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SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN IMPLICIT FORM

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Abstract. A method is described to determine exact solutions of general type in implicit form for the equations governing the steady two-dimensional motion of a viscous incompressible liquid.

Introduction. It is remarked in [1] that even when exact solutions of nonlinear differential equations are available it is not always possible to express the dependent variables as explicit functions of the independent variables. There are several instances in fluid mechanics where one or more integrations of the flow equations are possible and the net result is a representation for the independent variables as explicit functions of the velocity components. This results in a major simplification in spite of the fact that the quantities of physical interest are implicitly defined, and it may be desirable to carry out further analysis to determine an inverse function. One such example is the Jeffery-Hamel similarity solution [2] for the steady two-dimensional flow in a converging or diverging channel. Here two integrations of the flow equations are possible, and the result naturally expresses the angular space coordinate as an explicit function of the radial velocity in terms of an elliptic integral.

The starting point in [1] is a concise complex variable formulation for the equations governing the steady two-dimensional flow of a viscous incompressible liquid and the results of the analysis express a flow of general type in parametric form by means of twelve partial differential equations. These equations contain the stream function ψ and an auxiliary function ω related to the Bernoulli function and their first and second partial derivatives with respect to the space variables. The objective of this paper is to provide a substantial simplification of the parametric system by means of elimination of the derivatives and to express the space variables x, y as explicit functions ω and ψ, containing the arbitrary functions found through the integration process. Even with this simplified form of solution, the results are somewhat cumbersome, but may be more manageable for numerical computation when particular forms are chosen for the arbitrary functions to generate specific flow fields. As a check on the method, simple functions are chosen to recover a well-known

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potential solution. Finally it is noted that the elimination procedure is straightforward and it is not facilitated by the use of a symbolic computer program.

The flow equations and elimination procedure. It is necessary for the present analysis to recapture some of the results obtained in [1] expressed in the following way. If $N(\omega)$ is an arbitrary complex function of $\omega = \phi - \psi$, with $N'(\omega) \neq 0$ and ϕ, ψ both real, the equations

$$L' \equiv \omega_{\bar{z}} - \frac{1}{N'(\omega)} e^{(-N(\omega)-2\psi)/(2\nu i)} = 0, \tag{1}$$

$$\phi_{\bar{z}} + i\psi_{\bar{z}} = k = [1 - iBe^{(-\psi-F)/(2\nu i)}]e^{-\Gamma-\psi/(2\nu(1+i))}, \tag{2}$$

where

$$A + B = e^{\Gamma+\psi/(2\nu(1+i))} \neq 0, \tag{3}$$

and

$$\begin{aligned}[e^{(N+2\psi)/(2\nu i)}N' + i(A + B - BN')]&\left[\frac{BN'^2}{2\nu} - A_\phi - B_\phi(1 + iN')\right] \\ = [e^{(N+2\psi)/(2\nu i)}N' - A - B - iBN']&\left\{\frac{A}{2\nu} + \frac{B}{2\nu}(1 - N')^2 - iA_\psi - iB_\psi(1 - N')\right. \\ &\left. + (\Gamma_\phi + \Gamma_\psi)[e^{(N+2\psi)/(2\nu i)}N' + i(A + B - BN')]\right\}\end{aligned} \tag{4}$$

imply that

$$L = \omega_{\bar{z}\bar{z}} + (1 + i)\psi_{\bar{z}\bar{z}} + \frac{1}{2\nu}\psi_{\bar{z}}^2 = 0, \tag{5}$$

which is a concise complex variable formulation of the steady two-dimensional Navier-Stokes equations. Since A, B are complex, the complex function k is only partially determined from Eqs. (3), (4), and the function k is used to satisfy integrability conditions. Now it is shown in [1] that Eq. (1) implies

$$L'' \equiv y - x \cdot \frac{\omega_y}{\omega_x} - H\left(\frac{\omega_y}{\omega_x}\right) = 0, \tag{6}$$

where H is a arbitrary real function of ω_y/ω_x. Conversely Eq. (6) and

$$\frac{\omega_y}{\omega_x} = \frac{[\overline{N}'(\omega) - N'(\omega)e^{(N+\overline{N}+4\psi)/(2\nu i)}]}{i[e^{(N+\overline{N}+4\psi)/(2\nu i)}N'(\omega) + \overline{N}'(\omega)}} \tag{7}$$

imply Eq. (1). This may be shown in the following way: Eq. (6) can be written as

$$y - x\tan\chi = H(\tan\chi), \qquad \frac{\omega_y}{\omega_x} = \tan\chi, \tag{8}$$

where χ is real, differentiation with respect to x and y gives

$$1 = [x + H'(\tan\chi)]\sec^2\chi\chi_y, \qquad -\tan\chi = (x + H'\tan\chi)\sec^2\chi\chi_x, \tag{9}$$

and elimination of $(x + H'(\tan\chi))$ produces the equation

$$\sin\chi \cdot \chi_y + \cos\chi \cdot \chi_x = 0. \tag{10}$$

It follows from Eq. (10) that there exists W such that

$$W_x = \cos\chi, \qquad W_y = \sin\chi, \tag{11}$$

so that

$$\tan\chi = \frac{W_y}{W_x} = \frac{\omega_y}{\omega_x}, \tag{12}$$

which in turn implies that $W = F(\omega)$, where F is an arbitrary function of ω and

$$F'(\omega)\omega_x = \cos\chi, \qquad F'(\omega)\omega_y = \sin\chi, \tag{13}$$

and, by elimination of χ,

$$[F'(\omega)]^2(\omega_x^2 + \omega_y^2) = 1. \tag{14}$$

Since $F(\omega)$ is arbitrary, it is appropriate to set

$$[F'(\omega)]^2 = e^{(N-\overline{N})/(2\nu i)} N'(\omega)\overline{N}'(\omega), \tag{15}$$

in which case it follows that Eqs. (7), (14) imply the complex equation (1). Now from Eqs. (1), (2) it follows that

$$(1+i)\phi_{\overline{z}} = k + \frac{ie^{-(N+2\psi)/(2\nu i)}}{N'(\omega)} = A'(1+i), \tag{16}$$

$$(1+i)\psi_{\overline{z}} = k - ie^{-(N+2\psi)/(2\nu i)} = (1+i)B'. \tag{17}$$

The integrability conditions $\phi_{z\overline{z}} = \phi_{\overline{z}z}$, $\psi_{z\overline{z}} = \psi_{\overline{z}z}$, with ϕ, ψ real, require

$$A'_\phi \overline{A}' + A'_\psi \overline{B}' = \overline{A}'_\phi A' + \overline{A}'_\psi B', \tag{18}$$

$$B'_\phi \overline{A}' + B'_\psi \overline{B}' = \overline{B}'_\phi A' + \overline{B}'_\psi B', \tag{19}$$

which are essentially two real partial differential equations to determine k and $\overline{k}$. It follows that there are real functions ω, ψ that satisfy $L' = L'' = 0$ and also satisfy $L = 0$. To determine the solutions, integrability conditions are applied to Eqs. (1), (5), and (6) or, equivalently, $L = L' = L'' = 0$. In this case it is sufficient to consider the system of equations

$$L' = L'_{\overline{z}} = L'_z = L'_{z\overline{z}} - \overline{L}'_{\overline{z}\overline{z}} = L'' = L''_y = L = 0. \tag{20}$$

The symbolic form given by Eq. (20) gives a complete parametrization for the solution of the flow equations. The main object of the paper is to simplify the system by elimination of the first and second derivatives and to present the solution in a more informative manner.

It is convenient to write the equation $L' = 0$ in the form

$$\omega_{\overline{z}} = M(\omega)e^{-\psi/(\nu i)}, \qquad M(\omega) = \frac{1}{N'(\omega)}e^{-N(\omega)/(2\nu i)}, \tag{21}$$

so that $M(\omega)$ is an arbitrary complex function of ω. Differentiation of Eq. (21) with respect to $\bar{z}$ and z in turn gives

$$\omega_{\bar{z}\bar{z}} = MM'e^{-2\psi/(\nu i)} - \frac{M}{\nu i}\psi_{\bar{z}}e^{-\psi/(\nu i)}, \tag{22}$$

$$\omega_{z\bar{z}} = M'\overline{M} - \frac{M\psi_z}{\nu i}e^{-\psi/(\nu i)} = \overline{M}'M + \frac{\overline{M}}{\nu i}\psi_{\bar{z}}e^{\psi/(\nu i)}. \tag{23}$$

Further differentiation of Eq. (23) with respect to $\bar{z}$ produces the complex equation

$$\begin{aligned}(M''\overline{M} - \overline{M}''M)Me^{-\psi/(\nu i)} - \frac{MM'}{\nu i}\psi_z e^{-2\psi/(\nu i)} - Me^{-\psi/(\nu i)}\left[\frac{\psi_{z\bar{z}}}{\nu i} + \frac{\psi_z\psi_{\bar{z}}}{\nu^2}\right] \\ - \frac{M\overline{M}'}{\nu i}\psi_{\bar{z}} - \overline{M}e^{\psi/(\nu i)}\left[\frac{\psi_{\bar{z}\bar{z}}}{\nu i} - \frac{\psi_{\bar{z}}^2}{\nu^2}\right] = 0.\end{aligned} \tag{24}$$

Elimination of the vorticity $4\psi_{z\bar{z}}$ gives the real equation

$$\frac{-2\psi_z\psi_{\bar{z}}}{\nu^2} - \frac{\overline{M}}{M}e^{2\psi/(\nu i)}\left(\frac{\psi_{\bar{z}\bar{z}}}{\nu i} - \frac{\psi_{\bar{z}}^2}{\nu^2}\right) + \frac{M}{\overline{M}}e^{-2\psi/(\nu i)}\left(\frac{\psi_{zz}}{\nu i} + \frac{\psi_z^2}{\nu^2}\right) = 0. \tag{25}$$

Now the equation $L = 0$, together with Eq. (22) implies

$$MM'e^{-2\psi/(\nu i)} - \frac{M}{\nu i}e^{-\psi/(\nu i)}\psi_{\bar{z}} + (1+i)\psi_{\bar{z}\bar{z}} + \frac{1}{2\nu}\psi_{\bar{z}}^2 = 0, \tag{26}$$

and it remains to eliminate ψ_{zz}, $\psi_{\bar{z}\bar{z}}$ from Eq. (25) using Eq. (26) and its conjugate. This produces the real equation

$$\begin{aligned}-2\psi_z\psi_{\bar{z}} + \frac{\overline{M}(3-i)}{4M}e^{2\psi/(\nu i)}\psi_{\bar{z}}^2 + \frac{M(3+i)}{4\overline{M}}e^{-2\psi/(\nu i)}\psi_z^2 \\ - \frac{M\overline{M}'\nu}{(1+i)} - \frac{\overline{M}M'\nu}{1-i} + \frac{M\psi_z e^{-\psi(\nu i)}}{(1-i)} + \frac{\overline{M}\psi_{\bar{z}}e^{\psi/(\nu i)}}{(1+i)} = 0.\end{aligned} \tag{27}$$

If $\alpha = (a+ib) = \psi_z e^{-\psi/(\nu i)}$, then Eq. (27) can be written as

$$A\alpha^2 + \overline{A}\overline{\alpha}^2 + B\alpha + \overline{B}\overline{\alpha} + C\alpha\overline{\alpha} + D = 0, \tag{28}$$

where

$$A = \frac{M(3+i)}{4\overline{M}}, \qquad B = \frac{M}{1-i}, \qquad C = -2, \qquad D = -\frac{M\overline{M}'\nu}{(1+i)} - \frac{\overline{M}M'\nu}{(1-i)}, \tag{29}$$

and Eq. (23) can be written as

$$E\alpha + \overline{E}\overline{\alpha} + F = 0, \qquad E = M, \qquad F = \nu i(M\overline{M}' - \overline{M}M'). \tag{30}$$

Elimination of a from Eqs. (28) and (30) produces a quadratic equation for b of the form

$$A_1 b^2 + 2B_1 b + C_1 = 0, \tag{31}$$

where the coefficients A_1, B_1, C_1 are expressed by

$$A_1 = (C - A - \overline{A})(E+\overline{E})^2 - (E-\overline{E})^2(C + A + \overline{A}) + 2(A - \overline{A})(E^2 - \overline{E}^2), \tag{32}$$

$$B_1 = i(A + \overline{A} + C)F(E - \overline{E}) - iF(A - \overline{A})(E + \overline{E}) + i(B\overline{E} - \overline{B}E)(E + \overline{E}), \tag{33}$$

$$C_1 = F^2(A + \overline{A} + C) - F(B + \overline{B})(E + \overline{E}) + D(E + \overline{E})^2. \tag{34}$$

In terms of the complex functions $M(\omega)$, $\overline{M}(\omega)$, the coefficients A_1, B_1, C_1 are given by

$$A_1 = -14M\overline{M},$$

$$B_1 = \frac{7\nu}{2}(M-\overline{M})(M\overline{M}' - \overline{M}M') - (M+\overline{M})\left[M\overline{M} - \frac{i\nu}{2}(M\overline{M}' - \overline{M}M')\right], \tag{35}$$

$$C_1 = \nu^2(M\overline{M}' - \overline{M}M')^2\left[2 - \frac{M}{\overline{M}}\frac{(3+i)}{4} - \frac{\overline{M}}{M}\frac{(3-i)}{4}\right] - \nu M\overline{M}(M+\overline{M})(M'+\overline{M}'). \tag{36}$$

Since b is real, it follows that $B_1^2 > A_1C_1$, which is equivalent to

$$\begin{aligned}&\left\{\frac{7\nu}{2}(M-\overline{M})(M\overline{M}' - \overline{M}M') - (M+\overline{M})[M\overline{M} - \frac{i\nu}{2}(M\overline{M}' - \overline{M}M')]\right\}^2\\ &\quad + 14M\overline{M}\nu^2(M\overline{M}' - \overline{M}M')^2\left[2 - \frac{M}{\overline{M}}\frac{(3+i)}{4} - \frac{\overline{M}}{M}\frac{(3-i)}{4}\right]\\ &\quad + 14\nu M^2\overline{M}^2(M+\overline{M})(M'+\overline{M}') > 0.\end{aligned} \tag{37}$$

It follows that

$$b = \frac{1}{2}\psi_x \sin\left(\frac{\psi}{\nu}\right) - \frac{1}{2}\psi_y \cos\left(\frac{\psi}{\nu}\right) = -\frac{B_1}{A_1} \pm \frac{(B_1^2 - A_1C_1)^{1/2}}{A_1}, \tag{38}$$

and again from Eq. (23) there is a second relation between ψ_x, ψ_y given by

$$2\nu i(M'\overline{M} - M\overline{M}') = [Me^{i\psi/\nu} + \overline{M}e^{-i\psi/\nu}]\psi_x + i[\overline{M}e^{-\psi/\nu} - Me^{i\psi/\nu}]\psi_y. \tag{39}$$

Solving explicitly for ψ_x, ψ_y produces the representations for the velocity components as follows:

$$\begin{aligned}\psi_x &= \frac{2\nu i(M'\overline{M} - M\overline{M}')}{M+\overline{M}}\cos\left(\frac{\psi}{\nu}\right)\\ &\quad + \frac{2i[\overline{M}e^{-i\psi/\nu} - Me^{i\psi/\nu}]}{M+\overline{M}}\left[-\frac{B_1}{A_1} \pm \frac{(B_1^2 - A_1C_1)^{1/2}}{A_1}\right],\end{aligned} \tag{40}$$

$$\begin{aligned}\psi_y &= \frac{2\nu i(M'\overline{M} - M\overline{M}')}{M+\overline{M}}\sin\left(\frac{\psi}{\nu}\right)\\ &\quad - \frac{2[Me^{+i\psi/\nu} + \overline{M}e^{-i\psi/\nu}]}{M+\overline{M}}\left[-\frac{B_1}{A_1} \pm \frac{(B_1^2 - A_1C_1)^{1/2}}{A_1}\right].\end{aligned} \tag{41}$$

Finally, it remains to consider the equation $L'' = 0$. First it is noted from Eq. (21) that ω_x, ω_y can be expressed as

$$\omega_x = \frac{1}{2}\left[Me^{-\psi/(\nu i)} + \overline{M}e^{\psi/(\nu i)}\right], \qquad \omega_y = \frac{1}{2i}\left[Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}\right], \tag{42}$$

and $L'' = 0$ can be written in the form

$$y - \frac{x[Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}]}{i[Me^{-\psi/(\nu i)} + \overline{M}e^{\psi(\nu i)}]} = H\left\{\frac{Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}}{iMe^{-\psi/(\nu i)} + i\overline{M}e^{\psi(\nu i)}}\right\}. \tag{43}$$

Differentiation of Eq. (43) with respect to y, elimination of ω_y, and solving for x and y, gives the representations

$$x = \frac{[Me^{-\psi/(\nu i)} + \overline{M}e^{\psi(\nu i)}]^2}{[(M\overline{M}' - \overline{M}M')(Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}) + \frac{4}{\nu}M\overline{M}\psi_y]} - H'\left\{\frac{Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}}{iMe^{-\psi(\nu i)} + i\overline{M}e^{\psi/(\nu i)}}\right\}, \tag{44}$$

$$\begin{aligned} y = {} & \frac{[M^2e^{-2\psi/(\nu i)} - \overline{M}^2e^{2\psi/(\nu i)}]}{[(M\overline{M}' - \overline{M}M')(Me^{-\psi/(\nu i)} - \overline{M}e^{\psi(\nu i)}) + \frac{4}{\nu}M\overline{M}\psi_y]i} \\ & + H\left\{\frac{Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}}{iMe^{-\psi/(\nu i)} + i\overline{M}e^{\psi/(\nu i)}}\right\} \\ & - \left\{\frac{Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}}{iMe^{-\psi/(\nu i)} + i\overline{M}e^{\psi/(\nu i)}}\right\} H'\left\{\frac{Me^{-\psi/(\nu i)} - \overline{M}e^{\psi(\nu i)}]}{i[Me^{-\psi/(\nu i)} + i\overline{M}e^{\psi/(\nu i)}]}\right\}. \end{aligned} \tag{45}$$

It is a routine but cumbersome calculation to eliminate ψ_y, A_1, B_1, C_1 from Eqs. (35), (36), (41), (44), and (45) to determine x, y as functions of ω and ψ containing the arbitrary functions $M(\omega)$, $\overline{M}(\omega)$, and H. In this case the resulting equations for x, y expressed as functions of ϕ, ψ are general solutions of the Navier-Stokes equations for steady two-dimensional flow. In order that there are real functions ϕ, ψ of (x, y), at least locally in a neighbourhood of a point, it is necessary that the first derivatives x_ω, y_ω, x_ψ, y_ψ are continuous and the Jacobian $\partial(x, y)/\partial(\phi, \psi)$ is neither nonvanishing nor infinite. This excludes flows with stagnation point.

Another way of expressing the solution in a more concise implicit form is

$$y - x\left(\frac{Me^{-\psi/(\nu i)} - \overline{M}e^{\psi(\nu i)}}{iMe^{-\psi/(\nu i)} + i\overline{M}e^{\psi(\nu i)}}\right) = H\left(\frac{Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}}{iMe^{-\psi/(\nu i)} + i\overline{M}e^{\psi/(\nu i)}}\right) \tag{46}$$

and

$$\begin{aligned} & -14\nu M\overline{M}b^2 \\ & + b\{7\nu(M - \overline{M})(M\overline{M}' - \overline{M}M') - (M + \overline{M})[2M\overline{M} - i\nu(M\overline{M}' - \overline{M}M')]\} \\ & + \nu^2(M\overline{M}' - \overline{M}M')^2\left[2 - \frac{M}{\overline{M}}\frac{(3+i)}{4} - \frac{\overline{M}}{M}\frac{(3-i)}{4}\right] \\ & - \nu M\overline{M}(M + \overline{M})(M' + \overline{M}') = 0, \end{aligned} \tag{47}$$

where b is defined by

$$\begin{aligned} b = {} & \frac{\nu i \sin(\psi/\nu)(\overline{M}M' - M\overline{M}')}{[Me^{i\psi/\nu} + \overline{M}e^{-i\psi/\nu}]} - \frac{\nu(M + \overline{M})}{8M\overline{M}[Me^{i\psi/\nu} + \overline{M}e^{-i\psi/\nu}]} \\ & \times \left\{\frac{(Me^{-\psi/(\nu i)} + \overline{M}e^{\psi/(\nu i)})^2}{[x + H'((Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)})/(iMe^{-\psi/(\nu i)} + i\overline{M}e^{\psi/(\nu i)}))]}\right. \\ & \left. - (M\overline{M}' - M'\overline{M})[Me^{-\psi/(\nu i)} - \overline{M}e^{\psi/(\nu i)}]\right\}. \end{aligned} \tag{48}$$

Equations (46)–(48) define the stream function implicitly and the system is suitable for numerical computation for special forms of the arbitrary functions $M(\omega)$, $\overline{M}(\omega)$, and H. As a simple check of the procedure described here, consider the particular forms for M and H given by

$$M(\omega) = k\omega, \quad M(\beta) = c\beta, \tag{49}$$

with k, c real constants. It is found that

$$x + c = \frac{\nu \cos^2(\psi/\nu)}{\psi_y}, \qquad y = \frac{-\nu \sin(\psi/\nu) \cdot \cos(\psi/\nu)}{\psi_y}, \tag{50}$$

where

$$\psi_y = \frac{2}{7} k\omega \cos\left(\frac{\psi}{\nu}\right)[1 \pm \sqrt{1 - 14\nu}]. \tag{51}$$

Elimination of ψ_y gives

$$-\tan\left(\frac{\psi}{\nu}\right) = \frac{y}{x + c}, \qquad \nu < \frac{1}{14}, \tag{52}$$

which corresponds to the flow produced by a potential line source. To present a new viscous solution, let polar coordinates be defined by $x + c = r\cos\theta$, $y = r\sin\theta$ and choose $M(\omega) = e^{i\omega/\nu}$, $H(s) = cs$, where c is a real constant. It follows from Eq. (46) that $\omega + \psi = \nu\theta$ and Eqs. (46), (47) can be decoupled to give

$$14\nu b^2 - 28b\sin(\theta - \psi/\nu) + 8 - 6\cos(2\theta - 2\psi/\nu) = 0. \tag{53}$$

Also from Eq. (48),

$$b = \frac{-\sin(\psi/\nu)}{\cos\theta} - \frac{\nu\cos(\theta - \psi/\nu)}{2\cos\theta}\left[\frac{\cos\theta}{r} - \frac{\sin\theta}{\nu}\right]. \tag{54}$$

The requirement that b is real from Eq. (53) is expressed by $\cos(2\theta - 2\psi/\nu) < (7 - 8\nu)/(7 - 6\nu)$ with $0 < \nu < \frac{7}{8}$ so that the flow only exists in a restricted region of the (x, y)-plane. Equation (53) can be solved for b to give

$$b = \frac{1}{\nu}\sin(\theta - \psi/\nu) \pm \frac{1}{\nu}\left\{\frac{7 - 8\nu - (7 - 6\nu)\cos(2\theta - 2\psi/\nu)}{14}\right\}^{1/2}, \tag{55}$$

and elimination of b from Eqs. (53), (54) determines an implicit equation for the stream function ψ given by

$$\begin{aligned} &\frac{\nu}{2}\frac{\cos(\theta - \psi/\nu)}{\cos\theta}\left[\frac{\sin\theta}{\nu} - \frac{\cos\theta}{r}\right] - \frac{\sin(\psi/\nu)}{\cos\theta} \\ &\quad = \frac{1}{\nu}\sin(\theta - \psi/\nu) \pm \frac{1}{\nu}\left\{\frac{7 - 8\nu - (7 - 6\nu)\cos(2\theta - 2\psi/\nu)}{14}\right\}^{1/2}. \end{aligned} \tag{56}$$

It is not straightforward to calculate ψ explicitly as a function of r, θ, but the equations of the streamlines, $\psi = K$, a constant, are readily found in polar form as follows to be

$$\begin{aligned} &\frac{\nu}{2}\frac{\cos(\theta - K/\nu)}{\cos\theta}\left[\frac{\sin\theta}{\nu} - \frac{\cos\theta}{r}\right] - \frac{\sin(K/\nu)}{\cos\theta} \\ &\quad = \frac{1}{\nu}\sin(\theta - K/\nu) \pm \frac{1}{\nu}\left\{\frac{7 - 8\nu - (7 - 6\nu)\cos(2\theta - 2K/\nu)}{14}\right\}^{1/2}, \end{aligned} \tag{57}$$

which exist for $\cos(2\theta - 2K/\nu) < (7-8\nu)/(7-6\nu)$ with $0 < \nu < 7/8$. A slightly more general solution is found by choosing $M(\omega) = e^{i\alpha\omega}$, where α is a real constant. The Eqs. (46), (47) can still be decoupled to provide an implicit equation for the stream function.

References

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