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ON THE SOLUTION TO A CLASS OF STRONGLY SINGULAR LINEAR INTEGRAL EQUATIONS

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Abstract. A strongly singular linear integral equation containing a small positive parameter δ is considered. This equation is transformed into a Fredholm integral equation of the second kind with a continuous kernel. The rate of convergence of the Neumann series for this integral equation is shown to be $o(\delta^2)$. An example from fracture mechanics is considered in detail.

1. Introduction. We consider the following integral equation

$$u(x) + \delta L_x L_x^* \int_0^1 m(x-y)u(y)\,dy = f_0(x), \qquad 0 < x < 1, \tag{1.1a}$$

$$u(0) = u(1) = 0, \tag{1.1b}$$

where

$$L_x = d/dx - i\alpha, \qquad L_x^* = -d/dx - i\alpha, \tag{1.2}$$

δ is a small positive parameter, and α is a constant with $\operatorname{Im}\{\alpha\} \geq 0$. The kernel $m(x)$ is assumed to have a singularity at $x = 0$. Thus the integral equation is hypersingular. The linearity of (1.1) plays an integral part in the method of solution developed herein.

A recent theoretical investigation by Nemat-Nasser and Hori [3] of the bridging of cracks in brittle materials through fiber reinforcement has provided a formulation of the problem as an integral equation. For the special case of a crack normal to the fibers with constant bridging stiffness, the integral equation takes the form (1.1) with $\alpha = 0$ and $m(x) = -\pi^{-1}\log|x|$. Olmstead and Gautesen [4], Hori and Nemat-Nasser [2], and Willis and Nemat-Nasser [5] have considered this example asymptotically for small δ. In the first two papers the authors give the leading order term in the asymptotic expansion of the solution u without any indication of how to obtain higher-order terms. Only in the latter work is a systematic method for obtaining the higher terms given. They use the method of inner and outer expansions to obtain the solution to $o(\delta^2)$. By this method, however, proceeding to higher-order terms is laborious. In addition, when their asymptotic approximation for u to any

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order in δ is substituted into (1.1a), singularities occur at $x = 0$ and 1. Our method does not have this difficulty.

The main result of this work is to show that under suitable assumptions about the kernel $k(x)$, $u(x)$ satisfies the Fredholm integral equation of the second kind

$$u(x) + \delta \int_0^1 m_1(x, y) u(y)\, dy = F_0(x), \qquad 0 < x < 1, \tag{1.3}$$

where $m_1(x, y)$ and $F_0(x)$ are given by (2.44) and (2.47) below. These functions are continuous functions of their variables and vanish when $x = 0, 1$. For δ small, $m_1(x, y) \sim o(\delta)$ uniformly in x and y. Thus, we immediately have the approximation $u(x) \sim F_0(x) + o(\delta^2)$. For the aforementioned example the error in this approximation is $O(\delta^3 \log \delta)$.

We make the following assumptions about the Fourier transform $\hat{m}(\xi)$ of the kernel $m(x)$, which is defined by

$$\hat{m}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} m(x)\, dx. \tag{1.4}$$

First we assume that

$$\hat{m}(\xi) = \hat{m}(-\xi). \tag{1.5}$$

Then on the region in the complex ξ-plane, which is the union of $|\mathrm{Im}\{\xi\}| < 2\tau_i$ and $|\mathrm{Re}\{\xi\}| > \frac{1}{2}\sigma_r$, we assume that

$$\hat{m}(\xi) \text{ is analytic;} \tag{1.6}$$

$$\hat{m}(\xi) \sim O(|\xi|^{-1+\gamma}), \quad \text{as } \xi \to \infty, \ 0 \le \gamma < 1; \tag{1.7}$$

$$1 + \delta(\xi^2 - \alpha^2)\hat{m}(\xi) \neq 0. \tag{1.8}$$

The first assumption (1.5) implies that the kernel is symmetric,

$$m(x) = m(-x). \tag{1.9}$$

The kernel $m(x)$ is given by

$$m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} \hat{m}(\xi)\, d\xi. \tag{1.10}$$

For $x > 0$, assumptions (1.6) and (1.7) allow us to deform the ξ-contour in (1.10) to the contour C_ξ where for $\xi = \sigma + i\tau$,

$$C_\xi = \{\sigma = -\sigma_r,\ \tau \le -\tau_i\} \cup \{|\sigma| < \sigma_r,\ \tau = -\tau_i\} \cup \{\sigma = \sigma_r,\ \tau \le -\tau_i\}. \tag{1.11}$$

Thus

$$m(x) = \frac{1}{2\pi} \int_{C_\xi} e^{-i\xi x} \hat{m}(\xi)\, d\xi, \qquad x > 0. \tag{1.12}$$

From (1.12) it follows that $m(x)$ and all its derivatives are continuous for $x > 0$. At $x = 0$ the kernel $m(x)$ is singular, which follows from (1.10) and (1.7).

We need a Wiener-Hopf factorization of

$$\hat{n}(\xi) = c_0[1 + \delta(\xi^2 - \alpha^2)\hat{m}(\xi)]^{-1}, \tag{1.13}$$

where

$$c_0 = 1 - \delta\alpha^2 \hat{m}(0). \tag{1.14}$$

By (1.5) we can write

$$\hat{n}(\xi) = \hat{b}(\xi)\hat{b}(-\xi), \tag{1.15}$$

where $\hat{b}(\xi)$ is analytic for $\mathrm{Im}\{\xi\} > -\tau_i$. An explicit expression for $\hat{b}(z)$ follows from assumptions (1.6)–(1.8) as

$$\begin{aligned}\log \hat{b}(z) &= \frac{1}{2\pi i}\int_{-\infty - i\tau_i}^{\infty - i\tau_i} \log \hat{n}(\xi)\left[\frac{1}{\xi - z} - \frac{1}{\xi}\right] d\xi \\ &= \frac{i}{2\pi}\int_{C_\xi} \log[1 + \delta(\xi^2 - \alpha^2)\hat{m}(\xi)]\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right) d\xi. \end{aligned}\tag{1.16}$$

Note that

$$\hat{b}(0) = 1 \tag{1.17}$$

and that

$$\lim_{\delta \to 0} \hat{b}(\xi) = 1. \tag{1.18}$$

For small δ we need an asymptotic expression for $b(x)$, the inverse Fourier transform of $\hat{b}(\xi)$. We can write

$$\begin{aligned} b(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[-ixz + \log \hat{b}(z)]\, dz \\ &= \frac{1}{2\pi}\int_{C_z} \exp[-ixz + \log \hat{b}(z)]\, dz, \qquad x > 0, \end{aligned}\tag{1.19}$$

where the contour C_z lies just outside of the contour C_ξ defined by (1.11). For $|z|$ large, the integrand in (1.19) has exponential decay. Thus for small δ and $x > 0$, we write

$$b(x) \sim \frac{1}{2\pi}\int_{C_z} e^{-ixz}\{1 + \log \hat{b}(z) + \cdots\}\, dz. \tag{1.20}$$

The integral of the first term is zero and as a function of z, $\log \hat{b}(z)$ has a pole at $z = \xi$. Thus (1.20) becomes

$$\begin{aligned} b(x) &\sim -\frac{1}{2\pi}\int_{C_\xi} e^{-ix\xi} \log[1 + \delta(\xi^2 - \alpha^2)\hat{m}(\xi)]\, d\xi \\ &\sim -\frac{\delta}{2\pi}\int_{C_\xi} e^{-ix\xi}(\xi^2 - \alpha^2)\hat{m}(\xi)\, d\xi. \end{aligned}\tag{1.21}$$

The last integral follows from (1.12). Thus we assume we can write

$$b(x) \sim \delta b_1(x) + o(\delta), \qquad x > 0, \tag{1.22}$$

where

$$b_1(x) = m''(x) + \alpha^2 m(x) \tag{1.23}$$

and $m''(x)$ denotes the second derivative.

In the next section we derive our main result (1.3). In the last section we show that $m_1(x, y) \sim o(\delta)$ and consider in detail the example from fracture mechanics. Also, we briefly indicate the changes required when the first term $u(x)$ in (1.1a) is replaced by $q(x)u(x)$.

2. Integral identities. We can also write the integral equation (1.1) as

$$Ku = L_x L_x^* \int_0^1 k(x-y)u(y)\,dy = f(x), \qquad 0 < x < 1, \tag{2.1a}$$

$$u(1) = u(0) = 0, \tag{2.1b}$$

where

$$k(x) = -(2i\alpha c_0)^{-1}\exp[i\alpha|x|] + \delta m_0(x), \tag{2.2}$$

$$m_0(x) = m(x)/c_0, \qquad f(x) = f_0(x)/c_0, \tag{2.3}$$

and c_0 is given by (1.14), and without loss of generality, we can assume that

$$f(x) = \varepsilon f(1-x), \qquad \varepsilon = 1 \text{ or } -1. \tag{2.4}$$

If f does not satisfy (2.4), we write $f(x) = f_1(x) + f_2(x)$ where $f_1(x) = \frac{1}{2}(f(x) + f(1-x))$ and $f_2(x) = \frac{1}{2}(f(x) - f(1-x))$. Then f_1 and f_2 satisfy (2.4). Since K is linear, we can write $u = u_1 + u_2$ where $Ku_i = f_i$. As a consequence of (2.4), we have

$$u(x) = \varepsilon u(1-x). \tag{2.5}$$

We show below that (2.1) can also be expressed as

$$u(x) = F(x) - U(x) - \varepsilon U(x_0), \tag{2.6}$$

where

$$x_0 = (1-x); \tag{2.7}$$

$$F(x) = F_1(x) + \varepsilon F_1(x_0) - Nf; \tag{2.8}$$

$$Nf = \int_0^1 n(x-y)f(y)\,dy; \tag{2.9}$$

$$n(x-y) = \int_{\max(x,y)}^{\infty} b(s-x)b(s-y)\,ds = \int_{-\infty}^{\min(x,y)} b(x-s)b(y-s)\,ds; \tag{2.10}$$

$$F_1(x) = \int_0^x b(x-s)\int_s^1 b(y-s)f(y)\,dy\,ds; \tag{2.11}$$

$$U(x) = \int_0^\infty b(x+s)U_1(s)\,ds; \tag{2.12}$$

$$U_1(s) = \delta\int_0^1\int_0^\infty b(1+s+t)L_t L_t^* m_0(1-y+t)u(y)\,dt\,dy; \tag{2.13}$$

and $b(x)$ is given by (1.19). We shall now write (2.6) in a different form. By repeated integration by parts we can also express $U_1(s)$ as

$$U_1(s) = \delta\int_0^1\int_0^\infty m_0(1-y+t)L_t L_t^* b(1+s+t)u(y)\,dt\,dy$$
$$- b'(1+s)C_1 - b(1+s)C_2, \tag{2.14}$$

where the prime denotes the derivative and

$$C_1 = \delta \int_0^1 m_0(1-y)u(y)\,dy, \tag{2.15}$$

$$C_2 = -\delta \int_0^1 m_0(1-y)u'(y)\,dy. \tag{2.16}$$

Then

$$U(x) + \varepsilon U(x_0) = V(x) - V_1(x)C_1 - V_2(x)C_2, \tag{2.17}$$

where

$$\begin{aligned} V(x) = \delta \int_0^1 \int_0^\infty \{ & m_0(1-y+t)L_t L_t^* g(x, 1+t) \\ & + m_0(y+t)L_t L_t^* g(x_0, 1+t)\} u(y)\,dt\,dy; \end{aligned} \tag{2.18}$$

$$V_1(x) = g_t(x, 1) + \varepsilon g_t(x_0, 1); \tag{2.19}$$

$$V_2(x) = g(x, 1) + \varepsilon g(x_0, 1); \tag{2.20}$$

$$g(x, t) = \int_0^\infty b(x+s)b(s+t)\,ds; \tag{2.21}$$

and $g_t(x, t)$ denotes the partial derivative of $g(x, t)$ with respect to t. Also, it can be shown that

$$\begin{aligned} F(x) = & \int_0^x \int_0^{x_0} b(s)b(y)f(x+y-s)\,dy\,ds \\ & - \int_{x_0}^1 g(x, y)f(y)\,dy - \int_0^{x_0} g(x_0, 1-y)f(y)\,dy. \end{aligned} \tag{2.22}$$

Thus (2.6) becomes

$$u(x) = F(x) - V(x) + C_1 V_1(x) + C_2 V_2(x). \tag{2.23}$$

We shall now obtain an alternate expression for the constants C_1 and C_2. Since $u(1) = 0$, (2.23) yields

$$0 = F(1) - V(1) + C_1 V_1(1) + C_2 V_2(1). \tag{2.24}$$

We remark that (2.24) also assures that $u(0) = 0$. Near $x = 0$,

$$u(x) \sim C_3 B(x) + C_4 x + o(x), \tag{2.25}$$

where

$$B(x) = \int_0^x b(t)\,dt; \tag{2.26}$$

$$C_3 = \int_0^1 b(y)f(y)\,dy - b'(1)C_1 - b(1)C_2 + C_5; \tag{2.27}$$

$$\begin{aligned} C_4 = & \int_0^1 g_x(1, y)f(y)\,dy - (g_{tt}(0, 1) + \varepsilon g_{xt}(1, 1))C_1 \\ & - (g_t(0, 1) + \varepsilon g_x(1, 1))C_2 + C_6; \end{aligned} \tag{2.28}$$

$$C_5 = \delta \int_0^1 \int_0^\infty m_0(1-y+t)L_tL_t^* b(1+t)u(y)\,dt\,dy; \tag{2.29}$$

$$C_6 = \delta \int_0^1 \int_0^\infty \{m_0(1-y+t)L_tL_t^* g_t(0\,,\,1+t) + m_0(y+t)L_tL_t^* g_x(1\,,\,1+t)\}u(y)\,dt\,dy. \tag{2.30}$$

To establish (2.25) it is sufficient to show that

$$\lim_{x\to 0}(u'(x) - C_3 b(x)) = C_4, \tag{2.31}$$

where $u'(x)$ denotes the derivative of $u(x)$. This relation follows by differentiating (2.23) and using the identity

$$g_x(x\,,\,t) = -b(x)b(t) - g_t(x\,,\,t), \tag{2.32}$$

where the subscripts denote the partial derivative. Substituting $u(x) = B(x)$ into (2.1a) and noting the identity

$$L_xL_x^* \int_0^\infty k(x-y)B(y)\,dy = 1, \qquad x > 0, \tag{2.33}$$

we find that the resulting function of x is regular near $x = 0$. However, when $u = x$, (2.1a) becomes

$$Ku \sim -k(x) \quad \text{as } x \to 0, \tag{2.34}$$

which is singular at $x = 0$. Thus by (2.25) it follows that

$$C_4 = 0. \tag{2.35}$$

Instead of solving (2.24) and (2.28) for C_1 and C_2, we find it convenient to define

$$C_1V_1(x) + C_2V_2(x) = D_1W_1(x) + D_2W_2(x), \tag{2.36}$$

where

$$2V_1(1)V_2(1)W_1(x) = V_2(1)V_1(x) + V_1(1)V_2(x) - [V_1(1)g_1 + V_2(1)g_2]W_2(x); \tag{2.37}$$

$$W_2(x) = [V_2(1)V_1(x) - V_1(1)V_2(x)]/[V_2(1)g_2 - V_1(1)g_1]; \tag{2.38}$$

$$g_1 = g_t(0\,,\,1) + \varepsilon g_x(1\,,\,1); \tag{2.39}$$

$$g_2 = g_{tt}(0\,,\,1) + \varepsilon g_{xt}(1\,,\,1). \tag{2.40}$$

Then (2.24) and (2.28), respectively, become

$$D_1 = V(1) - F(1), \tag{2.41}$$

$$D_2 = C_6 + \int_0^1 g_x(1\,,\,y)f(y)\,dy. \tag{2.42}$$

Finally, upon substitution from (2.36), (2.41), (2.42), and (2.18), we find that (2.23) becomes

$$u(x) + \delta \int_0^1 m_1(x\,,\,y)u(y)\,dy = F_0(x), \qquad 0 < x < 1, \tag{2.43}$$

where

$$m_1(x, y) = m_2(x, y) - m_2(1, y)W_1(x) - m_3(y)W_2(x); \tag{2.44}$$

$$m_2(x, y) = \int_0^\infty \{m_0(1 - y + t)L_t L_t^* g(x, 1 + t) + m_0(y + t)L_t L_t^* g(x_0, 1 + t)\}\, dt; \tag{2.45}$$

$$m_3(y) = \int_0^\infty \{m_0(1 - y + t)L_t L_t^* g_t(0, 1 + t) + m_0(y + t)L_t L_t^* g_x(1, 1 + t)\}\, dt; \tag{2.46}$$

$$F_0(x) = F(x) + \int_0^1 g(1, y)f(y)\, dy W_1(x) + \int_0^1 g_x(1, y)f(y)\, dy W_2(x). \tag{2.47}$$

Integral equation (2.43) is a Fredholm integral equation of the second kind whose kernel $m_1(x, y)$ is a continuous function of x and y. It has the properties that $m_1(0, y) = m_1(1, y) = 0$ and that

$$m_1(x, y) \sim \text{const}\, B(x) + o(x) \quad \text{as } x \to 0, \tag{2.48a}$$

$$m_1(x, y) \sim \text{const}\, B(1 - x) + o(1 - x) \quad \text{as } x \to 1 \tag{2.48b}$$

uniformly in y. Similarly, $F_0(x)$ is continuous, vanishes at $x = 0, 1$, and satisfies (2.48) when $m_1(x, y)$ is replaced by $F_0(x)$. If the Neumann series for (2.43) converges, then we can approximate u by a finite number of terms in this series. When this approximation is substituted into the original integral equation (1.1a), the resulting function of x is continuous.

We have yet to establish (2.6). To this end, we first show that

$$u(x) = Nf + U_2(x_0) + \varepsilon U_2(x), \qquad 0 < x < 1, \tag{2.49}$$

where

$$U_2(x) = \delta \int_0^1 \int_0^\infty n(x + t)L_t L_t^* m_0(1 - y + t)u(y)\, dt\, dy, \tag{2.50}$$

and then show that

$$u(x) = F_1(x) + U_2(x_0) - U(x). \tag{2.51}$$

Subtracting (2.49) and (2.51) gives

$$U_2(x) = \varepsilon(F_1(x) - Nf - U(x)). \tag{2.52}$$

Using (2.52) to eliminate $U_2(x_0)$ in (2.51) yields (2.6) upon noting that $(Nf)(x) = \varepsilon(Nf)(x_0)$.

To establish (2.49) we define $u(x) = 0$ and $f(x) = 0$ for $x \notin (0, 1)$ and define a function $U_3(x)$ by $U_3(x) = 0$, $0 < x < 1$ and

$$U_3(x) = L_x L_x^* \int_0^1 k(x - y)u(y)\, dy = \delta L_x L_x^* \int_0^1 m_0(x - y)u(y)\, dy, \qquad x \notin (0, 1). \tag{2.53}$$

Then (2.1) can be expressed as

$$L_x L_x^* \int_{-\infty}^\infty k(x - y)u(y)\, dy = f(x) + U_3(x), \qquad -\infty < x < \infty. \tag{2.54}$$

The inverse of the operator on the left side of (2.54) is the operator $\int_{-\infty}^{\infty} dz\, n(z-x)$. Thus (2.54) can also be expressed as

$$u(x) = Nf + \int_1^\infty n(s-x)U_3(s)\,ds + \int_0^\infty n(x+s)U_3(-s)\,ds, \qquad 0 < x < 1. \quad (2.55)$$

In the first integral we let $s = 1+t$ to find that it is given by $U_2(x_0)$. From (2.53) it follows that $U_3(-x) = \varepsilon U_3(1+x)$. The last integral, then, is $\varepsilon U_2(x)$. We have thus proved (2.49).

To establish (2.51), we shall use an identity from Gautesen [1]. His equation (1.2) with $\lambda = 1$ is identical to our equation (2.1). Thus, we can use his identity (2.28), which is

$$L_s \int_0^s a(s-y)u(y)\,dy = \int_s^1 b(y-s)f(y)\,dy + U_1(-s), \qquad 0 < s < 1, \qquad (2.56)$$

where U_1 is defined by (2.13). The Fourier transform of $a(x)$ is

$$\hat{a}(\xi) = i/[\hat{b}(\xi)(\xi+\alpha)]. \qquad (2.57)$$

It has the property that $\hat{a}(\xi)\hat{a}(-\xi) = \hat{k}(\xi)$, where $\hat{k}(\xi)$ is the Fourier transform of the kernel $k(x)$ in (2.1). Operating on (2.56) with $\int_0^x ds\, b(x-s)$ yields

$$u(x) = F_1(x) + \int_{-\infty}^x b(x-s)U_1(-s)\,ds - \int_{-\infty}^0 b(x-s)U_1(-s)\,ds, \qquad (2.58)$$

where $F_1(x)$ is given by (2.11). Letting $s = -t$ in the last integral in (2.58) we find by (2.12) that it is $U(x)$. Next we substitute for $U_1(-s)$ from (2.13) in the first integral in (2.58) and use (2.10) to perform the s integration. The result is that this integral is $U_2(x_0)$, where $U_2(x)$ is given by (2.50). Thus we have proved (2.51).

3. Asymptotics and example. In this section we consider the integral equation (2.43) when δ is small and the example from fracture mechanics. We shall show that the kernel $m_1(x, y)$, as defined by (2.44), is $o(\delta)$. This implies that the rate of convergence of the Neumann series for this integral equation is $o(\delta^2)$. Hence

$$u \sim F_0(x) + o(\delta^2), \qquad (3.1)$$

where $F_0(x)$ is defined by (2.47). For the fracture mechanics example we give $F_0(x)$ and show that the order estimate in (3.1) is actually $O(\delta^3 \log \delta)$.

The kernel $m_1(x, y)$ only involves δ through the function

$$g(x, t) = \int_0^\infty b(x+s)b(t+s)\,ds, \qquad t \geq 1,\ 0 < x < 1. \qquad (3.2)$$

We show below that

$$g(x, t) \sim \delta b_1(t)\phi(x) + o(\delta), \qquad t \geq 1, \qquad (3.3)$$

where

$$\phi(x) = 1 - B(x) = \int_x^\infty b(t)\,dt, \qquad (3.4)$$

$b_1(x)$ is defined by (1.23), and $B(x)$ is given by (2.26). When this approximation is substituted into (2.44), the result is $m_1(x, y) \sim o(\delta)$. Thus, we have the approximation (3.1).

To establish (3.3), we obtain an alternate representation of $g(x, t)$. From (2.10) and $n(t) = n(-t)$, we have

$$g(0, t) = n(t). \tag{3.5}$$

Integration of (2.32) with respect to x from 0 to x yields

$$g(x, t) = n(t)\phi(x) + [n(t) - b(t)]B(x) - \int_0^x g_t(x_1, t)\, dz dx_1, \tag{3.6}$$

where we have used the relation $\phi(0) = \hat{b}(0) = 1$, which follows from (3.4) and (1.17). To obtain an asymptotic expression for $n(x)$, $x > 0$, we note that

$$\begin{aligned} n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{n}(\xi)\, d\xi = \frac{1}{2\pi}\int_{C_\xi} e^{-i\xi x}\hat{n}(\xi)\, d\xi \\ &\sim \frac{1}{2\pi}\int_{C_\xi} e^{-i\xi x}\{c_0 - \delta(\xi^2 - \alpha^2)\hat{m}(\xi) + O(\delta^2)\}\, d\xi \sim \delta b_1(x) + O(\delta^2), \end{aligned} \tag{3.7}$$

where C_ξ is the contour defined by (1.11), $b_1(x)$ is given by (1.23), and the relations (1.13) and (1.14) have been used. Thus, by (1.22) and (3.7), we have

$$n(t) - b(t) \sim o(\delta), \qquad t \geq 1. \tag{3.8}$$

By (3.2) and (1.22) we find that for x near the boundary layer at $x = 0$,

$$g_t(x, t) \sim \int_0^{\infty} b(x+s)b_1'(t+s)\, ds \sim O(\delta), \qquad t \geq 1 \tag{3.9}$$

and that for x outside the boundary layer,

$$g_t(x, t) \sim \delta^2 \int_0^{\infty} b_1(x+s)b_1'(t+s)\, ds \sim O(\delta^2), \qquad t \geq 1. \tag{3.10}$$

Thus

$$\int_0^x g_t(x_1, t)\, dx_1 \sim o(\delta). \tag{3.11}$$

Substitution of (3.7), (3.8), and (3.11) into (3.6) yields (3.3).

Thus we can make the approximation

$$u(x) \sim F_0(x) + o(\delta^2), \tag{3.12}$$

where $F_0(x)$ is given by (2.47). Also, if we substitute the approximation (3.3) into (2.47), we obtain the simpler approximation

$$u(x) \sim \int_0^x \int_0^{x_0} b(s)b(y)f(x+y-s)\, dy\, ds + o(\delta). \tag{3.13}$$

If $f(x)$ is a polynomial or an exponential, the expression defining $F_0(x)$ simplifies. For example, if

$$f(x) = 1 \tag{3.14}$$

then

$$F_0(x) = B(x)B(x_0) - [\phi(1) - \phi(x)][\phi(1) - \phi(x_0)] + \left\{ \int_0^x - \int_{x_0}^1 \right\} g(1, y)\, dy - f_1 + f_1 W_1(x) + f_2 W_2(x), \tag{3.15}$$

where

$$f_1 = -\frac{1}{2}\phi^2(1) + \int_1^\infty [b(y) - n(y)]\, dy, \tag{3.16}$$

$$f_2 = \phi(1)b(1) - g(1, 1) + n(1) - b(1). \tag{3.17}$$

Let us now consider the example from fracture mechanics: $m(x) = -\pi^{-1} \log|x|$ and $\alpha = 0$. Then integral equation (1.1a) becomes

$$u(x) + \frac{\delta}{\pi} \frac{d^2}{dx^2} \int_0^1 \log|x - y| u(y)\, dy = f_0(x), \qquad 0 < x < 1. \tag{3.18}$$

Since $\alpha = 0$, $c_0 = 1$ and $f_0(x) = f(x)$. Also,

$$\hat{n}(\xi) = 1 + \delta(-i\xi)^{1/2}(i\xi)^{1/2}, \tag{3.19}$$

where the branch of the square-root function $\sqrt{\xi}$ is the half line: $\mathrm{Re}\{\xi\} < 0$ and $\mathrm{Im}\{\xi\} = 0$. Then

$$n(x) = n(-x) = \frac{1}{\pi\delta} \int_0^\infty t(1 + t^2)^{-1} e^{-xt/\delta}\, dt, \qquad x > 0, \tag{3.20}$$

$$b(x) = \frac{1}{\pi\delta} \int_0^\infty \frac{t}{(1 + t^2)^{3/4}} \exp\left[-xt/\delta - \frac{1}{\pi} \int_0^t \frac{\log s}{1 + s^2}\, ds \right] dt. \tag{3.21}$$

From (3.20) and (3.21), we obtain the following asymptotic expressions:

$$n(x) = \delta b_1(x) + O(\delta^3), \qquad x \gg \delta; \tag{3.22}$$

$$b(x) = \delta b_1(x) + \delta^2 b_2(x) + O(\delta^3), \qquad x \gg \delta, \tag{3.23}$$

where

$$b_1(x) = (\pi x^2)^{-1} = m''(x), \tag{3.24}$$

$$b_2(x) = \pi^{-2} x^{-3} [2\gamma_e - 1 + 2\log(x/\delta)], \tag{3.25}$$

and $\gamma_e = 0.577\ldots$ is Euler's constant. Note that (3.24) is consistent with (1.22). For this example, the approximation (3.3) becomes

$$g(x, t) \sim \delta b_1(t) + O(\delta^2 \log\delta). \tag{3.26}$$

Thus $m_1(x, t) \sim O(\delta^2 - \log\delta)$ and the rate of convergence of the Neumann series for the integral equation (2.43) is $O(\delta^3 \log\delta)$. Hence we have the approximations

$$u \sim F_0(x) + O(\delta^3 \log\delta), \tag{3.27}$$

$$u \sim \int_0^x \int_0^{x_0} b(s)b(y)f(x + y - s)\, dy\, ds + O(\delta^2 \log\delta), \tag{3.28}$$

where F_0 is given by (2.47).

For the purpose of comparison, we offer explicit expansion of u inside and outside the boundary layer for the case where $f_0(x) = 1$. Using (3.15) to compute $F_0(x)$, we find that

$$u(x) \sim \left\{1 - \frac{\delta}{\pi} - \frac{2\delta^2}{\pi^2}\left[\gamma_e - \log\delta - \frac{1}{4}\right]\right\} B(x) - \frac{\delta}{\pi}\int_0^x (x-y)b(y)\,dy + O(\delta^3\log\delta), \qquad x = O(\delta); \tag{3.29}$$

$$u(x) \sim 1 - \frac{\delta}{\pi x x_0} + \frac{\delta^2}{\pi^2}\left[\frac{2}{x x_0} - \{\gamma_e - \log\delta\}\left\{\frac{1}{x^2} + \frac{1}{x_0^2}\right\} + \left\{\frac{1}{x_0^2} - \frac{1}{x^2}\right\}\log(x/x_0)\right] + O(\delta^3\log\delta), \qquad x x_0 \gg \delta. \tag{3.30}$$

To compare our result with that of Willis and Nemat-Nasser [5] we need to introduce the function

$$v(x) = \int_0^x (x-y)b(y)\,dy - x + \frac{\delta}{\pi}(\gamma_e + 1)B(x). \tag{3.31}$$

It is easily shown that $\lim_{x\to\infty}[v(x) + \frac{\delta}{\pi}\log\frac{x}{\delta}] = 0$ and

$$\delta\frac{d^2}{dx^2}\int_0^\infty \log|x-y|v(y)\,dy + v(x) = -\frac{\delta}{\pi}\log\frac{x}{\delta}, \qquad x > 0. \tag{3.32}$$

This function is related to the function $V_2(x)$ defined by Willis and Nemat-Nasser [5] by some scalar factors. Our asymptotic expansions (3.29) and (3.30) are in agreement with those of Willis and Nemat-Nasser [5]. For this example, we also have

$$u(x) \sim F_0(x) + \int_0^1 m_1(x, y)B(y)B(1-y)\,dy + O(\delta^5(\log\delta)^2). \tag{3.33}$$

We can also consider the equation

$$q(x)u(x) + \delta L_x L_x^* \int_0^1 m(x-y)u(y)\,dy = f_0(x), \qquad 0 < x < 1, \tag{3.34}$$

with the conditions $u(0) = u(1) = 0$, where $q(x)$ has the properties that $q(x) > 0$, $0 \le x \le 1$, $q(x) = q(1-x)$, and $q(0) = 1$. We define

$$\tilde{f}_0(x) = f_0 + (1 - q(x))u(x). \tag{3.35}$$

Then (3.34) becomes

$$u(x) + \delta L_x L_x^* \int_0^1 m(x-y)u(t)\,dy = \tilde{f}_0(x). \tag{3.36}$$

We can now use the result (2.43) to obtain

$$q(x)u(x) + \delta\int_0^1 m_1(x, y)u(y)\,dy + M_1 u + M_2 u = F_0(x), \qquad 0 < x < 1, \tag{3.37}$$

where m_1 and F_0 are given by (2.44) and (2.47), and

$$c_0 M_1 u = -\int_0^x \int_0^{x_0} b(s)b(y)[1 - q(x+y-s)]u(x+y-s)\,dy\,ds + c_0[1 - q(x)]u(x); \tag{3.38}$$

$$\begin{aligned} c_0 M_2 u = & \int_{x_0}^1 g(x, y)[1 - q(y)]u(y)\,dy + \int_0^x g(x_0, 1-y)[1 - q(y)]u(y)\,dy \\ & - \int_0^1 g(1, y)[1 - q(y)]u(y)\,dy W_1(x) \\ & - \int_0^1 g_x(1, y)[1 - q(y)]u(y)\,dy W_2(x). \end{aligned} \tag{3.39}$$

As $\delta \to 0$, it can be shown that $M_1 u \sim o(1)$ and $M_2 u \sim o(\delta)$. Thus the Neumann series for (3.37) converges. Then we immediately have the approximation

$$q(x)u(x) \sim \int_0^x \int_0^{x_0} b(s)b(y)f(x+y-s)\,dy\,ds + o(1). \tag{3.40}$$

For the fracture mechanics example ($\alpha = 0$, $m(x) = -\frac{1}{\pi}\log|x|$, and $f_0(x) = 1$), we can write

$$u(x) \sim u_1(x) + u_2(x) + O(\delta^2(\log\delta)^2), \tag{3.41}$$

where

$$q(x)u_1(x) = B(x)B(x_0), \tag{3.42}$$

$$q(x)u_2(x) = [q(x) - 1]u_1(x) + \int_0^x \int_0^{x_0} b(s)b(y)[1 - q(x+y-s)]u_1(x+y-s)\,dy\,ds. \tag{3.43}$$

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References

[1] A. K. Gautesen, *On the asymptotic solution to a class of linear integral equations*, SIAM J. Appl. Math. **48**, 294–306 (1988)

[2] M. Hori and S. Nemat-Nasser, *Asymptotic solution to a class of strongly singular integral equations*, SIAM J. Appl. Math. **50**, 716–725 (1990)

[3] S. Nemat-Nasser and M. Hori, *Toughening by partial or full bridging of cracks in ceramics and fiber reinforced composites*, Mechics of Materials **6**, 245–269 (1987)

[4] W. E. Olmstead and A. K. Gautesen, *Asymptotic solution of some singularly perturbed Fredholm integral equations*, Z. Angew. Math. Phys. **40**, 230–244 (1989)

[5] J. R. Willis and S. Nemat-Nasser, *Singular perturbation solution of a class of singular integral equations*, Quart. Appl. Math. **48**, 741–753 (1990)