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THE STEADY STATES OF ONE-DIMENSIONAL SIVASHINSKY EQUATIONS

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Abstract. Phase plane analysis is used to calculate the number of steady states for two equations which arise in the context of directional solidification: the Sivashinsky equation and the modified Sivashinsky equation.

1. Introduction. In a number of physical contexts, in particular in the context of first-order phase transitions, equations of the form

$$u_t = \Delta(f(u) - \Delta u) - \alpha u \tag{1.1}$$

arise. Notable among these are the *Sivashinsky equation* [S] in which

$$f(u) = -2u + u^2 \quad \text{and} \quad \alpha > 0 \tag{1.2}$$

and the *Cahn-Hilliard equation* [CH] with

$$f(u) = a_1 u + a_2 u^2 + a_3 u^3, \qquad a_3 > 0 \quad \text{and} \quad \alpha = 0. \tag{1.3}$$

It has been pointed out [HNCR] that it might be equally valid to consider the Sivashinsky equation in which the Laplacian of u was replaced by the mean curvature of u so that (1.1) becomes

$$u_t = \Delta(f(u) + \kappa[u]) - \alpha u. \tag{1.4}$$

Since the biharmonic term in the Cahn-Hilliard equation arose as the first term in a higher-order correction, it might be conceivable to consider the Cahn-Hilliard variant of (1.4).

In this paper we study the one-dimensional steady states of equations (1.1) and (1.4) on a finite interval $[-L, L]$ in the limit of vanishing α. In the context of the Sivashinsky equation, α is the segregation coefficient. Thus we consider the problems

$$(f(u) - u'')'' = 0, \qquad -L < x < L,$$

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$$\text{(I)} \qquad \begin{gathered} u'(-L) = u'(L) = 0 \quad \text{and} \quad u'''(-L) = u'''(L) = 0, \\ \frac{1}{2L}\int_{-L}^{L} u(x)\,dx = m, \end{gathered} \tag{1.5}$$

where the primes denote differentiation with respect to x and m is a prescribed number, and

$$\text{(II)} \qquad \begin{gathered} \left(f(u) - \frac{u''}{(1+u'^2)^{3/2}}\right)'' = 0, \qquad -L < x < L, \\ u'(-L) = u'(L) = 0 \quad \text{and} \quad u'''(-L) = u'''(L) = 0, \\ \frac{1}{2L}\int_{-L}^{L} u(x)\,dx = m. \end{gathered} \tag{1.6}$$

For the function f one can choose either (1.2) or (1.3). The question we wish to address here is for which values of the parameters L and m do problems (I) and (II) have *simple* solutions, i.e., solutions which are bounded and strictly monotone. Likewise we address the more general question as to for which values of these parameters do there exist solutions that are merely bounded. Since some work has been done on problem (I) with f a cubic function [CGS, Z], and since justification for problem (II) with a cubic f is a little dubious, we shall focus here on the study of problems (I) and (II) with f of the form given in (1.2).

As noted, problems (I) and (II) with $f(u) = -2u + u^2$ correspond to the Sivashinsky equation and to the modified Sivashinsky equation in which $u(x, t)$ denotes the solid-liquid interface in the solidification of a dilute binary alloy. These equations are in fact formally derived asymptotically in [S] and [HNCR] from the system

$$\begin{gathered} \Delta c + c_{zz} + c_z = c_t, \qquad z > u, \\ [1 + (k-1)c][1 + u_t] = c_z - \nabla u \cdot \nabla c, \qquad z = u, \\ c - M^{-1}u + \Gamma\nabla \cdot [\nabla u/(1 + |\nabla u|^2)^{1/2}] = 0, \qquad z = u, \\ c \to 1 \quad \text{as } z \to \infty. \end{gathered} \tag{1.7}$$

This system of equations reflects an infinite one-sided model which neglects diffusion of solute in the solid and assumes that the thermal conductivities, densities and specific heats of the solid and the liquid are equal. Here $c(x, y, z, t)$ is the impurity concentration, and k, M, and Γ are three dimensionless parameters, respectively the segregation coefficient, a morphological parameter, and a scaled surface energy parameter. For derivations of the Sivashinsky equation in other physical contexts, see for example [K] and [KT].

Numerical evidence, e.g., [UB, MBC], shows that, for certain parameter values, (1.7) is capable of evolving toward a periodic steady state. Apparently this behavior is not carried over into the asymptotically derived Sivashinsky and modified Sivashinsky equation. In particular, it has been shown that if there are periodic steady states for the Sivashinsky equation then they are unstable [NC1]. Similar results have been found recently for the modified Sivashinsky equation [NC2]. For both equations, for certain values of the dimensionless parameters it is possible to prescribe initial conditions so that blow up will occur in finite time [NC2, EZ]. Furthermore, if $\alpha = 0$ it has been proven that solutions with sufficiently large mean and sufficiently small

variation about the mean decay to their mean [NC3]. Numerically, only blow up or decay to the mean has been found for the Sivashinsky and the modified Sivashinsky equations [HNR, KT], although for the modified Sivashinsky equation the blow up occurs in the gradient of the solution while the amplitude of the solution remains finite. One of our goals is to reconcile the behavior of these evolution equations with the set of possible equilibrium solutions. In particular, it is important to study the simple solutions since these may be stable. In passing we note that recently, using alternative distinguished limits, new asymptotic equations have been derived from (1.7) [BD, NC4, RD] which more readily support periodic structures.

Let us recall some of the work that has been done on the problem of determining the steady state solutions of the Cahn-Hilliard equation, i.e., problem (I) with $f(u)$ taken as in (1.3) and $a_3 > 0$. Notably, in [CGS] (where in fact $f(u)$ is taken to be any C^5 function with "cubic" structure) it is proved that if L is sufficiently large then there exist simple solutions for all $\alpha + \delta(L) < m < \beta - \delta(L)$ for some $\delta(L)$ where $\delta(L) \to 0$ as $L \to \infty$ and where α and β are the Maxwell construction concentrations; i.e.,

$$f(\alpha) = f(\beta) = (F(\beta) - F(\alpha)) \cdot (\beta - \alpha)^{-1},$$

where $F(x) = \int^x f(u)\,du$. In [Z], it is proved that for any $L > 0$ and for $m = 0$, there are a finite number of steady state solutions. Thus, another of our goals is to see which of the above results are preserved under change of nonlinearity and when the curvature replaces the second derivative.

In Sec. 2 we study problem (I). Our method of approach is similar to that of Zheng [Z] but relies less on symmetry considerations. We prove

THEOREM A. For any $L > 0$, there exists a simple solution for every $m \geq -\pi^2/8L^2$; if $m \geq 0$ this simple solution is unique. For any $L > 0$ and for any $m \in (-\infty, \infty)$, there exists a countable infinity of (not necessarily simple) steady state solutions.

In Sec. 3 we discuss problem (II). We demonstrate

THEOREM B. For any $L > 0$, there exist at most a countable number of intervals $(m^i_-(L), m^i_+(L))$ such that for $m \in \bigcup(m^i_-(L), m^i_+(L))$ there exist simple solutions to problem (II). For any $L > 0$ there exists an $\hat{m}$ such that if $m < \hat{m}$, then there exists a solution to problem (II).

In conclusion we note that our results parallel the results of Carr, Gurtin and Slemrod [CGS]. There it is shown that for any $L > 0$, there exist nontrivial steady state solutions for $m \in [m_-(L), m_+(L)]$. Here we find that for the Sivashinsky equation for any $L > 0$ there exist nontrivial steady state solutions if m is sufficiently large and for the modified Sivashinsky equation there exist nontrivial steady state solutions only if m is sufficiently small.

2. Problem I. Let L and m be given numbers. In this section we shall establish conditions on L and m that guarantee the existence of nonconstant solutions of problem (I) when f is given by (1.2).

Integrating Eq. (1.5) twice we obtain the equivalent problem

$$\text{(Ia)} \qquad \begin{cases} u'' = f(u) - \sigma, & -L < x < L, \\ u'(-L) = u'(L) = 0, \\ \int_{-L}^{L} u(x)\,dx = 2mL, \end{cases}$$

where σ is a constant of integration, which we are free to choose appropriately.

For convenience we shift the origin and write

$$\tilde{u} = u - 1, \quad \tilde{\sigma} = \sigma + 1, \quad \tilde{m} = m - 1.$$

This yields upon substitution the problem

$$\text{(Ib)} \qquad \begin{cases} u'' = u^2 - \sigma, & -L < x < L, \quad \sigma \in \mathbf{R}, & (2.1) \\ u'(-L) = u'(L) = 0, & & (2.2) \\ \displaystyle\int_{-L}^{L} u(x)\,dx = 2mL, & & (2.3) \end{cases}$$

where we have omitted the tilde again.

Suppose $u(x)$ is a solution of (Ib). Then, if we multiply (2.2) by u' and integrate we find that $u(x)$ must satisfy the equation

$$\frac{1}{2}u'^2(x) = \phi(u(x), \sigma) - b, \tag{2.4}$$

where

$$\phi(u, \sigma) \stackrel{\text{def}}{=} \frac{1}{3}u^3 - \sigma u, \tag{2.5}$$

for some constant of integration b.

In view of the boundary condition (2.2), the equation

$$\phi(u(x), \sigma) - b = 0 \tag{2.6}$$

must be satisfied at $x = \pm L$. If u is a simple solution, $u(-L) \neq u(L)$, then the equation $\phi(u, \sigma) - b = 0$ must have at least two different roots, u_1 and u_2 $(u_1 < u_2)$, such that

$$\phi(u, \sigma) - b > 0 \quad \text{for } u_1 < u < u_2. \tag{2.7}$$

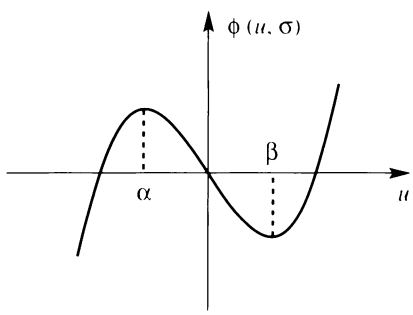

FIG. 1. The function $\phi(u, \sigma)$ for $\sigma > 0$.

In view of the particular form of ϕ (see Fig. 1), this means that $\sigma > 0$ and that

$$\phi(\beta, \sigma) \leq b < \phi(\alpha, \sigma), \tag{2.8}$$

where α and β are the values of u where $\phi(u, \sigma)$ reaches its unique local minimum and maximum:

$$\alpha = -\sqrt{\sigma}, \qquad \beta = +\sqrt{\sigma}.$$

The set of pairs (σ, b) that are thus *admissible* will be denoted by Σ:

$$\Sigma = \left\{(\sigma, b) \colon \sigma > 0,\ -\frac{2}{3}\sigma^{3/2} \leq b < \frac{2}{3}\sigma^{3/2}\right\}$$

or, if we define $\gamma = b/\sigma^{3/2}$,

$$\Sigma^* = \{(\sigma, \gamma) \colon \sigma > 0,\ -2/3 \leq \gamma < 2/3\}.$$

They are sketched in Fig. 2.

Now suppose again that $u(x)$ is a simple solution. Without loss of generality we may assume by (2.4) and (2.7) that $u' > 0$ on $(-L, L)$. Then

$$u'(x) = \sqrt{2\{\phi(u(x, \sigma)) - b\}} \quad \text{for } -L < x < L$$

and, upon integration,

$$\int_{u_1}^{u_2} \frac{du}{\sqrt{\phi(u, \sigma) - b}} = 2\sqrt{2}L,$$

where $u_1 = u(-L)$ and $u_2 = u(L)$. In a similar manner we find that

$$\int_{u_1}^{u_2} \frac{u\,du}{\sqrt{\phi(u, \sigma) - b}} = 2\sqrt{2}mL.$$

Thus if we define the functions

$$\mathscr{L}(\sigma, b) \stackrel{\text{def}}{=} \frac{1}{2\sqrt{2}} \int_{u_1}^{u_2} \frac{du}{\sqrt{\phi(u, \sigma) - b}}, \tag{2.9}$$

$$\mathscr{M}(\sigma, b) \stackrel{\text{def}}{=} \frac{1}{2\sqrt{2}} \int_{u_1}^{u_2} \frac{u\,du}{\sqrt{\phi(u, \sigma) - b}}, \tag{2.10}$$

where $u_i = u_i(\sigma, b)$, $i = 1, 2$, the question of the existence of a solution of problem (Ib) is reduced to finding a pair of numbers $(\sigma, b) \in \Sigma$ so that

$$\mathscr{L}(\sigma, b) = L \quad \text{and} \quad \mathscr{M}(\sigma, b) = mL. \tag{2.11}$$

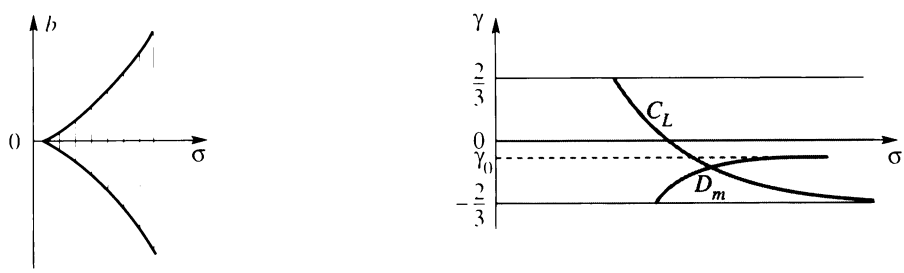

FIG. 2. The admissible sets Σ and Σ^*.

To solve (2.11) we shall study the functions $\mathscr{L}$ and $\mathscr{M}$ in Σ. Plainly they are well defined and continuous in the interior of Σ. By the homogeneity properties of ϕ it is convenient to study $\mathscr{L}$ and $\mathscr{M}$ along arcs

$$\Gamma_\gamma = \{(\sigma, b) : \sigma > 0,\ b = \gamma\sigma^{3/2}\}, \qquad |\gamma| \le \frac{2}{3}.$$

On such an arc, we have, writing $u = \sqrt{\sigma}t$,

$$\phi(\sqrt{\sigma}t, \sigma) - \gamma\sigma^{3/2} = \sigma^{3/2}(\varphi(t) - \gamma),$$

where

$$\varphi(t) = \frac{1}{3}t^3 - t.$$

Thus, $\mathscr{L}$ and $\mathscr{M}$ can be written as

$$\mathscr{L}(\sigma, b) = \mathscr{L}^*(\sigma, \gamma) = \frac{1}{2\sqrt{2}}\sigma^{-1/4}\int_{t_1}^{t_2}\frac{dt}{\sqrt{\varphi(t) - \gamma}}, \tag{2.12}$$

$$\mathscr{M}(\sigma, b) = \mathscr{M}^*(\sigma, \gamma) = \frac{1}{2\sqrt{2}}\sigma^{1/4}\int_{t_1}^{t_2}\frac{t\,dt}{\sqrt{\varphi(t) - \gamma}}, \tag{2.13}$$

where $\varphi(t_1) = \varphi(t_2) = \gamma$. Defining $m^* = \mathscr{M}^*(\sigma, \gamma)/\mathscr{L}^*(\sigma, \gamma)$, we thus deduce that

$$\frac{\partial\mathscr{L}^*}{\partial\sigma} < 0 \quad \text{and} \quad \frac{\partial|m^*|}{\partial\sigma} > 0 \quad \text{if } m^* \ne 0. \tag{2.14}$$

We now examine the behaviour of $\mathscr{L}^*$ and $\mathscr{M}^*$ with respect to the parameter γ.

LEMMA 2.1. If $|\gamma| < 2/3$, then

(a) $\partial\mathscr{L}^*(\sigma, \gamma)/\partial\gamma < 0$, (2.15)

(b) $\partial m^*(\sigma, \gamma)/\partial\gamma < 0$. (2.16)

Proof. From (2.12) it is clear that we have to examine the integrals

$$\ell(\gamma) = \int_{t_1}^{t_2}\frac{dt}{\sqrt{\varphi(t) - \gamma}} \quad \text{and} \quad p(\gamma) = \int_{t_1}^{t_2}\frac{t\,dt}{\sqrt{\varphi(t) - \gamma}}.$$

Setting $s = t + 1$, we can write them as

$$\ell(\gamma) = \int_{s_1}^{s_2}\{F(s) - F(s_1)\}^{-1/2}\,ds$$

and

$$p(\gamma) = \int_{s_1}^{s_2}(s - 1)\{F(s) - F(s_1)\}^{-1/2}\,ds,$$

where

$$F(x) = \frac{1}{3}x^3 - x^2, \tag{2.17}$$

and where s_1 and s_2 are the smaller roots of the equation

$$F(x) = \gamma - \frac{2}{3}. \tag{2.18}$$

In particular, we can write $p(\gamma) = b(\gamma) - \ell(\gamma)$, in which

$$b(\gamma) = \int_{s_1}^{s_2} s\{F(s) - F(s_1)\}^{-1/2}\, ds.$$

In order to prove (2.15) and (2.16) it is sufficient to show that

$$\ell'(\gamma) < 0 \tag{2.19}$$

and that

$$b'(\gamma)\ell(\gamma) - \ell'(\gamma)b(\gamma) < 0. \tag{2.20}$$

Note that it is possible to write

$$\ell(\gamma) = \ell_2(\gamma) - \ell_1(\gamma) \quad \text{and} \quad b(\gamma) = b_2(\gamma) - b_1(\gamma) \tag{2.21}$$

which in turn we can write, with $s = s_k u$, as

$$\ell_k(\gamma) = s_k \int_0^1 \{F(s_k u) - F(s_k)\}^{-1/2}\, du$$

and

$$b_k(\gamma) = s_k^2 \int_0^1 u\{F(s_k u) - F(s_k)\}^{-1/2}\, du.$$

Before proceeding we prove the following useful proposition. Define

$$\Delta F(x) = F(xu) - F(x). \tag{2.22}$$

We recall that s_1 and s_2 are the smaller roots of the equation (2.18). It follows that when $|\gamma| < 2/3$,

$$s_1 < 0 < s_2 < 2. \tag{2.23}$$

PROPOSITION 2.2. If $s_1 < s_2$, then

$$\Delta F(s_1 u) \geq \Delta F(s_2 u) \quad \text{for all } 0 \leq u \leq 1.$$

Proof of Proposition 2.2. Since $F(s_1) = F(s_2)$ it follows from (2.17) that if $s_2 \neq s_1$, then

$$-\frac{1}{3}(s_1^2 + s_1 s_2 + s_2^2) + (s_1 + s_2) = 0.$$

Therefore, if $u \in [0, 1]$, then

$$-\frac{1}{3}(s_1^2 + s_1 s_2 + s_2^2)u + (s_1 + s_2) \geq 0.$$

Multiplying by $s_2 - s_1 > 0$, we obtain that

$$F(s_1 u) \geq F(s_2 u) \quad \text{for all } 0 \leq u \leq 1.$$

We now return to the proof of Lemma 2.1. In order to demonstrate (2.19) we check by direct calculation that

$$\ell_k' = \frac{1}{6} s_k^3 s_k' \int_0^1 (1 - u^3)(\Delta F(s_k))^{-3/2}\, du. \tag{2.24}$$

From (2.21) it follows that

$$\ell'(\gamma) = \ell_2'(\gamma) - \ell_1'(\gamma). \tag{2.25}$$

Differentiating (2.18) we obtain $(s_k^2 - 2s_k)\,ds_k/d\gamma = 1$, or

$$s_k s_k' = -\frac{1}{2 - s_k}. \tag{2.26}$$

Therefore, according to (2.23),

$$s_k s_k' < 0 \quad \text{for } k = 1, 2 \tag{2.27}$$

and

$$|s_2 s_2'| > |s_1 s_1'|. \tag{2.28}$$

From Proposition 2.2 and (2.23) we deduce that $\Delta F(s_1) \geq \Delta F(s_2)$ and from Proposition 2.2 with $u = 0$ we obtain that $s_2^2 > s_1^2$. Thus it follows that $\ell_2' < \ell_1'$ and hence by (2.25) that $\ell'(\gamma) < 0$.

We now turn to prove (2.20). By direct computation we find that

$$\begin{aligned} b_k' &= s_k s_k' \int_0^1 u(\Delta F(s_k))^{-1/2}\,du + \frac{1}{6} s_k^4 s_k' \int_0^1 u(1-u^3)(\Delta F(s_k))^{-3/2}\,du \\ &= A_k + B_k. \end{aligned} \tag{2.29}$$

In order to prove (2.20) we note that by Proposition 2.2 and (2.23),

$$\ell(\gamma) > 0 \quad \text{for } |\gamma| < \frac{2}{3} \tag{2.30}$$

and by Proposition 2.2, together with (2.27) and (2.28),

$$A_2(\gamma) - A_1(\gamma) < 0 \quad \text{for } |\gamma| < \frac{2}{3}. \tag{2.31}$$

Combining (2.30) and (2.31), we conclude that (2.20) is proved once we demonstrate that

$$(B_2(\gamma) - B_1(\gamma))\ell(\gamma) - b(\gamma)\ell'(\gamma) \leq 0. \tag{2.32}$$

From (2.23) and (2.27) it follows that

$$B_1(\gamma) > 0, \quad B_2(\gamma) < 0, \quad \ell_1(\gamma) < 0, \quad \ell_2(\gamma) > 0$$

and in particular that

$$B_2(\gamma)\ell_1(\gamma) > 0 \quad \text{and} \quad B_1(\gamma)\ell_2(\gamma) > 0. \tag{2.33}$$

By (2.27)

$$\ell_k'(\gamma) < 0 \quad \text{and} \quad b_k(\gamma) > 0, \qquad k = 1, 2. \tag{2.34}$$

Using (2.33) and (2.34) in order to prove (2.32) it now suffices to show that

$$B_k(\gamma)\ell_k(\gamma) - b_k(\gamma)\ell_k'(\gamma) \leq 0. \tag{2.35}$$

Calculating explicitly we obtain

$$\begin{aligned} &B_k\ell_k - b_k\ell_k' \\ &\quad = \frac{1}{6} s_k' s_k^7 \left\{ \int_0^1 u(1-u^3)(\Delta F(s_k))^{-3/2}\,du \int_0^1 (1-u^2)(\Delta F(s_k))^{-3/2}\,du \right. \\ &\qquad \left. - \int_0^1 (1-u^3)(\Delta F(s_k))^{-3/2}\,du \int_0^1 u(1-u^2)(\Delta F(s_k))^{-3/2}\,du \right\}. \end{aligned}$$

Since by (2.27) $s_k' s_k^7 < 0$, it remains to show that

$$\begin{aligned}\int_0^1 u(1-u^3)(\Delta F(s_k))^{-3/2}\,du \int_0^1 (1-u^2)(\Delta F(s_k))^{-3/2}\,du \\ \geq \int_0^1 (1-u^3)(\Delta F(s_k))^{-3/2}\,du \int_0^1 u(1-u^2)(\Delta F(s_k))^{-3/2}\,du.\end{aligned} \tag{2.36}$$

Noting that

$$1-u^3 = (1-u)(1+u+u^2) \quad \text{and} \quad 1-u^2 = (1-u)(1+u),$$

and writing $g(u) = (1-u)(\Delta F(s_k))^{-3/2}$, (2.36) reduces to

$$\int_0^1 u^3 g(u)\,du \int_0^1 (1+u)g(u)\,du \geq \int_0^1 u^2 g(u)\,du \int_0^1 u(1+u)g(u)\,du. \tag{2.37}$$

By the Cauchy-Schwarz inequality,

$$\int_0^1 ug(u)\,du \leq \left(\int_0^1 g(u)\,du\right)^{1/2} \left(\int_0^1 u^2 g(u)\,du\right)^{1/2}$$

and

$$\int_0^1 u^2 g(u)\,du \leq \left(\int_0^1 ug(u)\,du\right)^{1/2} \left(\int_0^1 u^3 g(u)\,du\right)^{1/2}.$$

Therefore

$$\frac{\int_0^1 (u+u^2)g(u)\,du}{\int_0^1 (1+u)g(u)\,du} \leq \frac{\int_0^1 u^3 g(u)\,du}{\int_0^1 u^2 g(u)\,du}$$

from which (2.37) follows.

In order to complete the picture of the behaviour of $m^*(\sigma, \gamma)$ and $\mathscr{L}^*(\sigma, \gamma)$ for $(\sigma, \gamma) \in \Sigma^*$, we ascertain the asymptotic behaviour of m^* and $\mathscr{L}^*$ along the boundaries of Σ^*.

LEMMA 2.3. Let σ be fixed. Then

(a)
$$\begin{cases} \mathscr{L}^*(\sigma, \gamma) = \dfrac{\sigma^{-1/4}}{4\sqrt{2}} \log \dfrac{1}{\gamma + (2/3)} + O(1) & \text{as } \gamma \downarrow -\dfrac{2}{3}, \\ m^*(\sigma, \gamma) = \sigma^{1/2} + o(1) & \text{as } \gamma \downarrow -\dfrac{2}{3}, \end{cases}$$

and

(b)
$$\begin{cases} \mathscr{L}^*(\sigma, \gamma) \to \dfrac{\pi}{2\sqrt{2}} \sigma^{-1/4} & \text{as } \gamma \uparrow \dfrac{2}{3}, \\ m^*(\sigma, \gamma) \to -\sigma^{1/2} & \text{as } \gamma \uparrow \dfrac{2}{3}. \end{cases}$$

Proof. Plainly it is enough to study the behaviour of the integrals $\ell(\gamma)$ and $p(\gamma)$ as $\gamma \to -2/3$ and $\gamma \to 2/3$.

(a) Set $\gamma = -(2/3) + \varepsilon^2$ and $t = 1 - \varepsilon s$. Then, if we write $t_i = 1 - \varepsilon s_i(\varepsilon)$, $i = 1, 2$, we find that

$$s_1(\varepsilon) = \frac{3}{\varepsilon} + O(\varepsilon) \quad \text{as } \varepsilon \to 0$$

and

$$s_2(\varepsilon) \to 1\,, \quad s_3(\varepsilon) \to -1 \qquad \text{as } \varepsilon \to 0\,.$$

The integral $\ell(\gamma)$ can now be written as

$$\begin{aligned}\ell\left(-\frac{2}{3}+\varepsilon^2\right) &= \int_{s_2}^{s_1} \frac{ds}{\sqrt{s^2-1-(\varepsilon/3)s^3}} \\ &= \int_{s_2}^{s_1} \frac{ds}{s} + \int_{s_2}^{s_1} \left(\frac{1}{\sqrt{s^2-1-(\varepsilon/3)s^3}} - \frac{1}{s}\right) ds \\ &= \log\frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \to 0\,.\end{aligned}$$

For $p(\gamma)$ we proceed in a similar manner, yielding the same result:

$$p\left(-\frac{2}{3}+\varepsilon^2\right) = \log\frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \to 0\,. \tag{2.38}$$

(b) Set $\gamma = (2/3) - \varepsilon^2$ and $t = -1 + \varepsilon s$. Then, if we now write $t_i = -1 + s_i(\varepsilon)$, $i = 1, 2$, we find that $s_1(\varepsilon) \to -1$ and $s_2(\varepsilon) \to 1$ as $\varepsilon \to 0$. Thus

$$\ell\left(\frac{2}{3}-\varepsilon^2\right) = \int_{s_1}^{s_2} \frac{ds}{\sqrt{1-s^2+(\varepsilon/3)s^3}} \to \int_{-1}^{1} \frac{ds}{\sqrt{1-s^2}} = \pi \quad \text{as } \varepsilon \to 0\,.$$

For $p(\gamma)$ we show again in a similar manner that

$$p(\gamma) \to -\pi \quad \text{as } \gamma \uparrow \frac{2}{3}\,.$$

The result is now immediate.

REMARK 2.4. Since $p(\gamma)$ is continuous and takes on negative as well as positive values on $(-2/3, 2/3)$ it follows that it must vanish somewhere. Actually, noting that if $\gamma = 0$, $t_1 < t_2 = 0$ and

$$\mathscr{M}^*(\sigma, 0) = \int_{t_1}^{0} \frac{t\,dt}{\sqrt{\varphi(t)}} < 0\,.$$

Hence by (2.13) $p(0) < 0$ and so, in view of (2.38), there exists a $\gamma_0 \in (-2/3, 0)$, such that $p(\gamma_0) = 0$. Using (2.13) again we conclude that

$$\mathscr{M}^*(\sigma, \gamma_0) = 0 \quad \text{for all } \sigma > 0\,. \tag{2.39}$$

Remembering from (2.31) that $\ell(\gamma) > 0$ in $(-2/3, 2/3)$ this also means that

$$m^*(\sigma, \gamma_0) = 0 \quad \text{for all } \sigma > 0\,. \tag{2.40}$$

In order to determine for which values of L and m there exist simple solutions and more general solutions, we introduce the families of curves

$$\mathscr{C}_L = \{(\sigma, \gamma) \in \Sigma^* : \mathscr{L}^*(\sigma, \gamma) = L\}\,,$$
$$\mathscr{D}_m = \{(\sigma, \gamma) \in \Sigma^* : m^*(\sigma, \gamma) = m\}\,.$$

From (2.14), Lemmas 2.1 and 2.3 and Remark 2.4 we obtain immediately

THEOREM 2.5. For any $L > 0$, $\mathscr{C}_L$ is a continuous decreasing curve which connects the points

$$A = \left(\left(\frac{\pi}{2L\sqrt{2}}\right)^4, \frac{2}{3}\right) \quad \text{and} \quad B = \left(\infty, -\frac{2}{3}\right).$$

Moreover there exists a unique $\gamma_0 \in (-2/3, 0)$ such that

$$m^*(\sigma, \gamma_0) = 0 \quad \text{for } \sigma > 0.$$

If $m > 0$ then $\mathscr{D}_m$ is a monotonically increasing curve connecting

$$A = \left(m^2, -\frac{2}{3}\right) \quad \text{and} \quad B = (\infty, \gamma_0),$$

and if $m < 0$, then $\mathscr{D}_m$ is a monotonically decreasing curve connecting

$$A = \left(-m^2, \frac{2}{3}\right) \quad \text{and} \quad B = (\infty, \gamma_0).$$

From Theorem 2.5 we then conclude

THEOREM 2.6. For any $L > 0$, there exists a simple solution for all $m \geq -\pi^2/8L^2$ and if $m \geq 0$ the simple solution is unique.

Lastly we prove in this section that

THEOREM 2.7. For any $L > 0$ and for any $m \in (-\infty, \infty)$ there exists a countable number of (not necessarily simple) steady state solutions.

Proof. In order to obtain the total number of (not necessarily simple) solutions for a given $L > 0$ and $m \in (-\infty, \infty)$, it is sufficient to determine $N = \sum N_k$, where

$$N_k = \left\{ \begin{array}{l} \text{total number of points } (\sigma, \gamma) \in \Sigma^* \text{ such that} \\ \mathscr{L}^*(\sigma, \gamma) = L/k \text{ and } m^*(\sigma, \gamma) = m, \ k = 1, 2, \ldots \end{array} \right\}.$$

If $m \geq 0$ Theorem 2.6 shows that $N_k = 1$ for $k = 1, 2, \ldots$. Hence, if $m \geq 0$ there is a countable infinity of solutions.

If $m < 0$, then Theorem 2.6 implies that $N_k \geq 1$ for k sufficiently large. The proof is complete if we can show that for any k, N_k is finite. This we prove by contradiction.

Suppose that for some k, N_k is infinite. Then, by the properties of $\mathscr{C}_{L/k}$ and $\mathscr{D}_m$ proved in Theorem 2.5, the set

$$\mathscr{S}_k = \{(\sigma, \gamma) \in \Sigma^* : (\sigma, \gamma) \in \mathscr{C}_{L/k} \cap \mathscr{D}_m\}$$

is compact. Therefore there must exist an accumulation point $(\sigma^*, \gamma^*) \in \mathscr{S}_k$. It is easy to see that all the derivatives of $\mathscr{L}^*(\sigma, \gamma)$ and $m^*(\sigma, \gamma)$ must coalesce at this point. Since the coefficients of $u(x)$ in (2.6) are analytic in (σ, b) and $b = \gamma\sigma^{3/2}$ it follows that $u = u(x, \sigma, \gamma)$ is analytic in the parameters (σ, γ) when $\sigma > 0$. Therefore $\mathscr{L}^*(\sigma, \gamma)$ and $m^*(\sigma, \gamma)$ must be analytic in the parameters (σ, γ) and thus must coalesce in the neighbourhood of the accumulation point. Hence $\mathscr{D}_m$ cannot oscillate about $\mathscr{C}_{L/k}$ and so we have a contradiction. This completes the proof of Theorem 2.7.

3. Problem II. For Problem II we proceed as for Problem I to obtain the problem

$$\text{(IIa)} \qquad \begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} = f(u) - \sigma, & -L < x < L, \\ u'(-L) = u'(L) = 0, \\ \displaystyle\int_{-L}^{L} u(x)\,dx = 2mL. \end{cases}$$

Again we shift the origin and write $\tilde{u} = u - 1$, $\tilde{\sigma} = \sigma + 1$, and $\tilde{m} = m - 1$ so that problem (IIa) becomes, after the tildes have been dropped,

$$\text{(IIb)} \qquad \begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} = u^2 - \sigma, & -L < x < L, \quad \sigma \in \mathbf{R}, & (3.1) \\ u'(-L) = u'(L) = 0, & & (3.2) \\ \displaystyle\int_{-L}^{L} u(x)\,dx = 2mL. & & (3.3) \end{cases}$$

If we multiply (3.1) by u' and integrate we now find that u must satisfy the equation

$$-\frac{1}{\sqrt{1+u'^2}} = \phi(u(x), \sigma) - b, \qquad b \in \mathbf{R}, \tag{3.4}$$

where as in Sec. 2,

$$\phi(u, \sigma) = \frac{1}{3}u^3 - \sigma u. \tag{3.5}$$

Again, in order to satisfy the boundary conditions (3.2), the equation

$$\phi(u, \sigma) = b - 1 \tag{3.6}$$

needs to have at least two different roots. This implies that $\sigma > 0$ and that

$$\phi(\beta, \sigma) \le b - 1 < \phi(\alpha, \sigma), \tag{3.7}$$

where $\alpha = -\sqrt{\sigma}$ and $\beta = \sqrt{\sigma}$ are the critical points of ϕ.

The inequalities (3.7) provide a *first constraint* for the admissible pairs (σ, b). Note that it is a translate of the admissible region we had found in Sec. 2. Thus, denoting the set of admissible pairs by Σ again, it follows from (3.7) that

$$\Sigma \subset \left\{(\sigma, b)\colon 1 - \frac{2}{3}\sigma^{3/2} \le b < 1 + \frac{2}{3}\sigma^{3/2}\right\}.$$

However, in contrast to problem (Ia), we now have a *second constraint*. By (3.4),

$$0 \le \phi(u(x), \sigma) - (b - 1) < 1, \tag{3.8}$$

so that if u_1, u_2, and u_3 are the roots of Eq. (3.6) in increasing order, then

$$u_1 \le u(x) \le u_2 \quad \text{for } -L \le x \le L$$

and

$$\{\phi(s)\colon u_1 \le s \le u_2\} = [b - 1, \phi(\alpha)].$$

Therefore, in order that (3.8) is satisfied for all $x \in [-L, L]$ we must require that

$$\phi(\alpha) < b, \tag{3.9}$$

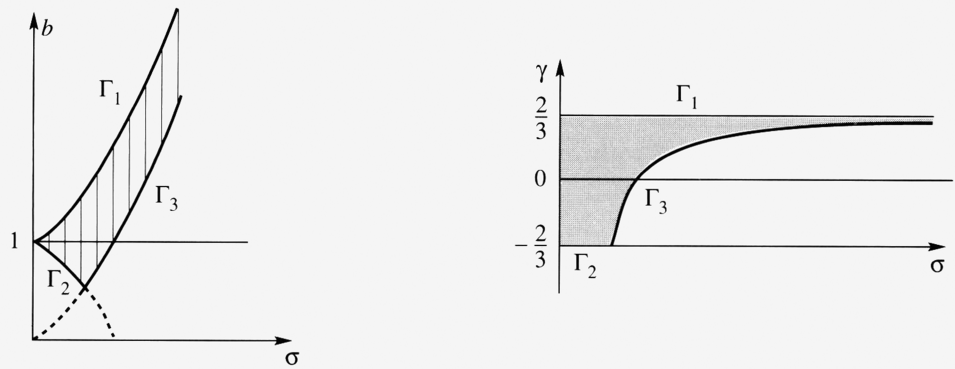

FIG. 3. The sets Σ and Σ^*.

or $b > 2\sigma^{3/2}/3$. Thus the set of admissible pairs (σ, b) is now (see Fig. 3)

$$\Sigma = \left\{(\sigma, b)\colon \sigma > 0,\ 1 - \frac{2}{3}\sigma^{3/2} \le b < 1 + \frac{2}{3}\sigma^{3/2},\ b > \frac{2}{3}\sigma^{3/2}\right\},$$

or, with $\gamma = (b-1)/\sigma^{3/2}$,

$$\Sigma^* = \left\{(\sigma, \gamma)\colon \sigma > 0,\ -\frac{2}{3} \le \gamma < \frac{2}{3},\ \gamma > \frac{2}{3} - \sigma^{-3/2}\right\}.$$

Continuing as in Sec. 2, we reduce the problem of finding a simple solution of problem (IIb) to finding a pair (σ, b) such that

$$\mathscr{L}(\sigma, b) = L \quad \text{and} \quad \mathscr{M}(\sigma, b) = mL, \tag{3.10}$$

where

$$\mathscr{L}(\sigma, b) = \frac{1}{2}\int_{u_1}^{u_2} \frac{du}{[\{b - \phi(u, \sigma)\}^{-2} - 1]^{1/2}}, \tag{3.11}$$

$$\mathscr{M}(\sigma, b) = \frac{1}{2}\int_{u_1}^{u_2} \frac{u\,du}{[\{b - \phi(u, \sigma)\}^{-2} - 1]^{1/2}}. \tag{3.12}$$

As before we inspect the behaviour of $\mathscr{L}$ and $\mathscr{M}$ in the set Σ and on its boundary, which consists of three arcs

$$\Gamma_1 = \left\{(\sigma, b)\colon \sigma > 0,\ b = 1 + \frac{2}{3}\sigma^{3/2}\right\},$$

$$\Gamma_2 = \left\{(\sigma, b)\colon 0 < \sigma \le (3/4)^{2/3},\ b = 1 - \frac{2}{3}\sigma^{3/2}\right\},$$

$$\Gamma_3 = \left\{(\sigma, b)\colon \sigma > (3/4)^{2/3},\ b = \frac{2}{3}\sigma^{3/2}\right\}.$$

For any $L > 0$ we define the set $\mathscr{C}_L = \{(\sigma, b)\colon \mathscr{L}(\sigma, b) = L\}$, about which we shall prove the following properties.

THEOREM 3.1. For any $L > 0$, the set $\mathscr{C}_L$ consists of at most a countable set of continuous line segments and, for $\bar{\sigma}$ large enough,

$$\mathscr{C}_L \cap \{\sigma \geq \underline{\sigma}\}$$

consists of a unique monotonically increasing curve in Σ. Moreover, $\mathscr{C}_L$ is nonempty and compact.

A number of lemmas precede the proof of Theorem 3.1. We define

$$\mathscr{L}^*(\sigma, \gamma) = \mathscr{L}(\sigma, 1 + \gamma\sigma^{3/2}) \quad \text{and} \quad \mathscr{M}^*(\sigma, \gamma) = \mathscr{M}(\sigma, 1 + \gamma\sigma^{3/2}). \tag{3.13}$$

LEMMA 3.2. We have

$$\frac{\partial}{\partial\sigma}\mathscr{L}^*(\sigma, \gamma) < 0 \quad \text{for } |\gamma| < \frac{2}{3}.$$

Proof. From (3.11) and (3.13) we obtain with $s = \sigma^{-1/2}u + 1$,

$$2\mathscr{L}^*(\sigma, \gamma) = \int_{s_1}^{s_2} \frac{\sigma^{1/2}\, ds}{[\{1 - \sigma^{3/2}(F(s) - F(s_1))\}^{-2} - 1]^{1/2}}, \tag{3.14}$$

where, as in Sec. 2,

$$F(x) = \frac{1}{3}x^3 - x^2$$

and s_1 and s_2 are the two smaller roots of the equation $F(s) = \gamma - 2/3$. From (3.14) we see that

$$2\mathscr{L}^*(\sigma, \gamma) = -\ell(s_1, \sigma^{1/2}) + \ell(s_2, \sigma^{1/2}), \tag{3.15}$$

where

$$\ell(x, q) \stackrel{\text{def}}{=} \int_0^1 \frac{qx\, du}{[(1 - q^3\Delta F(x))^{-2} - 1]^{1/2}}, \tag{3.16}$$

where, as before $\Delta F(x) = F(xu) - F(x)$.

To prove Lemma 3.2 we recall (2.23) and demonstrate that

$$\operatorname{sgn} \ell_q(x, q) = -\operatorname{sgn} x. \tag{3.17}$$

By direct calculation

$$\ell_q(x, q) = -q^3 x \int_0^1 \frac{(1 + 3q^3\Delta F - q^6(\Delta F)^2)\Delta F}{[(1 - q^3\Delta F)^{-2} - 1]^{3/2}(1 - q^3\Delta F)^3}\, du.$$

Since $0 \leq q^3\Delta F(x) < 1$ for $0 \leq u \leq 1$, (3.17) follows.

LEMMA 3.3. Along Γ_1 we have

$$\mathscr{L}(\sigma, b) = \frac{\pi}{2\sqrt{2}}\sigma^{-1/4}, \quad \mathscr{M}(\sigma, b) = -\frac{\pi}{2\sqrt{2}}\sigma^{1/4}, \qquad m = -\sigma^{1/2}.$$

Proof. From (3.15) we have

$$2\mathscr{L}(\sigma, b) = -\ell(s_1, \sigma^{1/2}) + \ell(s_2, \sigma^{1/2})$$

and likewise we can write

$$2\mathscr{M}(\sigma, b) = -p(s_1, \sigma^{1/2}) + p(s_2, \sigma^{1/2}) + \sigma^{1/2}\{\ell(s_1, \sigma^{1/2}) - \ell(s_2, \sigma^{1/2})\},$$

where

$$p(x, q) \stackrel{\text{def}}{=} \int_0^1 \frac{q^2 x^2 u\, du}{[(1 - q^3 \Delta F(x))^{-2} - 1]^{1/2}}. \tag{3.18}$$

As $(\sigma^{1/2}, b) \to \Gamma_1$, s_1 and s_2 both tend to 0 and so we must examine the behaviour of $\ell(x, q)$ and $p(x, q)$ as $x \to 0$. Noting that

$$\Delta F(x) = x^2 \left\{ (1 - u^2) - \frac{1}{3} x (1 - u^3) \right\},$$

we calculate that

$$\lim_{x \to 0} \ell(x, q) = \operatorname{sgn} x \int_0^1 \frac{q^{-1/2}\, du}{\sqrt{2(1 - u^2)}} = \operatorname{sgn} x q^{-1/2} \left(\frac{\pi}{2\sqrt{2}} \right)$$

and

$$\lim_{x \to 0} p(x, q) = 0,$$

from which the claim of the lemma follows.

In the following lemma we determine the behaviour of $\mathscr{L}^*$ and m^* along the arc Γ_2. The argumentation follows closely the one given to prove Lemma 2.3.

LEMMA 3.4. We have

(a) $$\mathscr{L}^*(\sigma, \gamma) = \frac{\sigma^{-1/4}}{4\sqrt{2}} \log \frac{1}{\gamma + 2/3} + O(1) \quad \text{as } \gamma \to -\frac{2}{3}$$

and

(b) $$m^* = \sigma^{1/2} + o(1) \quad \text{as } \gamma \to -\frac{2}{3}.$$

Proof. If we set $u = \sqrt{\sigma} t$ in (3.11) we obtain

$$\mathscr{L}^*(\sigma, \gamma) = \frac{\sigma^{-1/4}}{2} \int_{t_1}^{t_2} \frac{[1 + \sigma^{3/2}\{\gamma - \varphi(t)\}]\, dt}{\sqrt{\{\varphi(t) - \gamma\}[2 + \sigma^{3/2}\{\gamma - \varphi(t)\}]}},$$

where $\varphi(t) = t^3/3 - t$ and t_1 and t_2 are the smaller roots of the equation $\varphi(t) = \gamma$. Defining $\gamma = -2/3 + \varepsilon^2$ and $t = 1 - \varepsilon s$, we can write $\mathscr{L}^*$ as

$$\mathscr{L}^*(\sigma, \gamma) = \frac{\sigma^{-1/4}}{2} \int_{s_2}^{s_1} \frac{\{1 + \sigma^{3/2}\varepsilon^2(1 - s^2 + \varepsilon s^3/3)\}\, ds}{\sqrt{(-1 + s^2 - \varepsilon s^3/3)\{2 + \sigma^{3/2}\varepsilon^2(1 - s^2 + \varepsilon s^3/3)\}}}.$$

As in the proof of Lemma 2.3 we note that

$$s_1(\varepsilon) = \frac{3}{\varepsilon} + O(\varepsilon) \quad \text{and} \quad s_2(\varepsilon) \to 1 \text{ as } \varepsilon \to 0.$$

Thus

$$\begin{aligned}
\mathscr{L}^*(\sigma, \gamma) &= \frac{\sigma^{-1/4}}{2} \int_{s_2}^{s_1} \frac{\{1 + \sigma^{3/2}\varepsilon^2(1 - s^2 + \varepsilon s^3/3)\, ds}{s\sqrt{2 + \sigma^{3/2}\varepsilon^2(1 - s^2 + \varepsilon s^3/3)}} + O(1) \\
&= \frac{\sigma^{-1/4}}{2\sqrt{2}} \int_{s_2}^{s_1} \frac{ds}{s} + O(1) \\
&= \frac{\sigma^{-1/4}}{4\sqrt{2}} \log \frac{1}{\gamma + 2/3} + O(1) \quad \text{as } \gamma \downarrow -\frac{2}{3}.
\end{aligned}$$

A similar analysis shows that

$$\mathscr{M}^*(\sigma, \gamma) = \frac{\sigma^{1/4}}{4\sqrt{2}} \log \frac{1}{\gamma + 2/3} + O(1) \quad \text{as } \gamma \downarrow -\frac{2}{3}.$$

The second assertion is now immediate.

The next two lemmas give information concerning the behaviour along Γ_3.

LEMMA 3.5. $\mathscr{L}(\sigma, b)$ does not vanish along Γ_3.

Proof. This is an immediate consequence of the positivity of the integrand in (3.11) on the interval (u_1, u_2).

In the following lemma we give a characterisation of $\mathscr{L}^*(\sigma, b)$ in terms of coordinates parallel to Γ_1 to Γ_3. Define

$$\xi = b - \frac{2}{3}\sigma^{3/2} \quad \text{for } (\sigma, b) \in \overline{\Sigma}$$

and $\widehat{\mathscr{L}}(\sigma, \xi) = \mathscr{L}(\sigma, b)$. Note that $0 \le \xi \le 1$.

LEMMA 3.6. For $\xi \in (0, 1)$ we have

$$\widehat{\mathscr{L}}(\sigma, \xi) = r(\xi)\sigma^{-1/4} + O(\sigma^{-1/2}) \quad \text{as } \sigma \to \infty, \tag{3.19}$$

in which $r \in C([0, 1])$, $r > 0$ on $[0, 1]$ and $r' > 0$ on $(0, 1)$, and

$$m^* = -\sigma^{1/2} + O(\sigma^{-1/2}). \tag{3.20}$$

Proof. Let $0 \le \xi < 1$. For σ large, the smaller roots of (3.6) are given by

$$u_1 = -\sigma^{1/2} - \left(\frac{1-\xi}{\sqrt{\sigma}}\right)^{1/2} + \frac{1-\xi}{6\sigma} + O(\sigma^{-7/4}),$$

$$u_2 = -\sigma^{1/2} + \left(\frac{1-\xi}{\sqrt{\sigma}}\right)^{1/2} + \frac{1-\xi}{6\sigma} + O(\sigma^{-7/4}).$$

Accordingly we define the new variable

$$w = \left(\frac{1-\xi}{\sqrt{\sigma}}\right)^{-1/2} (u + \sigma^{1/2}).$$

Thus, in terms of w

$$2\widehat{\mathscr{L}}(\sigma, \xi) = \left(\frac{1-\xi}{\sqrt{\sigma}}\right)^{1/2} \int_{w_1}^{w_2} \frac{dw}{\sqrt{[\xi + (1-\xi)w^2 - \frac{1}{3}((1-\xi)/\sqrt{\sigma})^{3/2} w^3]^{-2} - 1}},$$

where $w_1 = -1 + O(\sigma^{-3/4})$ and $w_2 = 1 + O(\sigma^{-3/4})$. Hence, for σ large

$$\widehat{\mathscr{L}}(\sigma, \xi) = r(\xi)\sigma^{-1/4} + O(\sigma^{-1/2}),$$

where

$$r(\xi) = \frac{1}{2} \int_{-1}^{1} \frac{[\xi + (1-\xi)w^2]\, dw}{\sqrt{(1-w^2)[(1+\xi) + (1-\xi)w^2]}}.$$

Note that $r \in C^1([0, 1])$ and that

$$r(0) = \frac{1}{2}\int_{-1}^{1} \frac{w^2\,dw}{\sqrt{1-w^4}},$$

while

$$r(1) = \frac{1}{2\sqrt{2}}\int_{-1}^{1} \frac{dw}{\sqrt{1-w^2}} = \frac{\pi}{2\sqrt{2}},$$

which, substituted into (3.19), yields the result of Lemma 3.3. Moreover, for $\xi \in [0, 1]$

$$r'(\xi) = \frac{1}{2}\int_{-1}^{1} \frac{(1+\xi/2)+(1-\xi)w^2/2}{\{(1+\xi)+(1-\xi)w^2\}^{3/2}}(1-w^2)^{1/2}\,dw > 0.$$

Similarly we define

$$\mathscr{M}(\sigma, b) \equiv \widehat{\mathscr{M}}(\sigma, \xi) = \left(\frac{1-\xi}{\sqrt{\sigma}}\right)^{1/2}\int_{w_1}^{w_2} \frac{[-\sqrt{\sigma}+((1-\xi)/\sqrt{\sigma})^{1/2}w]\,dw}{\sqrt{[\xi+(1-\xi)w^2-\frac{1}{3}((1-\xi)/\sqrt{\sigma})^{3/2}w^3]^{-2}-1}}$$

from which (b) follows.

Lemmas 3.2–3.6 conclude the proof of Theorem 3.1. From this theorem a number of results readily follow. We begin with an existence theorem.

THEOREM 3.7. For any $L > 0$, on the ith component of $\mathscr{C}_L$, there exist $m_-^i(L)$ and $m_+^i(L)$, $m_-^i(L) < m_+^i(L)$ such that for all $m \in (m_-^i(L), m_+^i(L))$ there exist simple solutions of problem (II).

Proof. The assertion is an easy consequence of Theorem 3.1 and the continuous dependence of $m(\sigma, b)$ on $(\sigma, b) \in \overline{\Sigma}\backslash\Gamma_2$.

In the following result we observe that there exists a uniform upper bound for m^*.

THEOREM 3.8. There exists a number m^* so that for all $L > 0$ and for all i, $m_+^i(L) < m^*$.

Proof. It suffices to prove that $m(\sigma, b) < m^*$ for some m^* for all $(\sigma, b) \in \Sigma$. For σ sufficiently large, $m < 0$ (Lemma 3.6). Hence we need only study $m(\sigma, b)$ on the set $\Sigma_- = \Sigma \cap \{(\sigma, b): \sigma < \overline{\sigma}\}$ for some large $\overline{\sigma}$.

From the definitions of $\mathscr{L}(\sigma, b)$ and $\mathscr{M}(\sigma, b)$ given in (3.10) and (3.11) it follows that $m(\sigma, b) < u_2(\sigma, b)$. Because $u_2(\sigma, b)$ is a continuous function of (σ, b) and Σ_- is a compact region, we may conclude that $u_2(\sigma, b)$ is bounded from above on Σ_-, and hence, so is $m(\sigma, b)$.

Lastly we prove

THEOREM 3.9. For any $L > 0$ there exists a constant $\hat{m}$, such that if $m < \hat{m}$, then there exists a bounded solution of problem (II).

Proof. For any $L > 0$ existence of bounded solutions on the interval $[-L, L]$ is equivalent to existence of simple solutions on the interval $[-L/N, L/N]$ for some $N \in \mathbf{N}^+$. From Lemma 3.6, it follows that $\mathscr{L}(\sigma, b) \to 0$ and $m(\sigma, b) \to -\infty$ as $\sigma \to \infty$ for $(\sigma, b) \in \Sigma$. Therefore, the set

$$T = \{m_-(L/k): k = 1, 2, \dots\}$$

is not bounded from below. The theorem is proved once we demonstrate that the intervals $(m_-(L/N), m_+(L/N))$, $N = 1, 2, \ldots$, start to overlap when N becomes large, so that for any m large enough negative, it is possible to find an integer N_m such that $m \in (m_-(L/N_m), m_+(L/N_m))$.

Thus, we need to show that there exists an integer N^* such that for any $N > N^*$

$$(m_-(L/N), m_+(L/N)) \cap (m_-(L/(N+1)), m_+(L/(N+1))) \neq \varnothing. \tag{3.21}$$

Let σ_1 and σ_2 be such that

$$\widehat{\mathscr{L}}(\sigma_1, 1) = \frac{L}{N} \quad \text{and} \quad \widehat{\mathscr{L}}(\sigma_2, 0) = \frac{L}{N}.$$

Then by Lemma 3.3,

$$\hat{m}(\sigma_1, 1) = -\frac{1}{2}\left(\frac{N\pi}{2L}\right)^2 = -\left(\frac{Nr(1)}{L}\right)^2 \tag{3.22}$$

and by (3.19) and (3.20) in Lemma 3.6 we obtain the estimate

$$\hat{m}(\sigma_2, 0) = -\left(\frac{Nr(0)}{L}\right)^2 + O\left(\frac{Nr(0)}{L}\right) \quad \text{as } N \to \infty. \tag{3.23}$$

Because $0 < r(0) < r(1)$ by Lemma 3.6, we find that for N large enough $\hat{m}(\sigma_1, 1) < \hat{m}(\sigma_2, 0)$. Let σ_3 and σ_4 be such that

$$\widehat{\mathscr{L}}(\sigma_3, 1) = \frac{L}{N+1} \quad \text{and} \quad \widehat{\mathscr{L}}(\sigma_4, 0) = \frac{L}{N+1}.$$

Then (3.21) will be satisfied if

$$\hat{m}(\sigma_4, 0) > \hat{m}(\sigma_1, 1) \tag{3.24}$$

and

$$\hat{m}(\sigma_2, 0) > \hat{m}(\sigma_3, 1). \tag{3.25}$$

According to (3.22) and (3.23), the inequalities (3.24) and (3.25) become asymptotically for large N,

$$-\left(\frac{(N+1)r(0)}{L}\right)^2 + O\left(\frac{(N+1)r(0)}{L}\right) > -\left(\frac{Nr(1)}{L}\right)^2$$

and

$$-\left(\frac{Nr(0)}{L}\right)^2 + O\left(\frac{Nr(0)}{L}\right) > -\left(\frac{(N+1)r(1)}{L}\right)^2,$$

or

$$r^2(0) < \left(\frac{N}{N+1}\right)^2 r^2(1) + O\left(\frac{1}{N}\right)$$

and

$$r^2(0) < \left(\frac{N+1}{N}\right)^2 r^2(1) + O\left(\frac{1}{N}\right).$$

Since $0 < r(0) < r(1)$, these inequalities are indeed satisfied when N is large enough. This completes the proof of Theorem 3.9.

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