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BENDING ENERGY OF HIGHLY ELASTIC MEMBRANES

BY

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Abstract. For a membrane composed of elastic material with strain energy W per unit initial volume, approximations to the energy per unit initial area are obtained by integrating W through the thickness. The usual stretching energy $M(\mathbf{r}_{,a})$ is modified by including a bending energy term $\alpha B(\mathbf{r}_{,a}, \mathbf{r}_{,ab})$ that is quadratic in the second derivatives $\mathbf{r}_{,ab}$. If W is the strain energy function for a stable material, M need not satisfy the Legendre-Hadamard material stability conditions, but the modified energy $M + \alpha B$ does satisfy these conditions. The special form that B takes when the membrane is isotropic is given.

1. Introduction. Problems in the theory of large elastic deformations of membranes [1] can be formulated by using the minimum energy principle, with a strain energy function $M(\mathbf{r}_{,a})$ that is a function of the first derivatives of the deformation $\mathbf{r}(x_1, x_2)$. Membranes defined by such a strain energy function have no bending stiffness, and as a consequence, there are problems in which the energy functional has no minimizer. One way to remedy this defect is to replace $M(\mathbf{r}_{,a})$ by an associated *relaxed* energy density $M_r(\mathbf{r}_{,a})$ [2]. A different approach is to use a strain energy function $E(\mathbf{r}_{,a}, \mathbf{r}_{,ab})$ that accounts for the bending stiffness of the membrane through the dependence of E on the second derivatives $\mathbf{r}_{,ab}$ [3]. As Ball, Currie, and Olver [4] have remarked, it can then be expected that minimum energy problems will have solutions if E is quasiconvex in its dependence on the second derivatives.

In the present paper we consider initially flat membranes of uniform thickness, composed of a homogeneous elastic material with strain energy W per unit initial volume. We derive from W an approximate expression for the energy per unit initial area of the form $E = M + \alpha B$, where M is the membrane energy and αB is a lowest-order approximation to the bending energy. We show that if W is the strain energy function for a stable material, then the approximate bending energy $\alpha B(\mathbf{r}_{,a}, \mathbf{r}_{ab})$ that we derive from it is indeed quasiconvex with respect to $\mathbf{r}_{,ab}$.

The energy per unit initial area is the integral of W through the thickness of the membrane, and exact evaluation of this energy requires knowledge of the details of the deformation through the thickness. However, we seek only lowest-order

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approximations to the stretching and bending energies, and for this purpose it is sufficient to use an approximate deformation that varies linearly in the thickness direction, as Kirchhoff [5] did in formulating classical plate theory. In Sec. 3 we derive the general form of the stretching energy $M(\mathbf{r}_{,a})$ for a material of arbitrary anisotropy, and in Sec. 4 we obtain the lowest-order approximation to the bending energy, $\alpha B(\mathbf{r}_{,a}, \mathbf{r}_{,ab})$. Here α is a small parameter and B is a quadratic function of the second derivatives.

Material stability conditions are discussed in Sec. 6. We restrict attention to cases in which W is the strain energy density for a stable material. In particular, this means that W satisfies the Legendre-Hadamard and Weierstrass (Graves [6]) stability conditions for every value of the deformation gradient. The associated form of $M(\mathbf{r}_{,a})$ does not satisfy these inequalities for all values of $\mathbf{r}_{,a}$, and for this reason there are problems in membrane theory that have no stable solution at all. The object of adding a term $\alpha B(\mathbf{r}_{,a}, \mathbf{r}_{ab})$ to the energy is to rectify this situation, but of course nothing is gained by doing this unless the new energy function does satisfy the appropriate material stability conditions. We show that this is indeed the case, if αB is derived from the strain energy function of W of a stable material (Sec. 7).

The remainder of the paper concerns various subsidiary issues. We first note, in Sec. 7, that although B satisfies the weak form of the Legendre-Hadamard inequality, it does not satisfy the strong form, so the associated equilibrium equations are not totally elliptic as a system of fourth-order equations.

The restrictions on the form of the bending energy implied by isotropy of space and by reflectional symmetry of the material are given in Sec. 8. In Sec. 9 we restrict attention to isotropic materials and derive the expression for B in terms of $\overline{W}(\lambda_1, \lambda_2, \lambda_3)$, the strain energy expressed as a function of the principal stretches.

It is natural to suppose that the bending energy should be nonnegative, and in fact there is a theory of fabrics for which this is true [7]. The material stability conditions are satisfied if B is nonnegative definite, but this is not a necessary condition for material stability. In Sec. 10 we compare the material stability conditions with the conditions for nonnegative bending energy in the isotropic case, for which the form of B is relatively explicit. The additional conditions that W must satisfy in order to ensure that B is nonnegative do not appear to be unduly restrictive.

2. Notation. We consider a thin sheet of homogeneous elastic material that is initially bounded by the planes $x_3 = \pm h$ in a system of Cartesian coordinates x_A $(A = 1, 2, 3)$. If lengths are measured in units of a characteristic lateral dimension of the sheet, h is a small parameter. It is convenient to write $x_3 = z$, so that the particle labels are x_a $(a = 1, 2)$ and z. Generally, subscripts $A, B, \ldots$ have the range $1, 2, 3$, and subscripts $a, b, \ldots$ have the range $1, 2$.

In a deformation of the sheet, the particle initially at x_A moves to the place $\mathbf{r}^*(x_A)$. In terms of components with respect to an orthonormal system of base vectors $\mathbf{e}_i$, $\mathbf{r}^*$ is $r_i^* \mathbf{e}_i$. Components with respect to a basis $\mathbf{e}_i$ are numbered with subscripts $i, j, \ldots$ that have the range $1, 2, 3$. We usually suppress such subscripts by using vector and dyadic notation.

The strain energy W per unit initial volume is a function of the deformation gradients $\mathbf{r}^*_{,A}$. For derivatives of W with respect to the deformation gradients we use the notation

$$\begin{aligned} \mathbf{W}_A &= \mathbf{e}_i W_{iA}, \qquad W_{iA} = \partial W/\partial r^*_{i,A}, \\ \mathbf{W}_{AB} &= \mathbf{e}_i \mathbf{e}_j W_{iAjB}, \qquad W_{iAjB} = \partial^2 W/\partial r^*_{i,A}\partial \mathbf{r}^*_{j,B}, \end{aligned} \tag{2.1}$$

and for short, we may write

$$\mathbf{W}_A = \partial W/\partial \mathbf{r}^*_{,A}, \qquad \mathbf{W}_{AB} = \partial^2 W/\partial \mathbf{r}^*_{,A}\partial \mathbf{r}^*_{,B}. \tag{2.2}$$

The vector $\mathbf{W}_A$ is the force per unit initial area on an element whose normal was initially in the x_A-direction. For each AB, $\mathbf{W}_{AB}$ is a matrix of stiffness coefficients with elements W_{iAjB}. This is not the same thing as $\mathbf{W}_{A,B}$, the derivative of $\mathbf{W}_A$ with respect to x_B, which is the vector

$$\mathbf{W}_{A,B} = \mathbf{W}_{AC} \cdot \mathbf{r}^*_{,CB}. \tag{2.3}$$

Notation similar to (2.1) is also used for other functions of the deformation gradients.

3. The membrane strain energy function. The strain energy per unit initial *area* of the deformed sheet is

$$E = \int_{-h}^{h} W(\mathbf{r}^*_{,a}, \mathbf{r}^*_{,3})\, dz. \tag{3.1}$$

In *membrane theory*, E is approximated by a function $M(\mathbf{r}_{,a})$ of the gradients of the midplane $(z = 0)$ deformation,

$$\mathbf{r}(x_a) = \mathbf{r}^*(x_a, 0). \tag{3.2}$$

To find the expression for M in terms of W it is sufficient to consider homogeneous deformations, for which the second derivatives $\mathbf{r}^*_{,AB}$ are zero,

$$\mathbf{r}^*(x_A) = \mathbf{r}_0 + \mathbf{r}_{,a} x_a + \mathbf{f} z. \tag{3.3}$$

Here $\mathbf{r}_0$, $\mathbf{r}_{,a}$ and $\mathbf{f}$ are constants. For such deformations

$$E = 2hW(\mathbf{r}_{,a}, \mathbf{f}), \tag{3.4}$$

and we take M to be the minimum of E with respect to $\mathbf{f}$. At the minimizing value $\mathbf{f} = \mathbf{f}(\mathbf{r}_{,a})$, W is stationary,

$$\mathbf{W}_3[\mathbf{r}_{,a}, \mathbf{f}(\mathbf{r}_{,a})] = \mathbf{0}, \tag{3.5}$$

so the minimizing value of $\mathbf{f}$ is a value that makes the stress on surfaces $z = \text{constant}$ equal to zero. At the minimizing value, W has a local minimum in particular, so

$$\delta \mathbf{f} \cdot \mathbf{W}_{33}[\mathbf{r}_{,a}, \mathbf{f}(\mathbf{r}_{,a})] \cdot \delta \mathbf{f} \geq 0, \tag{3.6}$$

i.e., the matrix $\mathbf{W}_{33}$ is nonnegative definite. When $\mathbf{f}$ has been determined, the membrane energy function M is

$$M(\mathbf{r}_{,a}) = 2hW(\mathbf{r}_{,a}, \mathbf{f}(\mathbf{r}_{,a})). \tag{3.7}$$

In notation like (2.1), the derivative of M with respect to $\mathbf{r}_{,a}$ is

$$\mathbf{M}_a = 2h\mathbf{W}_a + 2h\mathbf{W}_3 \cdot \mathbf{F}_a = 2h\mathbf{W}_a, \tag{3.8}$$

since $\mathbf{W}_3 = \mathbf{0}$. Here we have used the notation $\mathbf{F}_a$ for the matrix

$$\mathbf{F}_a = \mathbf{e}_i \mathbf{e}_j F_{ija}, \qquad F_{ija} = \partial f_i / \partial r_{j,a}. \tag{3.9}$$

Differentiating (3.8) gives

$$\mathbf{M}_{ab} = 2h(\mathbf{W}_{ab} + \mathbf{W}_{a3} \cdot \mathbf{F}_b). \tag{3.10}$$

By differentiating (3.5) we find that

$$\mathbf{W}_{3a} + \mathbf{W}_{33} \cdot \mathbf{F}_a = \mathbf{0}, \tag{3.11}$$

and using this in (3.10) gives

$$\mathbf{M}_{ab} = 2h(\mathbf{W}_{ab} - \mathbf{F}_a^{\mathrm{t}} \cdot \mathbf{W}_{33} \cdot \mathbf{F}_b), \tag{3.12}$$

where $\mathbf{F}_a^{\mathrm{t}}$ is the transpose of $\mathbf{F}_a$.

4. Bending energy. Because the membrane energy function $M(\mathbf{r}_{,a})$ does not depend on the second derivatives $\mathbf{r}_{,ab}$, which determine the curvature of the deformed sheet, membrane theory treats the sheet as having no bending stiffness. To obtain a lowest-order estimate of the bending energy of the sheet we suppose that the deformation has the form

$$\mathbf{r}^*(x_a, z) = \mathbf{r}(x_a) + z\mathbf{f}(\mathbf{r}_{,a}), \tag{4.1}$$

where $\mathbf{f}(\mathbf{r}_{,a})$ is the value of $\mathbf{r}^*_{,3}$ used in the determination of M. Then

$$E = \int_{-h}^{h} W(\mathbf{r}_{,a} + z\mathbf{f}_{,a}, \mathbf{f})\, dz. \tag{4.2}$$

By expanding the integrand in powers of z we obtain

$$E = 2hW(\mathbf{r}_{,a}, \mathbf{f}) + (h^3/3)\mathbf{f}_{,a} \cdot \mathbf{W}_{ab} \cdot \mathbf{f}_{,b} + O(h^5). \tag{4.3}$$

We see that the leading term is $M(\mathbf{r}_{,a})$, the lowest-order approximation to the stretching energy. The next term is the lowest-order approximation to the bending energy. Since

$$\mathbf{f}_{,a} = \mathbf{F}_b \cdot \mathbf{r}_{,ba}, \tag{4.4}$$

where $\mathbf{F}_b$ is defined by (3.9), the second term in (4.3) is proportional to

$$\begin{aligned} B &= \mathbf{f}_{,a} \cdot \mathbf{W}_{ab} \cdot \mathbf{f}_{,b} \\ &= \mathbf{r}_{,ab} \cdot (\mathbf{F}_a^{\mathrm{t}} \cdot \mathbf{W}_{bc} \cdot \mathbf{F}_d) \cdot \mathbf{r}_{,cd}, \end{aligned} \tag{4.5}$$

a quadratic function of the second derivatives $\mathbf{r}_{,ab}$.

It must be recognized that (4.3) does *not* give the value of E in equilibrium, accurate to order h^3. The stretching energy of the sheet is not M exactly, but rather $M + O(h^3)$, and the $O(h^3)$ perturbation of the stretching energy is not included in (4.3). Nevertheless, in the remainder of this paper we take E to have the form

$$E = M(\mathbf{r}_{,a}) + \alpha B(\mathbf{r}_{,a}, \mathbf{r}_{,ab}), \tag{4.6}$$

where α is the small parameter $h^3/3$ and B is defined in (4.5). This is the sum of the leading term of the stretching energy and the leading term of the bending energy. For the simpler and more explicit case of infinitesimal deformations, the theory based on an energy of the form (4.6) returns the usual equations of classical plate theory even though the exact energy per unit area, which is known, is not exactly of the form (4.6) (Love [8, Articles 301 and 304]). In the classical theory, third-derivative terms that are $O(h^3)$ are simply omitted [3].

5. Geometric formulas. We now treat the sheet as a surface $\mathbf{r}(x_a)$. The vectors $\mathbf{r}_{,a}$ are tangential to the deformed sheet. The normal vector $\mathbf{N}$, the area per unit initial area A, and the unit normal $\mathbf{n}$ are defined by

$$\mathbf{N} = \mathbf{r}_{,1} \times \mathbf{r}_{,2}, \qquad A = |\mathbf{N}|, \qquad n = \mathbf{N}/A. \tag{5.1}$$

Let G_{ab} and G^{ab} be defined by

$$G_{ab} = \mathbf{r}_{,a} \cdot \mathbf{r}_{,b}, \qquad G^{ab}G_{bc} = \delta_{ac}, \tag{5.2}$$

and let

$$\mathbf{r}^a = G^{ab}\mathbf{r}_{,b}. \tag{5.3}$$

Then

$$\mathbf{r}^a \cdot \mathbf{r}_{,b} = \delta_{ab} \quad \text{and} \quad \mathbf{r}^a \cdot \mathbf{n} = 0. \tag{5.4}$$

Any vector $\mathbf{v}$ can be expressed as a linear combination of the base vectors $(\mathbf{r}_{,a}, \mathbf{n})$ and equally well as a linear combination of the reciprocal base vectors $(\mathbf{r}^a, \mathbf{n})$,

$$\mathbf{v} = \mathbf{r}_{,a}(\mathbf{r}^a \cdot \mathbf{v}) + \mathbf{n}(\mathbf{n} \cdot \mathbf{v}) = \mathbf{r}^a(\mathbf{r}_{,a} \cdot \mathbf{v}) + \mathbf{n}(\mathbf{n} \cdot \mathbf{v}). \tag{5.5}$$

This implies that the identity operator $\mathbf{I}$ can be expressed in dyadic form as

$$\mathbf{I} = \mathbf{r}_{,a}\mathbf{r}^a + \mathbf{nn} = \mathbf{r}^a\mathbf{r}_{,a} + \mathbf{nn}, \tag{5.6}$$

and also as

$$\mathbf{I} = G^{ab}\mathbf{r}_{,a}\mathbf{r}_{,b} + \mathbf{nn}, \tag{5.7}$$

where we have used (5.3).

By differentiating $\mathbf{n} \cdot \mathbf{n} = 1$ and $\mathbf{n} \cdot \mathbf{r}_{,b} = 0$ we find that

$$\mathbf{n} \cdot \mathbf{n}_{,a} = 0 \quad \text{and} \quad \mathbf{r}_{,b} \cdot \mathbf{n}_{,a} = -\mathbf{n} \cdot \mathbf{r}_{,ab}. \tag{5.8}$$

Then setting $\mathbf{v} = \mathbf{n}_{,a}$ in (5.5) gives

$$\mathbf{n}_{,a} = -\mathbf{r}^b(\mathbf{n} \cdot \mathbf{r}_{,ba}). \tag{5.9}$$

For materials with reflectional symmetry, the function $\mathbf{f}(\mathbf{r}_{,a})$ has the form $\mathbf{f} = \phi\mathbf{n}$ (see Sec. 8). Then with (5.9),

$$\mathbf{f}_{,a} = (\mathbf{n}\phi_b - \phi\mathbf{r}^b\mathbf{n}) \cdot \mathbf{r}_{,ba}, \tag{5.10}$$

where

$$\phi_a = \partial\phi/\partial\mathbf{r}_{,a} \qquad (\phi_{ia} = \partial\phi/\partial r_{i,a}). \tag{5.11}$$

Comparison of (5.10) with (4.4) shows that

$$\mathbf{F}_b = \mathbf{n}\phi_b - \phi \mathbf{r}^b \mathbf{n}. \tag{5.12}$$

6. Material stability conditions. Problems in finite elasticity theory can be formulated as energy-minimization problems, with a total energy that includes the integral of $W(\mathbf{r}^*_{,A})$ over the body. An energy-minimizing deformation satisfies various necessary conditions, known from the calculus of variations. First, there are equilibrium conditions: the Euler-Lagrange equations, natural boundary conditions, and Erdmann corner conditions (the equilibrium conditions at discontinuities). These are the conditions that are used to determine equilibrium states, but, in fact, they are all consequences of *stationary* energy rather than minimum energy. For a deformation to furnish even a weak local minimum of the energy, it is necessary (but not sufficient) that at every point the matrix of coefficients $\mathbf{W}_{AB}$ satisfies the Legendre-Hadamard condition

$$\mathbf{u} \cdot (\mathbf{W}_{AB} v_A v_B) \cdot \mathbf{u} \geq 0 \qquad (W_{iAjB} u_i v_A u_j v_B \geq 0) \tag{6.1}$$

for all u_i and v_A. Although this is a stability condition, it does not make reference to the shape of the structure or its loading in any particular problem. If it fails to be satisfied for the value that $\mathbf{W}_{AB}$ takes at some particular deformation gradient $\mathbf{r}^*_{,A}$, then that value of the deformation gradient can never appear in any stable equilibrium state, and we say that the *material* is unstable at that value of $\mathbf{r}^*_{,A}$. Similarly, we say that the material is unstable at a particular value of the deformation gradient if it fails to satisfy the Weierstrass condition (which is actually due to Graves [6])

$$W(\mathbf{r}^*_{,A} + \mathbf{u}v_A) \geq W(\mathbf{r}^*_{,A}) + \mathbf{W}_A \cdot \mathbf{u}v_A. \tag{6.2}$$

This is a necessary (but not sufficient) condition for stability under large variations of $\mathbf{r}^*_{,A}$.

As part of the definition of a *stable material*, we require W to have such a form that (6.1) and (6.2) are satisfied for all admissible values of $\mathbf{r}^*_{,A}$. Then by making particular choices of v_A in (6.1) we find that

$$\begin{aligned} &\text{(a) } \mathbf{u} \cdot (\mathbf{W}_{ab} v_a v_b) \cdot \mathbf{u} \geq 0 \text{ and} \\ &\text{(b) } \mathbf{u} \cdot \mathbf{W}_{33} \cdot \mathbf{u} \geq 0. \end{aligned} \tag{6.3}$$

We shall in fact need these relations only in the special cases in which $\mathbf{r}^*_{,3} = \mathbf{f}(\mathbf{r}^*_{,a})$, and in that case (6.3)(b) is satisfied by definition of $\mathbf{f}$ (see (3.6)).

7. Material stability for membranes. In membrane theory, the Legendre-Hadamard condition is

$$\mathbf{u} \cdot (\mathbf{M}_{ab} v_a v_b) \cdot \mathbf{u} \geq 0. \tag{7.1}$$

By using (3.12) we can write this in terms of W as

$$\mathbf{u} \cdot (\mathbf{W}_{ab} v_a v_b) \cdot \mathbf{u} \geq \mathbf{U} \cdot \mathbf{W}_{33} \cdot \mathbf{U}, \tag{7.2}$$

where

$$\mathbf{U} = v_a \mathbf{F}_a \cdot \mathbf{u}. \tag{7.3}$$

For a stable material, the right-hand member of (7.2) is nonnegative by (6.3)(b), so (7.2) is more restrictive than the material stability condition (6.3)(a), and we cannot

expect that it will be satisfied at all values of $\mathbf{r}_{,a}$. Indeed, any value of $\mathbf{r}_{,a}$ that yields a compressive normal stress in some direction violates (7.1) [2]. This has the unwanted consequence that there are apparently well-set membrane problems in which the energy has no minimizer. That is, there is no equilibrium state that is even weakly stable.

For an energy density of the form $E(\mathbf{r}_{,a}, \mathbf{r}_{,ab})$, the Legendre-Hadamard and Weierstrass material stability conditions refer to the dependence of E on the second derivatives $\mathbf{r}_{,ab}$ [9]. With E of the form $M + \alpha B$, where B is quadratic in $\mathbf{r}_{,ab}$, the Legendre-Hadamard condition can be written as

$$B(\mathbf{r}_{,a}, \mathbf{u}v_a v_b) \geq 0. \tag{7.4}$$

That is, B is nonnegative whenever the second derivative has the special form $\mathbf{r}_{,ab} = \mathbf{u}v_a v_b$. The Weierstrass condition [9] is

$$B(\mathbf{r}_{,a}, \mathbf{r}_{,ab} + \mathbf{u}v_a v_b) \geq B(\mathbf{r}_{,a}, \mathbf{r}_{,ab}) + (\partial B/\partial \mathbf{r}_{,ab}) \cdot \mathbf{u}v_a v_b. \tag{7.5}$$

The two conditions are in fact equivalent because B is quadratic in $\mathbf{r}_{,ab}$. Now, with B given in terms of W by (4.5), condition (7.4) is

$$\mathbf{U} \cdot (\mathbf{W}_{ab} v_a v_b) \cdot \mathbf{U} \geq 0, \tag{7.6}$$

where $\mathbf{U}$ is defined in (7.3). Then if W is the energy density for a stable material, so that (6.3)(a) is satisfied, (7.6) is satisfied for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{r}_{,a}$, including values of $\mathbf{r}_{,a}$ for which the membrane stress is compressive. Thus, adding the bending energy αB to M does restore material stability, even if α is very small.

Quasiconvexity is an even stronger material stability condition, which implies the Legendre-Hadamard and Weierstrass conditions [4, 9]. The converse is not known to be true in general, but it *is* true for quadratic functions of $\mathbf{r}_{,ab}$, according to van Hove's theorem [10]. Thus αB is quasi-convex, and we expect the theory based on the energy $M + \alpha B$ to be free from the nonexistence troubles that are met in pure membrane theory [4].

The stability condition (7.4) is satisfied if B is nonnegative for all values of $\mathbf{r}_{,ab}$, but this is not necessary for stability. In this connection, we note that it is not possible for B to be positive definite as a function of the derivatives $\mathbf{r}_{,ab}$. For, the relation (4.4) expresses the six quantities $f_{i,a}$ as linear combinations of the nine independent derivatives $r_{i,ab}$, so it is always possible to choose values $\mathbf{r}_{,ab}$, not all zero, such that $\mathbf{f}_{,a} = \mathbf{0}$. Then from (4.5), $B = 0$ even though the derivatives $\mathbf{r}_{,ab}$ are not all zero.

The partial differential equations of equilibrium would be called *strongly elliptic* if (7.4) were satisfied as a strict inequality whenever $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. This is not the case, even if the equivalent inequality (7.6) is strict whenever $\mathbf{U} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. For, $\mathbf{U} = \mathbf{0}$ does not necessarily imply that $\mathbf{u} = \mathbf{0}$, as we see from (7.3). The matrix $v_a \mathbf{F}_a$ may be singular. In fact, it is always singular if $\mathbf{F}_a$ has the form (5.12) appropriate for materials with reflectional symmetry. In such cases (7.3) is

$$\mathbf{U} = \mathbf{n}(v_a \boldsymbol{\phi}_a \cdot \mathbf{u}) - \phi v_a \mathbf{r}^a (\mathbf{n} \cdot \mathbf{u}), \tag{7.7}$$

so $\mathbf{U} = \mathbf{0}$ whenever $\mathbf{u}$ is orthogonal to both $\mathbf{n}$ and $v_a \boldsymbol{\phi}_a$.

8. Rotational invariance and reflectional symmetry. We assume that superposing a rigid rotation on a given deformation does not alter W. This implies that W can be expressed as a function of the inner products $G_{AB} = \mathbf{r}^*_{,A} \cdot \mathbf{r}^*_{,B}$. It is more convenient here to regard W as a function of the strain components E_{AB} defined by

$$2E_{AB} = G_{AB} - \delta_{AB}. \tag{8.1}$$

We use the notation

$$W_{AB} = \partial W/\partial E_{AB}, \qquad W_{ABCD} = \partial^2 W/\partial E_{AB}\partial E_{CD}, \tag{8.2}$$

and we stipulate that W is expressed symmetrically in terms of E_{AB} and E_{BA}, so that

$$W_{AB} = W_{BA} \quad \text{and} \quad W_{ABCD} = W_{CDAB} = W_{BACD}. \tag{8.3}$$

Then

$$\begin{aligned} &\text{(a) } \mathbf{W}_A = W_{AB}\mathbf{r}^*_{,B} \quad \text{and} \\ &\text{(b) } \mathbf{W}_{AC} = W_{ABCD}\mathbf{r}^*_{,B}\mathbf{r}^*_{,D} + W_{AC}\mathbf{I}. \end{aligned} \tag{8.4}$$

We now restrict attention to materials that are symmetric under reflection in the plane $x_3 = 0$. For such materials, W is an even function of the shearing strain components E_{3a}, so its derivatives W_{3a} are odd functions, and they vanish when $E_{3a} = 0$. In terms of $W(E_{ab}, E_{3a}, E_{33})$,

$$W_{3a}(E_{ab}, 0, E_{33}) = 0. \tag{8.5}$$

It follows that any second derivative W_{ABCD} with an odd number of subscripts equal to 3 is also zero when $E_{3a} = 0$.

For materials with this reflectional symmetry, the vector $\mathbf{f}(\mathbf{r}_{,a})$ is perpendicular to the deformed membrane at each point; $\mathbf{f} = \phi\mathbf{n}$, say. For, (8.5) and the material stability condition $W_{3a3b}v_a v_b \geq 0$ imply that $W(E_{ab}, E_{3a}, E_{33})$ is minimized with respect to E_{3a} at $E_{3a} = 0$, or thus $\mathbf{f}\cdot\mathbf{r}_{,a} = 0$. The factor ϕ is the deformed thickness per unit initial thickness. This is determined by minimizing $W(E_{ab}, 0, E_{33})$ with respect to E_{33},

$$W_{33}(E_{ab}, 0, E_{33}) = 0, \qquad 2E_{33} = \phi^2 - 1. \tag{8.6}$$

The material stability condition $W_{3333} \geq 0$ implies that a solution E_{33} of (8.6) is a minimizer, and the stronger condition $W_{3333} > 0$ guarantees that the minimizer is unique; we assume that this is the case. Thus ϕ is some function of E_{ab}, with derivatives ϕ_{ab}, say.

Hereafter we assume that $\mathbf{f} = \phi\mathbf{n}$. Then in the expressions for M and B, E_{3a} is zero. All derivatives W_{AB} and W_{ABCD} with an odd number of subscripts equal to 3 are zero, and in addition $W_{33} = 0$. It follows that

$$\mathbf{W}_a = W_{ab}\mathbf{r}_{,b} \quad \text{and} \quad \boldsymbol{\phi}_a = \phi_{ab}\mathbf{r}_{,b}, \tag{8.7}$$

and that

$$\mathbf{W}_{ac} = (W_{abcd} + W_{ac}G^{bd})\mathbf{r}_{,b}\mathbf{r}_{,d} + (\phi^2 W_{3a3c} + W_{ac})\mathbf{nn}. \tag{8.8}$$

To obtain the latter expression we have used the form of $\mathbf{I}$ given by (5.7) in (8.4)(b), and $\mathbf{r}^*_{,3} = \phi\mathbf{n}$. We also find that

$$\mathbf{W}_{33} = W_{3a3b}\mathbf{r}_{,a}\mathbf{r}_{,b} + \phi^2 W_{3333}\mathbf{nn}, \tag{8.9}$$

since $W_{33} = 0$.

By using (5.12), (8.8), and (5.4), we find that

$$\mathbf{F}_a^t \cdot \mathbf{W}_{bc} \cdot \mathbf{F}_d = \phi_a(\phi^2 W_{3b3c} + W_{bc})\phi_d + \phi^2(W_{abcd} + W_{bc}G^{ad})\mathbf{nn}. \tag{8.10}$$

Then from (4.5),

$$B = (\phi^2 W_{3b3c} + W_{bc})\phi_{,b}\phi_{,c} + \phi^2(W_{abcd} + W_{bc}G^{ad})(\mathbf{n}\cdot\mathbf{r}_{,ab})(\mathbf{n}\cdot\mathbf{r}_{,cd}), \tag{8.11}$$

since

$$\phi_{,b} = \phi_a \cdot \mathbf{r}_{,ab}. \tag{8.12}$$

With (8.9), the condition that $\mathbf{W}_{33}$ is non-negative definite implies that

$$\begin{aligned} &\text{(a) } W_{3a3b}v_a v_b \geq 0 \text{ and} \\ &\text{(b) } W_{3333} \geq 0, \end{aligned} \tag{8.13}$$

the former being satisfied for all v_a. With (8.8), the material stability condition (6.3)(a) implies that

$$(\phi^2 W_{3a3b} + W_{ab})v_a v_b \geq 0 \tag{8.14}$$

and

$$(W_{abcd} + W_{ac}G^{bd})v_a u_b v_c u_d \geq 0, \tag{8.15}$$

for arbitrary v_a and u_a. From (8.14) we observe that the first term of B in (8.11) is nonnegative, but the second term may be negative even though (8.15) is satisfied.

9. Isotropic materials. If the material of the membrane is isotropic, W can be expressed as a function of the isotropic invariants

$$I_1 = G_{AA}, \qquad I_2 = G_{AB}G_{BA}, \qquad I_3 = G_{AB}G_{BC}G_{CA}. \tag{9.1}$$

The principal stretches λ_α $(\alpha = 1, 2, 3)$ are the positive square roots of the principal values of the matrix G_{AB}. In terms of the stretches, the invariants (9.1) are

$$G_{AA} = \sum \lambda_\alpha^2, \qquad G_{AB}G_{BA} = \sum \lambda_\alpha^4, \qquad G_{AB}G_{BC}B_{CA} = \sum \lambda_\alpha^6. \tag{9.2}$$

The formula for B in terms of derivatives of W with respect to the invariants is of no immediate use in the present paper. However, some features of the bending energy can be seen a little more clearly by expressing B in terms of $\overline{W}(\lambda_1, \lambda_2, \lambda_3)$, the expression for the strain energy in terms of the stretches.

We shall obtain the expression for B when the coordinate axes are oriented along the principal directions at the point considered, so that

$$\mathbf{r}^*_{,\alpha} = \lambda_\alpha \mathbf{u}_\alpha, \qquad \mathbf{u}_\alpha \cdot \mathbf{u}_\beta = \delta_{\alpha\beta}. \tag{9.3}$$

(The summation convention is suspended for Greek subscripts.) Then

$$G_{\alpha\alpha} = 1/G^{\alpha\alpha} = \lambda_\alpha^2, \qquad G_{\alpha\beta} = G^{\alpha\beta} = 0 \qquad (\alpha \neq \beta). \tag{9.4}$$

We shall compute the derivatives

$$\lambda_{\alpha,AB} = \partial\lambda_\alpha/\partial E_{AB} \quad \text{and} \quad \lambda_{\alpha,ABCD} = \partial^2\lambda_\alpha/\partial E_{AB}\partial E_{CD}, \tag{9.5}$$

evaluated at the values of E_{AB} implied by (9.4) and (8.1). The derivatives $\lambda_{\alpha,ABCD}$ are taken to have index symmetries like (8.3). The derivatives are evaluated by differentiating (9.2), substituting (9.4) in the resulting expressions, and then solving for the derivatives (9.5). The nonzero derivatives of λ_1 are found to be

$$\lambda_{1,11} = 1/\lambda_1, \qquad \lambda_{1,1111} = -1/\lambda_1^3, \tag{9.6}$$

$$\lambda_{1,1212} = 1/\lambda_1(\lambda_1^2 - \lambda_2^2), \qquad \lambda_{1,1313} = 1/\lambda_1(\lambda_1^2 - \lambda_3^2). \tag{9.7}$$

Each of the derivatives (9.7) actually represents four of the quantities $\lambda_{1,ABCD}$. Similar formulas for the nonzero derivatives of λ_2 and λ_3 are obtained.

The derivatives of W with respect to E_{AB} can now be expressed in terms of derivatives with respect to the stretches,

$$\overline{W}_\alpha = \partial \overline{W}/\partial \lambda_\alpha, \qquad \overline{W}_{\alpha\beta} = \partial^2 \overline{W}/\partial \lambda_\alpha \partial \lambda_\beta, \tag{9.8}$$

by using the chain rule and formulas like (9.6) and (9.7). Typical results are

$$W_{11} = \overline{W}_1/\lambda_1, \quad W_{1122} = \overline{W}_{12}/\lambda_1\lambda_2, \tag{9.9}$$

$$W_{1111} = \overline{W}_{11}/\lambda_1^2 - \overline{W}_1/\lambda_1^3, \tag{9.10}$$

and

$$W_{1212} = (\overline{W}_1/\lambda_1 - \overline{W}_2/\lambda_2)/(\lambda_1^2 - \lambda_2^2). \tag{9.11}$$

Similar formulas with other choices of the subscripts are also valid. Derivatives that are not of these types are zero. That is, all of the derivatives W_{AB} and W_{ABCD} are zero except those that have subscripts equal in pairs. These results can also be obtained by an argument based on formulas provided by Ogden [11, Sec. 6.1.4].

For abbreviation we write

$$\mu_{\alpha\beta} = (\lambda_\alpha \overline{W}_\alpha - \lambda_\beta \overline{W}_\beta)/(\lambda_\alpha^2 - \lambda_\beta^2) \qquad (\alpha \neq \beta) \tag{9.12}$$

and

$$\nu_{\alpha\beta} = (\lambda_\beta \overline{W}_\alpha - \lambda_\alpha \overline{W}_\beta)/(\lambda_\alpha^2 - \lambda_\beta^2) \qquad (\alpha \neq \beta). \tag{9.13}$$

Then

$$W_{1212} = \nu_{12}/\lambda_1\lambda_2, \tag{9.14}$$

and since $\overline{W}_3 = 0$ $(W_{33} = 0)$,

$$W_{3131} = \mu_{31}/\lambda_1^2 \quad \text{and} \quad W_{3232} = \mu_{32}/\lambda_2^2. \tag{9.15}$$

Various combinations of derivatives arise in the expression (8.11) for B:

$$\phi^2 W_{3131} + W_{11} = \mu_{31}, \qquad \phi^2 W_{3232} + W_{22} = \mu_{32}, \tag{9.16}$$

$$W_{1212} + W_{11}G^{22} = \mu_{12}/\lambda_2^2, \quad W_{1212} + W_{22}G^{11} = \mu_{12}/\lambda_1^2, \tag{9.17}$$

$$W_{1111} + W_{11}G^{11} = \overline{W}_{11}/\lambda_1^2, \qquad W_{2222} + W_{22}G^{22} = \overline{W}_{22}/\lambda_2^2. \tag{9.18}$$

We also use the abbreviations

$$D = \mu_{12}(\lambda_1^{-2} + \lambda_2^{-2}) + 2\nu_{12}(\lambda_1\lambda_2)^{-1} \tag{9.19}$$

and

$$R_1 = \mathbf{n} \cdot \mathbf{r}_{,11}/\lambda_1, \qquad R_2 = \mathbf{n} \cdot \mathbf{r}_{,22}/\lambda_2. \tag{9.20}$$

In (8.11) we first simplify by omitting those coefficients whose subscripts are not equal in pairs, and then use the preceding notation for the remaining coefficients. Then B has the form

$$B = \mu_{31}\phi_{,1}^2 + \mu_{32}\phi_{,2}^2 + D(\phi \mathbf{n} \cdot \mathbf{r}_{,12})^2 + \phi^2(\overline{W}_{11}R_1^2 + 2\overline{W}_{12}R_1R_2 + \overline{W}_{22}R_2^2), \tag{9.21}$$

when the principal directions of strain are taken as the 1- and 2-directions.

10. Stability conditions for isotropic materials. The condition that $\mathbf{W}_{33}$ is nonnegative definite is equivalent to conditions (8.13). For isotropic materials, with the principal directions as the 1- and 2-directions, (8.13)(a) means that W_{3131} and W_{3232} are nonnegative; with (9.15),

$$\mu_{31} \geq 0 \quad \text{and} \quad \mu_{32} \geq 0. \tag{10.1}$$

These are two of the Baker-Ericksen inequalities. Since $\overline{W}_3 = 0$, they require $\overline{W}_a$ to have the same sign as $\lambda_a - \phi$, as we see from (9.12).

With (9.16), we see that for isotropic materials the stability condition (8.14) merely repeats conditions (10.1). With (9.17) and (9.18), condition (8.15) takes the form

$$\begin{aligned}(u_1/\lambda_1)^2(\overline{W}_{11}v_1^2 + \mu_{12}v_2^2) + (u_2/\lambda_2)^2(\mu_{12}v_1^2 + \overline{W}_{22}v_2^2) \\ + 2(u_1/\lambda_1)(u_2/\lambda_2)(\overline{W}_{12} + \nu_{12})v_1v_2 \geq 0,\end{aligned} \tag{10.2}$$

which is to be satisfied for all u_a and v_a. By various special choices of u_a and v_a, we see that

$$\overline{W}_{11} \geq 0, \qquad \overline{W}_{22} \geq 0, \quad \text{and} \quad \mu_{12} \geq 0. \tag{10.3}$$

It can also be shown [2, 12] that (10.2) implies that

$$[(\overline{W}_{11}\overline{W}_{22})^{1/2} + \mu_{12}]^2 \geq (\overline{W}_{12} + \nu_{12})^2, \tag{10.4}$$

and that if (10.3) and (10.4) are satisfied then (10.2) holds for all u_a and v_a.

Material stability does not require B to be nonnegative, but of course the condition $B \geq 0$ does imply the stability conditions in particular. In the expression (9.21) for B, the quantities $\phi_{,a}$ and $\mathbf{n} \cdot \mathbf{r}_{,ab}$ can be specified arbitrarily and independently by appropriate choices of $\mathbf{r}_{,ab}$. The terms that involve $\phi_{,a}$ are nonnegative by (10.1), but for the terms that involve the normal curvatures $\mathbf{n} \cdot \mathbf{r}_{,ab}$ to be nonnegative, it is necessary that

$$\overline{W}_{11} \geq 0, \qquad \overline{W}_{22} \geq 0, \qquad \overline{W}_{11}\overline{W}_{22} \geq \overline{W}_{12}^2, \quad \text{and} \quad D \geq 0. \tag{10.5}$$

The first three conditions require $\overline{W}$ to be locally convex as a function of λ_1 and λ_2 wherever $\lambda_3 = \phi$. As for the condition $D \geq 0$, by using (9.12) and (9.13) in (9.19) we get the expression

$$D\lambda_1^2\lambda_2^2(\lambda_1^2 - \lambda_2^2) = \lambda_1\overline{W}_1(\lambda_1^2 + 3\lambda_2^2) - \lambda_2\overline{W}_2(3\lambda_1^2 + \lambda_2^2). \tag{10.6}$$

Thus, the right-hand member must be nonnegative when $\lambda_1 \geq \lambda_2$ and $\lambda_3 = \phi$.

Although conditions (10.5) are not necessary for material stability, they do not appear to be impossibly restrictive. In experimenting with various simple stable forms of W, we found that conditions (10.5) were satisfied in every case.

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REFERENCES

[1] A. E. Green and W. Zerna, *Theoretical Elasticity*, Oxford Univ. Press, London and New York, 1954

[2] A. C. Pipkin, *The relaxed energy density for isotropic elastic membranes*, IMA J. Appl. Math. **36**, 85–99 (1986)

[3] M. G. Hilgers and A. C. Pipkin, *Elastic sheets with bending stiffness*, QJMAM, forthcoming.

[4] J. M. Ball, J. C. Currie, and P. J. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal. **41**, 135–174 (1981)

[5] G. Kirchhoff, *Uber das Gleichgewicht und die Bewegung einer elastischen Scheibe*, J. Reine Angew. Math. **40**, 51–88 (1850)

[6] L. M. Graves, *The Weierstrass condition for multiple integral variation problems*, Duke Math. J. **5**, 656–660 (1939)

[7] W.-B. Wang and A. C. Pipkin, *Inextensible networks with bending stiffness*, Quart. J. Mech. Appl. Math. **39**, 343–359 (1986)

[8] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Dover, New York, 1944

[9] M. G. Hilgers and A. C. Pipkin, *Energy-minimizing deformations of elastic sheets with bending stiffness*, J. Elast., forthcoming.

[10] L. van Hove, *Sur l'extension de la condition de Legendre du calcul des variations aux integrales multiples à plusieurs fonctions inconnues*, Nederl. Akad. Wetensch. Indag. Math. **50**, 18–23 (1947)

[11] R. W. Odgen, *Non-linear Elastic Deformations*, Ellis Horwood, Chichester, 1984

[12] J. K. Knowles and Eli Sternberg, *On the failure of ellipticity of the equations for finite elastostatic plane strain*, Arch. Rational Mech. Anal. **63**, 321–336 (1977)