{\nu=1}^{n} A_\nu \frac{\partial \bar{z}_j}{\partial x_\nu} - C\bar{z}_j, \qquad j = -1, 0, \ldots, \tag{3}$$

with

$$G\bar{z}_0 = 0. \tag{4}$$

Here the dispersion matrix $G(\bar{x}, t; \bar{k}, \omega)$ is given by

$$G(\bar{x}, t; \bar{k}, \omega) = \sum_{\nu=1}^{n} k_\nu A_\nu - iB - \omega A_0 \tag{5}$$

with k_ν (wave vector) and ω (frequency) given by

$$k_\nu = \frac{\partial \phi}{\partial x_\nu}, \quad \nu = 1, 2, \ldots, n; \qquad \omega = -\frac{\partial \phi}{\partial t} \tag{6}$$

and

$$\bar{z}_{-1}(\bar{x}, t) = 0.$$

For nontrivial solutions given by Eq. 2 to exist, the dispersion matrix must be singular, i.e.,

$$\det G(\bar{x}, t; \bar{k}, \omega) = 0, \tag{7}$$

for then from Eq. (7) we may construct solutions, beginning with the phase $\phi(\bar{x}, t)$. For specificity, let

$$\omega = h(\bar{x}, \bar{k}, t) \tag{8}$$

be a root of Eq. (7) of multiplicity q. Equation (8) may be regarded as a first-order partial differential equation for the phase $\phi(\bar{x}, t)$. This equation may be solved using the method of characteristics by introducing the characteristic (Hamilton's or ray) equations

$$\frac{dx_\nu}{dt} = \frac{\partial h}{\partial k_\nu} = g_\nu, \qquad \frac{dk_\nu}{dt} = -\frac{\partial h}{\partial x_\nu}, \tag{9}$$

where g_ν is the group velocity. Along the solution curves to these equations ω and $\phi(\bar{x}, t)$ are determined by noting that

$$\frac{d\omega}{dt} = \frac{\partial h}{\partial t}, \tag{10}$$

leading to

$$\frac{d\phi}{dt} = \sum_{\nu=1}^{n} k_\nu g_\nu - h. \tag{11}$$

Equation 11 may be solved for $\phi(\bar{x}, t)$ once initial values for $\bar{x}$, $\bar{k}$, ω, and $\phi(\bar{x}, 0)$ are specified.

To determine the amplitudes $\bar{z}_0$, we first note that if $\omega = h(\bar{x}, \bar{k}, t)$ is a root of Eq. (7) of multiplicity q, then there exist q linearly independent null vectors $\bar{r}_\alpha$ such that

$$G\bar{r}_\alpha = 0, \qquad \alpha = 1, 2, \ldots, q. \tag{12}$$

The sum of the multiplicities is equal to m and these m null vectors are linearly independent and may be chosen so that

$$(\overline{r}_\beta, A_0\overline{r}_\alpha) = \delta_{\beta\alpha}, \qquad \alpha, \beta = 1, 2, \dots, q. \tag{13}$$

From Eq. (4), $\overline{z}_0$ is in the null space of $G(\overline{x}, t; \overline{k}, \omega)$, hence may be expressed in the form

$$\overline{z}_0 = \sum_{\alpha=1}^{q} \sigma_\alpha(\overline{x}, t)\overline{r}_\alpha. \tag{14}$$

The σ_α (and hence $\overline{z}_0$) may be determined by first differentiating Eq. (5) and introducing the group velocity to obtain

$$\frac{\partial G}{\partial k_\nu} = A_\nu - g_\nu A_0, \qquad \nu = 1, 2, \dots, n. \tag{15}$$

Next, differentiating Eq. (12) and combining with Eq. (15) yields

$$G\frac{\partial \overline{r}_\alpha}{\partial k_\nu} + (A_\nu - g_\nu A_0)\overline{r}_\alpha = 0. \tag{16}$$

Then combining Eqs. (13) and (16) and noting the hermiticity of $G(\overline{x}, t; \overline{k}, \omega)$ determines

$$(\overline{r}_\beta, A_\nu\overline{r}_\alpha) = g_\nu\delta_{\beta\alpha}, \qquad \alpha, \beta = 1, 2, \dots, \quad \nu = 0, 1, \dots, \tag{17}$$

which includes A_0 by defining $g_0 = I$.

The $\sigma_\alpha(\overline{x}, t)$ in Eq. (14) may now be obtained by setting $j = 0$ in Eq. (3) and inserting Eq. (14) for z_0. Taking the inner product of the resulting equation with $\overline{r}_\beta$ leads to

$$\sum_{\alpha=1}^{q}\left\{\sum_{\nu=0}^{n}\left[(\overline{r}_\beta, A_\nu\overline{r}_\alpha)\frac{\partial\sigma_\alpha}{\partial x_\nu} + \left(\overline{r}_\beta, A_\nu\frac{\partial\overline{r}_\alpha}{\partial x_\nu}\right)\sigma_\alpha\right] + (\overline{r}_\beta, C\overline{r}_\alpha)\sigma_\alpha\right\} = 0. \tag{18}$$

Then inserting Eq. (17) and noting the definition of g_ν, with $x_0 = t$, allows us to write

$$\frac{d}{dt}\sigma(\overline{x}, t) = \frac{\partial\sigma}{\partial t} + \sum_{\nu=1}^{n}\frac{dx_\nu}{dt}\frac{\partial\sigma}{\partial x_\nu} = \sum_{\nu=0}^{n} g_\nu\frac{\partial\sigma}{\partial x_\nu} \tag{19}$$

and to rewrite Eq. (18) as

$$\frac{d\sigma_\beta}{dt} + \sum_{\alpha=1}^{q}\tau_{\beta\alpha}\sigma_\alpha = 0, \qquad \beta = 1, 2, \dots, q, \tag{20}$$

where

$$\tau_{\beta\alpha} = \sum_{\nu=0}^{n}\left(\overline{r}_\beta, A_\nu\frac{\partial\overline{r}_\alpha}{\partial x_\nu}\right) + (\overline{r}_\beta, C\overline{r}_\alpha).$$

The coefficients σ_β satisfy a system of q first-order linear ordinary (transport) equations, leading to amplitude $\overline{z}_0$. To solve this system, Granoff and Lewis introduce an auxiliary coordinate (ray) transformation, which leads to solutions for the σ_β, and hence $\overline{z}_0$. But, as they note, this transformation can become singular [2]. The locus of singular points is the caustic curve, for which a separate treatment must be considered [5]. One such approach is the Lagrange manifold formalism [4] which introduces a non-Hamiltonian flow to obtain transport equations near caustics associated with scalar differential equations. Our intent is to adapt the Lagrange manifold formalism for asymptotically conservative symmetric systems to develop a transport equation, analogous to Eq. (20), that applies at caustics.

3. Formalism. We consider asymptotically conservative symmetric systems of the form

$$A_0 \frac{\partial \overline{z}}{\partial t} + \sum_{\nu=1}^{n} A_\nu \frac{\partial \overline{z}}{\partial x_\nu} + \lambda B \overline{z} + C \overline{z} = 0, \tag{1}$$

with A_0, A_ν, B, C, and λ as noted above. Near caustics, ordinarily we would assume a solution of the form

$$\overline{z} \sim \int \overline{Z}(\overline{x}, \overline{k}, t, \lambda) \exp\{i\lambda(\overline{x} \cdot \overline{k} - S(\overline{k}, t))\}\, d\overline{k}, \tag{21}$$

where the phase $\phi(\overline{x}, \overline{k}, t)$ now has the explicit form

$$\phi(\overline{x}, \overline{k}, t) = \overline{x} \cdot \overline{k} - S(\overline{k}, t) \tag{22}$$

and

$$\overline{Z}(\overline{x}, \overline{k}, t, \lambda) = \sum_{j=0}^{\infty} (i\lambda)^{-j} z_j(\overline{x}, \overline{k}, t), \qquad z_{-1} = 0,$$

analogous to the ansatz that applies near caustics associated with scalar differential equations. (From Eq. (21), we note that the integral representation may be regarded as a continuous superposition of oscillatory solutions.) The usual algorithm then proceeds by substituting Eq. (21) into Eq. (1), followed by a regrouping by powers of $i\lambda$. Straightforward calculation, using Eqs. (6) (which apply here also), shows that this leads to equations identical to Eqs. (3), (4), and (5). Consequently, to simplify, we extend a device noted by Granoff and Lewis [2, p. 392] and introduce into Eq. (1) a coordinate transformation

$$\overline{z} = D\overline{u} \tag{23}$$

that simultaneously diagonalizes the Hermitian matrix $G(\overline{x}, t; \overline{k}, \omega)$ and the positive definite matrix A_0 [6, p. 120]. Thus we rewrite Eq. (1) as

$$\widehat{A}_0 \frac{\partial \overline{u}}{\partial t} + \sum_{\nu=1}^{n} \widehat{A}_\nu \frac{\partial \overline{u}}{\partial x_\nu} + \lambda \widehat{B} \overline{u} + C \overline{u} = 0, \tag{24}$$

where the carets indicate the resulting transformed coefficient matrices. Without loss of generality (to parallel the classical treatment) we may take $\widehat{A}_0 = I$, i.e., multiplication of Eq. (1) by A_0^{-1} may have preceded application of Eq. (23). The algorithm proceeds by assuming a solution of the form

$$\overline{u} \sim \int \overline{U}(\overline{x}, \overline{k}, t, \lambda) \exp\{i\lambda\phi(\overline{x}, \overline{k}, t)\}\, d\overline{k}, \tag{25}$$

where

$$\overline{U}(\overline{x}, \overline{k}, t, \lambda) = \sum_{j=0}^{\infty} (i\lambda)^{-j} \overline{u}_j(\overline{x}, \overline{k}, t), \qquad \overline{u}_{-1} = 0,$$

and the phase $\phi(\overline{x}, \overline{k}, t)$ has the form given in Eq. (22). Then substituting Eq. (25) into Eq. (24) and regrouping by powers of $i\lambda$ leads to

$$\int d\overline{k} \left\{ (-\omega \widehat{A}_0 + \sum_{\nu=1}^{n} \widehat{A}_\nu k_\nu - iB) i\lambda \overline{u} + \widehat{A}_0 \frac{\partial \overline{u}}{\partial t} + \sum_{\nu=1}^{n} \widehat{A}_\nu \frac{\partial \overline{u}}{\partial x_\nu} + C\overline{u} \right\} \exp\{i\lambda\phi(\overline{x}, \overline{k}, t)\} = O(\lambda^{-\infty}), \quad (26)$$

i.e., a solution for large λ. Analogous to the above, we define as our dispersion matrix

$$\widehat{G}(\overline{x}, t; \overline{k}, \omega) = \sum_{\nu=0}^{n} k_\nu \widehat{A}_\nu - i\widehat{B} - \omega \widehat{A}_0. \quad (27)$$

Evaluation of the integral at any caustic point $(\overline{x}_c)$ proceeds by invoking the stationary phase condition $(\nabla_k \phi(\overline{x}_c, \overline{k}, t) = 0)$ which determines the time-parameterized Lagrange manifold

$$\overline{x} = \nabla_k S(\overline{k}, t) \quad (28)$$

and turns Eq. (26) into

$$\int d\overline{k} \left\{ \widehat{G}(\overline{x}, t; \overline{k}, \omega) i\lambda \overline{u} + \widehat{A}_0 \frac{\partial \overline{u}}{\partial t} + \sum_{\nu=1}^{n} \widehat{A}_\nu \frac{\partial \overline{u}}{\partial x_\nu} + C\overline{u} \right\} \exp\{i\lambda\phi(\overline{x}, \overline{k}, t)\} = 0. \quad (29)$$

Next $\widehat{G}$ is Taylor expanded about the Lagrange manifold

$$\widehat{G}(\overline{x}, t; \overline{k}, \omega) = \widehat{G}_0(\nabla_k S, t; \overline{k}, \omega) + \sum_{\nu=1}^{n} \left(x_\nu - \frac{\partial S}{\partial k_\nu} \right) D^\nu, \quad (30)$$

where

$$D^\nu = -\int_0^1 \partial_{x_\nu} G(\gamma(\overline{x} - \nabla_k S) + \overline{x} - \nabla_k S, t; \overline{k}, \omega)\, d\gamma,$$

i.e., the remainder of the Taylor series less a factor of $(\overline{x} - \nabla_k S)$. Then inserting Eq. (30) into Eq. (29) and performing a partial integration leads to

$$\int d\overline{k} \left[i\lambda \widehat{G}_0 \overline{u} + \widehat{A}_0 \frac{\partial \overline{u}}{\partial t} + \sum_{\nu=1}^{n} \widehat{A}_\nu \frac{\partial \overline{u}}{\partial x_\nu} + \sum_{\nu=1}^{n} ((-D)^\nu) \frac{\partial \overline{u}}{\partial k_\nu} - \sum_{\nu=1} \frac{\partial D^\nu}{\partial k_\nu} \overline{u} + C\overline{u} \right] \exp\{i\lambda\phi\} = 0 \quad (31)$$

(cf. Eq. (3)). Here, for nontrivial solutions to exist, the matrix $\widehat{G}_0$ must be singular, i.e.,

$$\det \widehat{G}_0(\overline{x}, t; \overline{k}, \omega) = 0. \quad (32)$$

Then, following the algorithm above, if $\omega = h(\overline{x}, \overline{k}, t)$ is a root of Eq. (32) of multiplicity q, we find the phase from the characteristics

$$\begin{aligned} \frac{dx_\nu}{d\mu} &= \frac{\partial h}{\partial k_\nu} = g_\nu, & \frac{dk_\nu}{d\mu} &= -\frac{\partial h}{\partial x_\nu}, \\ \frac{dt}{d\mu} &= -\frac{\partial h}{\partial \omega}, & \frac{d\omega}{d\mu} &= \frac{\partial h}{\partial t}, \end{aligned} \tag{33}$$

where μ is a raypath parameter. These equations may be solved to obtain

$$\begin{aligned} x_\nu &= x_\nu(\mu, \overline{\sigma}), & k_\nu &= k_\nu(\mu, \overline{\sigma}), \\ t &= t(\mu, \overline{\sigma}), & \omega &= \omega(\mu, \overline{\sigma}), \end{aligned} \tag{34}$$

where σ are parameterized initial conditions, e.g., direction cosines. Then inversion of the wave vector and time transformations, followed by substitution into the coordinate space map, determines the Lagrange manifold (Eq. (28)) explicitly,

$$\overline{x} = \overline{x}(\mu(t, \overline{k}), \overline{\sigma}(t, \overline{k})) = \nabla_k S(\overline{k}, t). \tag{35}$$

An integration along the trajectories yields

$$S(\overline{k}, t) = \int_{\overline{k}_0}^{\overline{k}} \overline{x} \cdot d\overline{k} \tag{36}$$

and hence the phase

$$\phi(\overline{x}, \overline{k}, t) = \overline{x} \cdot \overline{k} - S(\overline{k}, t). \tag{37}$$

In the Lagrange manifold formalism the transport equation for the amplitudes $\overline{u}_j(\overline{x}, \overline{k}, t)$ is developed inside the integral by introducing a non-Hamiltonian flow. To develop such a transport equation that corresponds to Eq. (31), we begin by making Eq. (31) hold by requiring the nonexponential factor of the integrand to be zero. Hence the equation we consider is

$$i\lambda \widehat{G}_0 \overline{u} + \widehat{A}_0 \frac{\partial \overline{u}}{\partial t} + \sum_{\nu=1}^{n} \widehat{A}_\nu \frac{\partial \overline{u}}{\partial x_\nu} + \sum_{\nu=1}^{n} (-D^\nu) \frac{\partial \overline{u}}{\partial k_\nu} - \sum_{\nu=1}^{n} \frac{\partial D^\nu}{\partial k_\nu} \overline{u} + C\overline{u} = 0, \tag{38}$$

which may be written as the recursive system of equations

$$\widehat{G}_0 \overline{u}_{j+1} = -\widehat{A}_0 \frac{\partial \overline{u}_j}{\partial t} - \sum_{\nu=1}^{n} \widehat{A}_\nu \frac{\partial \overline{u}_j}{\partial x_\nu} + \sum_{\nu=1}^{n} D^\nu \frac{\partial \overline{u}_j}{\partial k_\nu} + \sum_{\nu=1}^{n} \frac{\partial D^\nu}{\partial k_\nu} \overline{u}_j + C\overline{u}_j \tag{39}$$

with

$$\widehat{G}_0 \overline{u}_0 = 0 \tag{40}$$

and $\overline{u}_{-1}(\overline{x}, \overline{k}, t) = 0$ (cf. Eqs. (3) and (4)). Then proceeding as in the classical approach, the null vectors $\overline{r}_\alpha = \overline{r}_\alpha(\overline{x}, \overline{k}, t)$ of $\widehat{G}_0$, i.e.,

$$\widehat{G}_0 \overline{r}_\alpha = 0, \qquad \alpha = 1, 2, \ldots, q, \tag{41}$$

may be chosen so that

$$(\overline{r}_\beta, \widehat{A}_0 \overline{r}_\alpha) = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3, \ldots, q \tag{42}$$

(cf. Eqs. (12) and (13)). The amplitude $\overline{u}_0(\overline{x}, \overline{k}, t)$ is obtained from these null vectors by first forming

$$\overline{u}_0 = \sum_{\alpha=1}^{q} \sigma_\alpha(\overline{x}, \overline{k}, t)\overline{r}_\alpha. \tag{43}$$

To solve for the $\sigma_\alpha(\overline{x}, \overline{k}, t)$ we first differentiate Eq. (27), then introduce the group velocity to obtain

$$\frac{\partial \widehat{G}_0}{\partial k_\nu} = A_\nu - g_\nu A_0. \tag{44}$$

Next, differentiation of Eq. (41) combined with Eq. (44) leads to

$$\widehat{G}_0 \frac{\partial \overline{r}_\alpha}{\partial k_\nu} + (\widehat{A}_\nu - g_\nu \widehat{A}_0)\overline{r}_\alpha = 0. \tag{45}$$

Then combining Eqs. (42) and (45) leads to

$$(\overline{r}_\beta, \widehat{A}_\nu \overline{r}_\alpha) = g_\nu \delta_{\beta\alpha}, \tag{46}$$

which follows from the hermiticity of $\widehat{G}$ and includes $\widehat{A}_0$ by defining $g_0 = I$. The $\sigma_\alpha(\overline{x}, \overline{k}, t)$ now proceed by setting $j = 0$ in Eq. (39) and inserting Eq. (43) for $\overline{u}_0$. Taking the inner product of the resulting equation with $\overline{r}_\beta$ followed by a regrouping leads to

$$\begin{aligned}\sum_{\alpha=1}^{q} \Bigg\{ &\sum_{\nu=0}^{n} (\overline{r}_\beta, \widehat{A}_\nu \overline{r}_\alpha)\frac{\partial \sigma_\alpha}{\partial x_\nu} - (\overline{r}_\beta, D^\nu \overline{r}_\alpha)\frac{\partial \sigma_\alpha}{\partial k_\nu} \\ &+ \left[\left(\overline{r}_\beta, \widehat{A}_\nu \frac{\partial \overline{r}_\alpha}{\partial x_\nu}\right) - \left(\overline{r}_\beta, D^\nu \frac{\partial \overline{r}_\alpha}{\partial k_\nu}\right) + \left(\overline{r}_\beta, \frac{\partial D^\nu}{\partial k_\nu}\overline{r}_\alpha\right) + (\overline{r}_\beta, C\overline{r}_\alpha)\right]\sigma_\alpha \Bigg\} = 0.\end{aligned} \tag{47}$$

The first term in this equation corresponds to the first term in Eq. (18), hence

$$(\overline{r}_\beta, \widehat{A}_\nu \overline{r}_\alpha)\frac{\partial \sigma_\alpha}{\partial x_\nu} = \frac{\partial \sigma_\beta}{\partial t} + \sum_{\nu=1}^{n} \frac{dx_\nu}{dt}\frac{\partial \sigma_\beta}{\partial x_\nu}. \tag{48}$$

Since D is diagonal, the second term may be written

$$-(\overline{r}_\beta, D^\nu \overline{r}_\alpha) = -\mathscr{D}^\nu \delta_{\alpha\beta}. \tag{49}$$

Hence by introducing a non-Hamiltonian flow analogous to that applying in scalar differential equations

$$\frac{dk_\nu}{dt} = -\mathscr{D}^\nu \tag{50}$$

we note that the first two terms in Eq. (47) may be written as

$$\frac{d\sigma_\beta}{dt} = \frac{\partial \sigma_\beta}{\partial t} + \sum_{\nu=1}^{n} \frac{dx_\nu}{dt}\frac{\partial \sigma_\beta}{\partial x_\nu} + \sum_{\nu=1}^{n} \frac{dk_\nu}{dt}\frac{\partial \sigma_\beta}{\partial k_\nu}. \tag{51}$$

Then by allowing $dk_0/dt = 0$, Eq. (51) becomes

$$\frac{d\sigma_\beta}{dt} + \sum_{\alpha=1}^{q} \tau_{\beta\alpha}\sigma_\alpha = 0, \tag{52}$$

where

$$\tau_{\beta\alpha} = \sum_{\nu=0}^{n} \left\{ \left(\bar{r}_\beta , A_\nu \frac{\partial \bar{r}_\alpha}{\partial x_\nu} \right) + \left(\bar{r}_\beta , D^\nu \frac{\partial \bar{r}_\alpha}{\partial k_\nu} \right) + (\bar{r}_\beta , C\bar{r}_\alpha) + \left(\bar{r}_\beta , \frac{\partial D^\nu}{\partial k_\nu} \bar{r}_\alpha \right) \right\} ,$$

i.e., a transport equation valid at caustics analogous to Eq. (20). This transport equation may be solved for the σ_β (and hence $\bar{u}_0$) using the classical procedure with the transformation $\bar{x} = \bar{x}(\mu(t, \bar{k}), \bar{\sigma}(t, \bar{k}))$ (cf. Eq. (35)) in place of the ray transformation. (Higher order $\bar{u}_j$'s require a treatment corresponding to that given in [2] and are not considered here.) The remaining computational considerations parallel the treatment for the scalar equation. As these have been detailed elsewhere [7], for brevity we do not repeat them

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