A NOTE ON THE EXISTENCE OF A WAITING TIME
FOR A TWO-PHASE STEFAN PROBLEM

BY
DOMINGO ALBERTO TARZIA (PROMAR, Instituto de Matemática “B. Levi”, Rosario, Argentina)

AND
CRISTINA VILMA TURNER (Universidad Nacional de Córdoba, Córdoba, Argentina)

Abstract. We consider a slab, represented by the interval $0 < x < x_0$, at the initial
temperature $\theta_0 = \theta_0(x) \geq 0$ (or $\phi_0 = \phi_0(x) \geq 0$) having a heat flux $q = q(t) > 0$
(or convective boundary condition with a heat transfer coefficient $h$) on the left face
$x = 0$ and a temperature condition $b(t) > 0$ on the right face $x = x_0$ ($x_0$ could
be also $+\infty$, i.e., a semi-infinite material). We consider the corresponding heat
conduction problem and assume that the phase-change temperature is $0^\circ$C.

We prove that certain conditions on the data are necessary or sufficient in order
to obtain the existence of a waiting-time at which a phase-change begins.

I. Introduction. We consider the following heat conduction problems ($0 < x_0 \leq +\infty$):

\begin{align}
(i) \quad & \rho c \theta_t - k \theta_{xx} = 0, \quad 0 < x < x_0, \quad t > 0; \\
(ii) \quad & \theta(x, 0) = \theta_0(x) > 0, \quad 0 \leq x \leq x_0; \\
(iii) \quad & k \theta_x(0, t) = q(t), \quad t > 0; \\
(iv) \quad & \theta(x_0, t) = b(t), \quad t > 0; \\
\end{align}

and

\begin{align}
(i) \quad & \rho c \phi_t - k \phi_{xx} = 0, \quad 0 < x < x_0, \quad t > 0; \\
(ii) \quad & \phi(x, 0) = \phi_0(x) > 0, \quad 0 \leq x \leq x_0; \\
(iii) \quad & k \phi_x(0, t) = h(D + \phi(0, t)), \quad t > 0; \\
(iv) \quad & \phi(x_0, t) = b(t), \quad t > 0; \\
\end{align}

where $\rho$ is the density, $k$ is the thermal conductivity, $c$ is the specific heat, $h$ is the
convective heat transfer coefficient from a fluid with ambient temperature $-D < 0$
flowing across the face $x = 0$. The function $b(t)$ represents the temperature at the
face $x = x_0 > 0$, and $\theta_0$ and $\phi_0$ are the initial temperatures for problems (1) and
(1') respectively.

We take, without loss of generality, the phase-change temperature as $0^\circ$C and
replace condition (1)(iv) by $\theta(+\infty, t) = \theta_0(+\infty) > 0$, $t > 0$ for the case $x_0 = +\infty$
We assume that the data satisfy the hypotheses that ensure the existence and uniqueness property of the solution of (1) and (1').

We consider the following possibilities:
(a) the heat conduction problem is defined for all \( t > 0 \) (waiting-time \( t^* = +\infty \));
(b) there exists a time \( t^* < +\infty \) such that another phase (i.e., the solid phase) appears for \( t \geq t^* \) (waiting-time \( 0 \leq t^* < +\infty \)) and then we have a two-phase Stefan problem for \( t > t^* \). In this case, there exists a free boundary \( x = s(t) \), which separates the liquid and solid phases with \( s(t^*) = 0 \).

We will separate the cases waiting-time \( t^* = 0 \) (i.e., there exists an instantaneous change of phase) and \( 0 < t^* < +\infty \). These possibilities depend on the data \( \theta_0, q, b \) for Problem (1) and the data \( \phi_0, h, b \) for Problem (1'). We try to clarify this dependence by finding necessary or sufficient conditions on \( \theta_0, q, b \) and \( \phi_0, h, b \) in order to have the different possibilities.

In [5, 8, 9] the one-phase Stefan problem with prescribed flux or convective boundary condition at \( x = 0 \) is studied.

This paper was motivated by [10, 12, 13] (see also [14]) and the term waiting-time was motivated by its correspondence to the term as used in the porous medium equation (see, for instance [1]).

In Sec. II we analyse problem (1) with a flux boundary condition at \( x = 0 \) and in Sec. III we study the problem (1') with a convective boundary condition at \( x = 0 \).

II. On some conduction problems with a flux boundary condition. We consider the following properties for the problem (1).

**Theorem 1.** If the data \( q = q(t), \theta_0 = \theta_0(x), \) and \( b = b(t) \) verify conditions

\[
\begin{align*}
(i) & \quad 0 < q(t) \leq q_0, \quad 0 \leq t \leq t_0 \text{ with } t_0 > 0; \\
(ii) & \quad \beta_0^l(x) \geq 0 \text{ and } \beta_1 \geq \theta_0(x) \geq \beta_0 > 0, \quad 0 \leq x \leq x_0 \text{ with } \beta_0 \leq \beta_1; \\
(iii) & \quad b(t) \geq \beta_1 \text{ and } b(t) \geq 0, \quad t > 0;
\end{align*}
\]

then there exists a waiting-time \( t^* > 0 \) for problem (1), (i.e., another phase could appear at \( t \geq t^* \)), where \( t^* \) verifies the following inequality:

\[
t^* \geq \text{Min}(t_0, t_0^*), \quad \text{where } t_0^* = \frac{\pi k \rho c \beta_0^2}{4 q_0^2}.
\]

**Proof.** It is sufficient to prove that \( \theta(x, t) \geq 0 \) for \( 0 \leq x \leq x_0 \) and \( 0 \leq t \leq t_0^* \). For the semi-infinite material \( x > 0 \), with the same thermal coefficients, we consider the following two problems:

\[
\begin{align*}
&p c T_t - k T_{xx} = 0, \quad x > 0, \quad 0 < t < t_0; \\
&k T_x(0, t) = q(t), \quad t > 0; \\
&T(x, 0) = T_0(x), \quad x \geq 0,
\end{align*}
\]

with

\[
T_0(x) = \begin{cases} 
\theta_0(x), & 0 \leq x \leq x_0, \\
\theta_0(x_0), & x > x_0,
\end{cases}
\]
and
\[ \rho c V_t - k V_{xx} = 0, \quad x > 0, \ t > 0; \]
\[ k V_x(0, t) = q_0 > 0, \quad t > 0; \]
\[ V(x, 0) = \beta_0 > 0, \quad x \geq 0, \]
whose solutions are given respectively by [3, 4]
\[ T(x, t) = \int_0^{+\infty} N(x, t; \xi, 0) T_0(\xi) d\xi - \frac{2a}{k} \int_0^t K(x, t; 0, \tau) q(\tau) d\tau, \]
and
\[ V(x, t) = \beta_0 - \frac{2a_0 a}{k} \sqrt{t} \text{erfc} \left( \frac{x}{2a\sqrt{t}} \right), \]
where
\[ a = \left( \frac{k}{\rho c} \right)^{1/2}; \quad K(x, t; \xi, \tau) = \frac{\exp(-(x-\xi)^2/4a^2(t-\tau))}{2a\sqrt{\pi}(t-\tau)}; \]
\[ N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau); \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt; \]
\[ \text{erfc}(x) = 1 - \text{erf}(x); \quad \text{ierrc}(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} - x \text{erfc}(x). \]

By the maximum principle [6, 7] we obtain
\[ V(x, t) \leq T(x, t), \quad x \geq 0, \ 0 \leq t \leq t_0, \]
\[ T(x, t) \leq \theta(x, t), \quad 0 \leq x \leq x_0, \ 0 \leq t \leq t_0, \]
because \( T(x_0, t) \leq \beta_1 \leq b(t) \) for \( 0 \leq t \leq t_0 \).

Let \( W \) be the function \( W = \theta_x \), which satisfies the following heat conduction problem:
\[ \rho c W_t - kW_{xx} = 0, \quad 0 < x < x_0, \ t > 0; \]
\[ W(x, 0) = \theta'_0(x), \quad 0 \leq x \leq x_0; \]
\[ W(0, t) = \frac{q(t)}{k}, \quad W_x(x_0, t) = \frac{\rho c}{k} b(t), \ t > 0. \]

By the maximum principle we have \( W = \theta_x \geq 0 \) for \( 0 \leq x \leq x_0, \ t \geq 0 \). Then, we deduce that
\[ \theta(x, t) \geq \theta(0, t) \geq V(0, t) = \beta_0 - \frac{2a_0 a}{k} \sqrt{\frac{t}{\pi}} \geq 0 \quad \text{for} \ 0 \leq t \leq t_0^*, \]
where \( t_0^* \) is defined by (3) proving our assertion.

**Remark 1.** When the data verify conditions (2), problem (1) represents a heat conduction problem for the initial phase (in our case, the liquid phase) for \( t \leq t^* \).

**Remark 2.** We can see that \( t_0^* \) does not depend on the length of the slab \( x_0 > 0 \).

**Corollary 2.** Under the hypotheses (2)(ii),(iii), a necessary condition in order to have an instantaneous change of phase (i.e., \( t^* = 0 \)) for problem (1) is given by
\[ q(0^+) = +\infty. \]
REMARK 3. If we consider the case
\[ x_0 = +\infty, \quad \theta_0(x) \geq \beta_0 > 0, \quad \forall x \geq 0, \]
(14)
then problem (1) is a heat conduction problem for the liquid phase for all \( t > 0 \) i.e., there is not a phase-change process for any \( t > 0 \) because we have
\[ \theta(x, t) \geq \theta q_0(x, t), \quad x \geq 0, \quad t > 0, \]
(15)
where \( \theta_0 \) is the solution of (1) with data: heat flux \( q_0(t) \) at \( x = 0, \ x_0 = +\infty \), and initial temperature \( \beta_0 \). It is given by [12]
\[ \theta q_0(x, t) = \beta_0 \text{erf} \left( \frac{x}{2a\sqrt{t}} \right) \geq 0, \quad x \geq 0, \quad t > 0. \]
(16)
Moreover, the particular case
\[ q(t) = \frac{\beta_0 k}{a\sqrt{\pi t}} = q_0(t), \quad t > 0, \]
(17)
shows us that condition (13) is not sufficient in order to have an instantaneous change of phase for problem (1).

REMARK 4. If \( x_0 = +\infty \) and \( \theta_0(x) \geq \beta_0 > 0 \) for \( x \geq 0 \), then a necessary condition for problem (1) to have an instantaneous change of phase (i.e., the waiting-time is \( t^* = 0 \)) is for there to exist a \( t_0 > 0 \) such that
\[ q(t_0) > \frac{\beta_0 k}{a\sqrt{\pi t_0}}. \]
(18)

THEOREM 3. If the data verify the conditions
\[ x_0 = +\infty; \quad 0 \leq \theta_0(x) \leq \beta_1 \quad \text{for} \quad x \geq 0, \]
\[ q(t) \geq \frac{q_0}{t^{\beta}}, \quad 0 < t < 1, \quad \text{with} \quad q_0 > 0 \quad \text{and} \quad \frac{1}{2} < \beta < 1, \]
(19)
then an instantaneous phase-change occurs, that is, the waiting-time is \( t^* = 0 \).

Proof. Let \( U = U(x, t) \) be the solution of the following heat conduction problem:
\[ \rho c U_t - k U_{xx} = 0, \quad x > 0, \quad t > 0; \]
\[ U(x, 0) = \beta_1, \quad x \geq 0; \]
\[ k U_x(0, t) = \frac{q_0}{t^\beta}, \quad t > 0, \]
(20)
which is given by [3]
\[ U(x, t) = \beta_1 - \frac{2a q_0}{k} \int_0^t \frac{K(x, t; 0, \tau)}{\tau^{\beta}} d\tau. \]
(21)
By using the maximum principle we have that \( \theta(x, t) \leq U(x, t) \) for \( x \geq 0, \ t > 0 \). Therefore, we obtain
\[ \theta(0, t) \leq U(0, t) \leq \beta_1 - \frac{a q_0}{k \sqrt{\pi}} \int_0^t \frac{d\tau}{\tau^{\beta} \sqrt{t - \tau}}. \]
(22)
and, for \( 0 < \varepsilon = t/2 < t < 1 \),
\[
\int_0^t \frac{d\tau}{\tau^\beta \sqrt{t - \tau}} \geq \int_0^\varepsilon \frac{d\tau}{\varepsilon^\beta \sqrt{t - \tau}} + \int_\varepsilon^t \frac{d\tau}{\tau^\beta \sqrt{t - \varepsilon}} = C_\beta \left( \frac{2}{t} \right)^{\beta - 1/2} \tag{23}
\]
Moreover, the temperature on the fixed face \( x = 0 \) verifies the inequality
\[
\theta(0, t) \leq \beta_1 - \frac{a q_0}{k \sqrt{\pi}} C_\beta \left( \frac{2}{t} \right)^{\beta - 1/2} < 0, \tag{24}
\]
for all \( t < \min(1, t_\beta) \), where
\[
t_\beta = 2 \left( \frac{a q_0}{k \beta_1 \sqrt{\pi}} C_\beta \right)^{1/(\beta - 1/2)} > 0, \tag{25}
\]
that is, the thesis is achieved.

**Remark 5.** If we consider the density jump under the phase of change, that is, \( \rho_1 \neq \rho_2 \), and the data verify the conditions
\[
x_0 = +\infty; \quad 0 \leq \theta_0(x) \leq \beta_1 \quad \text{for} \ x \geq 0, \tag{26}
\]
\[
q(t) \geq \frac{q_0}{\sqrt{t}} \quad \text{for} \ t > 0 \ \text{with} \ q_0 > \frac{\beta_1 k_2}{a_2 \sqrt{\pi}},
\]
where \( k_i, c_i, \rho_i, a_i = (k_i/\rho_i c_i)^{1/2} \) are the corresponding thermal coefficients for the phase \( i \) \((i = 2: \text{liquid phase}, \ i = 1: \text{solid phase})\), then the temperature \( \theta = \theta_{q, \theta_0} \), solution of problem (1), verifies the inequality \( \theta_{q, \theta_0}(x, t) \leq T_{q_0, \beta_i}(x, t), \ x \geq 0, \)
\( t > 0 \), where \( T_{q_0, \beta_i} \) is the solution of (1) with initial constant temperature \( \beta_1 \) and a flux condition of type \( q_0/\sqrt{t} \) on \( x = 0 \). Therefore, we obtain [2]
\[
\theta_{q, \theta_0}(0, t) \leq T_{q_0, \beta_i}(0, t) = \beta_1 - \frac{a q_0 a_2 \sqrt{\pi}}{k_2} < 0, \quad t > 0, \tag{27}
\]
that is, the waiting-time is \( t^* = 0 \) (i.e., we have an instantaneous two-phase Stefan problem) for data \( q \) and \( \theta_0 \). Moreover, its free boundary \( x = s_{q, \theta_0}(t) \) verifies \( s_{q, \theta_0}(0) = 0 \) and it is characterized by \( \theta_{q, \theta_0}(s_{q, \theta_0}(t), t) = 0 \) for all \( t > 0 \).

The free boundary \( x = s_{q_0, \beta_i}(t) \) corresponding to the temperature \( T_{q_0, \beta_i} \) is given by [2]
\[
s_{q_0, \beta_i} = 2 \omega \sqrt{t}, \tag{28}
\]
where \( \omega \) is the unique solution of the equation
\[
F_0(x) = x, \quad x > 0 \tag{29}
\]
with
\[
F_0(x) = \frac{a q_0}{h \rho_1} \exp \left( -\frac{x^2}{a_1^2} \right) - \frac{k_2 \beta_i}{h \rho_1 a_2 \sqrt{\pi}} \frac{\exp(-x^2/a_2)}{\text{erfc}(x/a_2)}, \tag{30}
\]
where \( h > 0 \) is the latent heat. Owing to
\[
\theta_{q, \theta_0}(s_{q_0, \beta_i}(t), t) \leq T_{q_0, \beta_i}(s_{q_0, \beta_i}(t), t) = 0, \quad \text{for all} \ t > 0, \tag{31}
\]
it follows that
\[ s_q, \theta_0(t) \geq s_{q_0}, \beta_1(t) = 2\omega \sqrt{t}, \quad t > 0. \]  

From now on we consider the particular case of constant temperature \( b(t) = b > 0, \ t > 0 \) at \( x = x_0 \) and constant heat flux \( q(t) = q > 0, \ t > 0 \) at \( x = 0 \) for problem (1). The steady-state solution is given by
\[ \theta_\infty(x) = \frac{q}{k}(x - x_0) + b, \]  
and a necessary and sufficient condition in order to have a two-phase steady-state Stefan problem is given by
\[ q > kb/x_0, \]  
where \( k \) is the thermal conductivity of the liquid phase [11]. (See [13, 14] for the general steady-state case for an \( n \)-dimensional domain).

Using the fact that \( \theta = \theta(x, t) \), the solution of problem (1) with data \( q > 0 \) and \( b > 0 \), converges to \( \theta_\infty = \theta_\infty(x) \) when \( t \) goes to \( +\infty \) [6], for any initial temperature \( \theta_0 = \theta_0(x) \), we can formulate the following problem: Find the relation between the heat flux \( q > 0 \) on \( x = 0 \) and a time \( t_1 \) such that another phase appears for \( t > t_1 \), and then we can reformulate problem (1) in a two-phase Stefan problem for \( t > t_1 \).

We obtain the following result.

**THEOREM 4.** Suppose the initial temperature verifies the conditions \( b \geq \theta_0 \geq 0 \) in \([0, x_0]\) and \( \theta_0(x_0) = b \). If the time \( t_1 > 0 \) and the constant heat flux \( q > 0 \) verify the inequality
\[ q > \frac{bk}{x_0(1 - \exp(-\alpha\pi^2t_1/4x_0^2))}, \quad \alpha = \frac{k}{\rho c}, \]  
then another phase (the solid phase) appears for \( t \geq t_1 \). Moreover, \( \theta(0, t) < 0 \) for all \( t \geq t_1 \) and the free boundary \( x = s(t) \) begins at a point \( (0, t') \) with \( 0 \leq t' < t_1 \).

**Proof.** The temperature \( \theta(x, t) \) is given by
\[ \theta(x, t) = \theta_\infty(x) + \sum_{n=0}^{\infty} C_n \cos(\sqrt{\lambda_n}x) \exp(-\alpha t\lambda_n), \]  
where
\[ \lambda_n = \left(n + \frac{1}{2}\right)^2 \pi^2 x_0^2, \quad n = 0, 1, 2, \ldots, \]  
\[ C_n = \frac{2}{x_0} \int_0^{x_0} \left[ \theta_0(x) - \theta_\infty(x) \right] \cos(\sqrt{\lambda_n}x) \, dx. \]  

Therefore, the temperature at \( x = 0 \) is given by
\[ \theta(0, t) = b - \frac{qx_0}{k}(1 + S(t)) + S_0(t) \]  
with
\[ S(t) = \frac{2}{x_0^2} \sum_{n=0}^{\infty} \exp(-\alpha t\lambda_n) \int_0^{x_0} (x - x_0) \cos(\sqrt{\lambda_n}x) \, dx = -\frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp(-\alpha t\lambda_n)}{(n + \frac{1}{2})^2}. \]
and
\[ S_0(t) = \frac{2}{x_0} \sum_{n=0}^{+\infty} \exp(-\alpha t \lambda_n) \int_0^{x_0} (\theta_0(x) - b) \cos(\sqrt{\lambda_n} x) \, dx. \] (41)

We get that \( S_0(t) = v(0, t) \leq 0 \) for all \( t \geq 0 \) because of the maximum principle, where the function \( v = v(x, t) \leq 0 \) is the solution of the problem
\[
\begin{align*}
\rho c v_t - k v_{xx} &= 0, \quad 0 < x < x_0, \; t > 0; \\
v_x(0, t) &= 0, \quad t > 0; \\
v(x_0, t) &= 0, \quad t > 0; \\
v(x, 0) &= \theta_0(x) - b \leq 0, \quad 0 \leq x \leq x_0.
\end{align*}
\] (42)

By some manipulations, it follows that
\[ 0 < S(t) = -S(t) \leq \exp \left[ -\frac{\alpha \pi^2 t}{4x_0^2} \right] < 1 \quad \text{for all } t \geq t_1. \] (43)

Therefore, if \( q \) and \( t_1 \) verify (35), we obtain
\[ \theta(0, t) \leq b - \frac{q x_0}{k} (1 + S(t)) < 0 \quad \text{for all } t \geq t_1, \] (44)
i.e., the thesis is achieved.

**Remark 6.** If \( \theta_0(x) = b \) in \([0, x_0]\), we deduce \( S_0(t) = 0 \) for \( t > 0 \). We remark that inequality (35) was obtained for this particular initial temperature because \( S_0(x) \leq 0 \) in \((0, +\infty)\) for \( \theta_0(x) \leq b \) in \([0, x_0]\).

**Corollary 5.** If we consider the \( t, q \) plane and define the following set
\[ Q = \{(t, q) \mid q > f(t), \; t > 0\}, \quad f(t) = \frac{bk}{x_0[1 - \exp(-\alpha \pi^2 t/4x_0^2)]}, \] (45)
then we have a two-phase problem for all \((t, q) \in Q\).

**III. On some conduction problems with a convective boundary condition.** Now we consider the same kind of techniques used in Sec. 2 for problem (1') corresponding to a heat conduction problem with a convective boundary condition at \( x = 0 \).

**Theorem 6.** If the data \( \phi_0 = \phi_0(x) \), \( b = b(t) \), and \( D \) verify the conditions
\[
\begin{align*}
\text{(i)} & \quad \phi'_0(x) \geq 0 \quad \text{and} \quad \beta_1 \geq \phi_0(x) \geq \beta_0 > 0, \quad 0 \leq x \leq x_0; \\
\text{(ii)} & \quad b(t) \geq \beta_1 \quad \text{and} \quad \dot{b} \geq 0, \quad t > 0; \\
\text{(iii)} & \quad D > 0,
\end{align*}
\] (46)
then there exists a waiting-time \( t^* > 0 \) for problem (1'), where \( t^* \) verifies the inequality
\[ t^* \geq t_1^*, \quad \text{where } t_1^* = \frac{k c \rho}{h^2} \left( F^{-1} \left( 1 + \frac{\beta_0}{D} \right) \right)^2, \] (47)
where \( F^{-1} \) is the inverse function of
\[ F(x) = \frac{\exp(-x^2)}{\text{erfc}(x)}, \quad x > 0. \] (48)
Proof. By using the maximum principle we get that
\[
\phi(x, t) \geq 0, \quad 0 \leq x \leq x_0, \quad t \geq 0,
\]
\[
\phi(x, t) \geq z(x, t), \quad 0 \leq x \leq x_0, \quad t \geq 0,
\]
where \( z = z(x, t) \) is the solution of the following heat conduction problem:
\[
\begin{align*}
(i) \quad & \rho c z_t - k z_{xx} = 0, \quad x > 0, \quad t > 0; \\
(ii) \quad & z(x, 0) = \beta_0, \quad x > 0; \\
(iii) \quad & k z_k(0, t) = h(D + z(0, t)), \quad t > 0,
\end{align*}
\]
which is given by \( a^2 = k / \rho c \):
\[
z(x, t) = (\beta_0 + D) \left[ \text{erfc} \left( \frac{x}{2a\sqrt{t}} \right) + \exp \left( \frac{hx}{k} + \eta^2 \right) \right] \text{erfc} \left( \frac{x}{2a\sqrt{t}} + \eta \right),
\]
x \geq 0, \quad t \geq 0,
where
\[
\eta = \frac{ha\sqrt{t}}{k}.
\]
Taking into account (47) and (49) we get
\[
\phi(x, t) \geq \phi(0, t) \geq z(0, t) = -D + (\beta_0 + D) \exp(\eta^2) \text{erfc}(\eta) \geq 0, \quad t \leq t_1^*,
\]
because the function \( F(x) \) verifies the conditions
\[
F(0) = 1, \quad F(+\infty) = +\infty, \quad F' > 0 \quad \text{in} \ R^+.
\]
Remark 7. We can see that \( t_1^* \) does not depend on the length of the slab \( x_0 > 0 \).

From now on we consider the particular case of constant temperature \( b(t) = b > 0, \ t > 0 \) at \( x = x_0 \) for problem \( (1') \). The corresponding steady-state solution is given by
\[
\phi_\infty(x) = b - \frac{h(D + b)}{k + hx_0}(x_0 - x), \quad 0 \leq x \leq x_0,
\]
and a necessary and sufficient condition, in order to have a two-phase steady-state Stefan problem is given by
\[
h > \frac{kb}{Dx_0}.
\]

We consider the following problem related to problem \( (1') \): Find the relation between the heat transfer coefficient \( h \) and a time \( t_2 \) such that another phase appears for \( t \geq t_2 \), and then we can reformulate problem \( (1') \) in a two-phase Stefan problem for \( t \geq t_2 \). We obtain the following result.

Theorem 7. Suppose the initial temperature verifies the conditions \( b \geq \theta_0 \geq 0 \) in \([0, x_0]\) and \( \theta_0(x_0) = b \). If the time \( t_2 > 0 \) and the constant heat transfer coefficient \( h > 0 \) verify the inequality
\[
h > g(t_2),
\]
then another phase (the solid phase) appears for \( t \geq t_2 \), where the function \( g = g(t) \) is defined implicitly by the equation

\[
\psi(t, g(t)) = 0, \quad t > 0,
\]

with

\[
\psi(t, h) = -D + \frac{k(D + b)}{k + h x_0} + \frac{2k(D + b)}{hx_0} g(t), \quad t > 0, \quad h > 0,
\]

\[
\gamma(t) = \sum_{n=1}^{\infty} \exp \left[ -\frac{(2n-1)^2 \pi^2 a^2 t}{x_0^2} \right], \quad t > 0.
\]

**Proof.** The solution of problem \((1')\) is given by

\[
\phi(x, t) = \phi_\infty(x) + \sum_{n=1}^{\infty} B_n \exp(-\mu_n^2 a^2 t) \left[ \sin(\mu_n x) + \frac{k \mu_n}{h} \cos(\mu_n x) \right],
\]

\[
0 \leq x \leq x_0, \quad t > 0,
\]

where

\[
B_n = \frac{2}{x_0} \int_0^{x_0} [\phi(x) - \phi_\infty(x)] \left[ \sin(\mu_n x) + \frac{k \mu_n}{h} \cos(\mu_n x) \right] dx,
\]

and \( \mu_n = \omega_n/x_0 \), where \( \omega_n \) is the \( n \)th root of the eigenvalue equation

\[
tg(\omega) = -\frac{k}{hx_0} \omega, \quad \omega > 0.
\]

Moreover, we get that

\[
(2n - 1)\frac{\pi}{2} < \omega_n < n\pi, \quad n \in \mathbb{N}.
\]

After some manipulation, we deduce that the temperature at \( x = 0 \) is bounded by

\[
\phi(0, t) \leq \psi(t, h), \quad t > 0,
\]

where the function \( \psi \) has been defined before.

We notice that the function \( g = g(t) \) is well defined since the functions \( \gamma = \gamma(t) \) and \( \psi = \psi(t, h) \) satisfy the properties

\[
\gamma(0^+) = +\infty, \quad \gamma(+\infty) = 0, \quad \gamma'(t) < 0, \quad \forall t > 0,
\]

\[
(a) \quad \psi(t, 0^+) = +\infty, \quad \psi(t, +\infty) = -D < 0, \quad t > 0;
\]

\[
(b) \quad \frac{\partial \psi}{\partial h}(t, h) < 0, \quad \frac{\partial \psi}{\partial t}(t, h) < 0, \quad t > 0, \quad h > 0.
\]

Therefore, the function \( g = g(t) \) satisfies the conditions

\[
g(0^+) = +\infty, \quad g(+\infty) = \frac{kb}{Dx_0}, \quad g'(t) < 0, \quad \forall t > 0.
\]

By using the inequality (65) we get the thesis.
Corollary 8. We consider in the plane $t, h$ the following set:

$$R_2 = \{(t, h) \mid h > g(t), \ t > 0\}; \quad (69)$$

then we have a two-phase problem for all $(t, h) \in R_2$.

Remark 8. If the initial temperature is given by $\phi_0(x) = b > 0$ in $[0, x_0]$, then

we have a heat conduction problem for the initial phase for all $(t, h) \in R_1$, where

$$R_1 = \left\{(t, h) \mid 0 < h < \text{Max} \left( \frac{kb}{Dx_0}, F^{-1} \left(1 + \frac{b}{D}\right) \sqrt{\frac{k\rho c}{t}} \right), \ t > 0 \right\}. \quad (70)$$

Acknowledgment. This paper has been partially sponsored by the Project "Problemas de Frontera Libre de la Fisica-Matemática" (CONICET-UNR, Argentina).

References


[12] D. A. Tarzia, An inequality for the coefficient $\sigma$ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem, Quart. Appl. Math. 39, 491–497 (1981–82)
