QUARTERLY OF APPLIED MATHEMATICS
VOLUME LI, NUMBER 1
MARCH 1993, PAGES 69-79

PLANE DEFORMATIONS OF MEMBRANES AND NETWORKS WITH CIRCULAR CORDS

By

JINGYU SHI

Nottingham University, U.K., and Queensland University, Australia

Abstract. Deformations of membranes and networks formed with two families of highly elastic cords are considered. The cords are along the radial and circumferential directions of concentric circles and are perfectly joined together at their intersection points so that there is no slip relative to each other. The deformations studied include: (1) symmetric deformation of circular arcs or annuli, (2) straightening of circular arcs and (3) inverse bending of circular arcs. It is found that the second is a universal deformation, which satisfies the equilibrium equations for any constitutive relations of the material of the cords.

1. Introduction. The infinitesimal plane deformation of membranes formed by two families of parallel straight cords has been discussed by Genensky and Rivlin [1]. It is assumed that the cords are perfectly flexible and the membrane cannot resist shearing. They have obtained solutions to the displacement boundary-value problem, the traction boundary-value problem and the mixed boundary-value problem. The finite deformations of such membranes have been considered by Green and Shi [2, 3]. Green and Shi [4] also investigated some deformations of discrete networks formed of straight elastic cords while Shi [5] showed some solutions to the deformations of the membranes with shearing resistance. Some other deformations and membranes are considered by Shi in [6].

In the present paper we will consider the plane deformations of membranes and networks formed of two families of perfectly flexible cords, with one family lying along the radial direction while the other lies along the circumferential (chord in case of network) direction. When we refer to membrane we assume that the cords are continuously distributed so that the theory of continuum mechanics can be employed, and that the radial cords have thinner central-ward end than outward end so that the numbers of radial cords crossing arcs with the same central angle interval but different radii are the same. The latter assumption might not be realistic but it makes mathematics convenient.

In Sec. 2 we discuss the circularly symmetrical deformation of the membrane. The analytic solutions are found in terms of the modified Bessel functions when

Received February 27, 1991.
1991 *Mathematics Subject Classification.* Primary 73G05.

the mechanical response functions are linear. As an example, a mixed boundary value problem is illustrated. The straightening and inverse bending of a circular arc of the membrane are discussed in Secs. 3 and 4 respectively. It is found that the straightening deformation satisfies the equilibrium conditions for any mechanical response functions. So it is a universal deformation. Then in Secs. 4, 5, and 6 the corresponding deformations of the networks are discussed.

2. Symmetrical deformation of membranes with circular cords. We will, in the present section, investigate the deformation of a circular (arc) membrane formed of two families of elastic cords lying in the radial and circumferential directions. The cords are joined together at their intersection points so that there is no slip. To form the membrane the cords must be continuously distributed and be thinner at the central-ward end than outward end while the mechanical property is the same along the cord; i.e., to obtain the same extensions at the central-ward and outward ends the same forces should be applied. We assume that the cords are perfectly flexible and the membrane cannot resist shearing. It is convenient to introduce circular polar coordinate systems (R, Θ) and (r, θ) with the centre of the circles as poles, for undeformed and deformed membranes respectively. The general expression for a plane deformation from (R, Θ) to (r, θ) is

$$r = r(R, \Theta), \qquad \theta = \theta(R, \Theta). \tag{2.1}$$

The corresponding deformation gradient tensor and right Cauchy-Green deformation tensor are (Spencer [7])

$$\mathbf{F}^* = \begin{pmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} \\ r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} \end{pmatrix}, \tag{2.2}$$

$$\mathbf{C}^* = \begin{pmatrix} \left[\dfrac{\partial r}{\partial R}\right]^2 + \left[r\dfrac{\partial \theta}{\partial R}\right]^2 & \dfrac{1}{R}\dfrac{\partial r}{\partial R}\dfrac{\partial r}{\partial \Theta} + \dfrac{r^2}{R}\dfrac{\partial \theta}{\partial \Theta}\dfrac{\partial \theta}{\partial R} \\ \dfrac{1}{R}\dfrac{\partial r}{\partial R}\dfrac{\partial r}{\partial \Theta} + \dfrac{r^2}{R}\dfrac{\partial \theta}{\partial \Theta}\dfrac{\partial \theta}{\partial R} & \left[\dfrac{1}{R}\dfrac{\partial r}{\partial \Theta}\right]^2 + \left[\dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta}\right]^2 \end{pmatrix}. \tag{2.3}$$

For simplicity, the deformation we consider here is one in which the radial and circumferential cords still lie along the radial and circumferential directions respectively, i.e.,

$$r = r(R), \qquad \theta = \theta(\Theta). \tag{2.4}$$

Then the deformation gradient and right Cauchy-Green deformation tensors are simply

$$\mathbf{F}^* = \begin{pmatrix} r' & 0 \\ 0 & r\theta'/R \end{pmatrix}, \tag{2.5}$$

$$\mathbf{C}^* = \begin{pmatrix} [r']^2 & 0 \\ 0 & [r\theta'/R]^2 \end{pmatrix}, \tag{2.6}$$

which give the stretch ratios

$$\lambda_R = r', \qquad \lambda_\Theta = r\theta'/R, \tag{2.7}$$

in radial and circumferential directions respectively, where the primes denote the derivatives with respect to their arguments. For this deformation, furthermore, the equilibrium of a small arc element enclosed by two segments of radial cords and two of circumferential ones gives

$$\frac{\partial T_\Theta}{\partial \Theta} = 0, \qquad \frac{\partial T_R}{\partial R} - \theta' T_\Theta = 0, \tag{2.8}$$

where T_Θ is tension carried by circumferential cords crossing per unit of initial length of radial cords, the usual stress definition, and T_R is tensile carried by radial cords located in unit central angle, being of the dimension of force. In addition, the response functions of the cords take the form

$$T_R = \begin{cases} T_R(\lambda_R) & \text{for } \lambda_R > 1, \\ 0 & \text{for } \lambda_R \le 1; \end{cases} \tag{2.9a}$$

$$T_\Theta = \begin{cases} T_\Theta(\lambda_\Theta) & \text{for } \lambda_\Theta > 1, \\ 0 & \text{for } \lambda_\Theta \le 1; \end{cases} \tag{2.9b}$$

or

$$\begin{aligned} \lambda_R &= \lambda_R(T_R) \quad \text{for } T_R > 0, \\ \lambda_\Theta &= \lambda_\Theta(T_\Theta) \quad \text{for } T_\Theta > 0. \end{aligned} \tag{2.10}$$

In the following we assume that $\lambda_R \ge 1$ and $\lambda_\Theta \ge 1$.

The first equation of (2.8) and the second equation of (2.10) indicate that

$$T_\Theta = T_\Theta(R) \quad \text{and} \quad \lambda_\Theta = \lambda_\Theta(R).$$

From the second equation of (2.7) we can see that θ' must be constant, say C, so

$$\theta = C\Theta + \theta_0, \tag{2.11}$$

in which θ_0 is a constant. Then we have

$$\lambda_\Theta = Cr/R. \tag{2.12}$$

The second of the equilibrium equation (2.8) yields the governing equation for $r = r(R)$

$$\frac{d}{dR}[T_R(r')] - CT_\Theta(Cr/R) = 0, \tag{2.13}$$

which is a nonlinear second order ordinary differential equation if the response functions are nonlinear and therefore it is generally very difficult to obtain an analytic solution. Even if the equations in (2.9) are of linear form

$$T_R = E_1(\lambda_R - 1), \qquad T_\Theta = E_2(\lambda_\Theta - 1) \tag{2.14}$$

and then (2.13) is also linear

$$r'' - C^2 k^2 r/R = -Ck^2, \tag{2.15}$$

with $k^2 = E_2/E_1$, we could still not obtain a general solution expressed in terms of simple functions.

In fact by some transformation, the solution may be expressed in terms of Bessel functions as follows (see Mclachlan [8, p. 12] and [9, p. 102])

$$r = R/C + R^{1/2}\{C_1 I_1(2CkR^{1/2}) + C_2 K_1(2CkR^{1/2})\}. \tag{2.16}$$

Here $I_1(z)$ is the modified Bessel function of the first kind of order one and $K_1(z)$ is the modified Bessel function of the second kind of order one.

The constant C is determined from the deformed and undeformed central angles. If the arc forms a whole circle then $C = 1$ and

$$r = R + R^{1/2}\{C_1 I_1(2kR^{1/2}) + C_2 K_1(2kR^{1/2})\}, \tag{2.17}$$

$$\lambda_\Theta = r/R, \tag{2.18}$$

and choosing $\theta_0 = 0$ gives

$$\theta = \Theta. \tag{2.19}$$

Using the series expansion form of the modified Bessel functions, we have

$$\begin{aligned} r = R &+ C_1 \sum_{m=0}^{\infty} \frac{k^{2m+1}}{m!(m+1)!} R^{m+1} \\ &+ C_2 \left\{ \frac{1}{2k} + kR\left[\ln(kR^{1/2}) + \gamma - \frac{1}{2}\right] \right. \\ &\quad + \sum_{m=1}^{\infty} \frac{k^{2m+1}}{m!(m+1)!}\left[\ln(kR^{1/2}) + \gamma - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right. \\ &\quad \left.\left. - \frac{1}{2(m+1)}\right] R^{m+1} \right\}, \end{aligned} \tag{2.20}$$

and the stretch ratio in the circumferential direction

$$\begin{aligned} \lambda_\theta = \frac{r}{R} = 1 &+ C_1 \sum_{m=0}^{\infty} \frac{k^{2m+1}}{m!(m+1)!} R^{m} \\ &+ C_2 \left\{ \frac{1}{2kR} + k\left[\ln(kR^{1/2}) + \gamma - \frac{1}{2}\right] \right. \\ &\quad + \sum_{m=1}^{\infty} \frac{k^{2m+1}}{m!(m+1)!}\left[\ln(kR^{1/2}) + \gamma - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right. \\ &\quad \left.\left. - \frac{1}{2(m+1)}\right] R^{m} \right\}. \end{aligned} \tag{2.21}$$

Here $\gamma = 0.5772\ldots$ is Euler's constant. Then carrying out the derivative of the expression in (2.20), we obtain the stretch ratio in radial direction

$$\begin{aligned} \lambda_R &= \frac{dr}{dR} \\ &= 1 + C_1 \sum_{m=0}^{\infty} \frac{k^{2m+1}}{(m!)^2} R^m + C_2 \left\{ k[\ln(kR^{1/2}) + \gamma] + \sum_{m=1}^{\infty} \frac{k^{2m+1}}{(m!)^2} \right. \\ &\quad \left. \times \left[\ln(kR^{1/2}) + \gamma - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right] R^m \right\}. \end{aligned} \tag{2.22}$$

The author [6] determined the constants C_1 and C_2 for a variety of boundary value problems. One of these is a circular annulus fixed on the internal edge $R = B$

and pulled on the external edge $R = A$ by tension T_A, i.e.,

$$r = B \quad \text{at } R = B \quad \text{and} \quad T_R = T_A \quad \text{at } R = A. \tag{2.23}$$

Then from (2.20), (2.22), and (2.14), we have

$$\begin{aligned} C_1 &= -\frac{T_A}{E_1} \frac{\widetilde{K}(k, B)}{\widetilde{I}(k, B)K^*(k, A) - I^*(k, A)\widetilde{K}(k, B)}, \\ C_2 &= \frac{T_A}{E_1} \frac{\widetilde{I}(k, B)}{\widetilde{I}(k, B)K^*(k, A) - I^*(k, A)\widetilde{K}(k, B)}, \end{aligned} \tag{2.24}$$

where

$$\widetilde{I}(k, B) = \sum_{m=0}^{\infty} \frac{k^{2m+1}}{m!(m+1)!} B^{m+1},$$

$$\begin{aligned} \widetilde{K}(k, B) = \frac{1}{2k} + kB\left[\ln(kB^{1/2}) + \gamma - \frac{1}{2}\right] + \sum_{m=1}^{\infty} \frac{k^{2m+1}}{m!(m+1)!} \\ \times \left[\ln(kB^{1/2}) + \gamma - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) - \frac{1}{2(m+1)}\right] B^{m+1}, \end{aligned}$$

$$I^*(k, A) = \sum_{m=0}^{\infty} \frac{k^{2m+1}}{(m!)^2} A^m,$$

$$\begin{aligned} K^*(k, A) &= k\ln(kA^{1/2}) + \gamma \\ &+ \sum_{m=1}^{\infty} \frac{k^{2m+1}}{(m!)^2} \left[\ln(kA^{1/2}) + \gamma - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right] A^m. \end{aligned} \tag{2.25}$$

3. Straightening a circular arc membrane. Now we discuss the deformation of straightening an arc with the same cords distribution as that in Sec. 2. We assume that after the deformation the radial cords lie in the x_1-direction and the circumferential cords in the x_2-direction; i.e., the deformation is of the form

$$x_1 = x_1(R), \qquad x_2 = x_2(\Theta). \tag{3.1}$$

The corresponding stretch ratios are given by

$$\lambda_R = x_1'(R), \qquad \lambda_\Theta = x_2'(\Theta)/R. \tag{3.2}$$

If we consider the equilibrium of a small rectangular element deformed from an arc element with boundaries along the cords, we find that the equilibrium conditions are

$$\frac{\partial T_\Theta}{R\partial\Theta} = 0, \qquad \frac{\partial T_R}{\partial R} = 0, \tag{3.3}$$

where T_Θ and T_R have the same meanings as in Sec. 2. Besides these we have the general relations in (2.9).

It follows, therefore from (2.9), (3.2), and (3.3), that

$$T_R = C_1, \tag{3.4}$$

$$T_\Theta = T_\Theta(x_2'(\Theta)/R) = T_\Theta^*(R), \tag{3.5}$$

where C_1 is a constant and $T^*_\Theta(R)$ is function of R only. The last equality in (3.5) implies that

$$x_2'(\Theta) = C_2, \tag{3.6}$$

with C_2 being another constant. Thus (3.5) and the second equation of (3.2) yield

$$T_\Theta = T_\Theta(C_2/R), \tag{3.7}$$

$$\lambda_\Theta = C_2/R. \tag{3.8}$$

The equation (3.4) with the first equation of (2.10) implies that λ_R is also a constant, given by

$$\lambda_R = \lambda_R(C_1). \tag{3.9}$$

Integrating the first equation of (3.2), with the aid of (3.9) and (3.6), we have

$$x_1 = \lambda_R(C_1)R + x_{10}, \qquad x_2 = C_2\Theta + x_{20}. \tag{3.10}$$

Here x_{10} and x_{20} are constants representing rigid body displacement.

We can see, from the above procedure, that the deformation (3.10) satisfies the equilibrium equations (3.3) for any deformation-stress relation (2.9). So it is a universal deformation. This deformation can be maintained by the tractions on the boundary given by (3.4) and (3.7). The traction T_R does not depend on the constitutive relation as T_Θ does. If T_Θ increases as λ_Θ increases, then it decreases as R increases.

4. Inverse bending of circular arcs. In the previous section, the arcs are straightened. In this section we will discuss the deformation in which the straightened arcs are further bent inversely; i.e., the central-ward and outward ends of the radial cords before the total deformation point outward and central-ward respectively after the deformation and the cords along the circumferential direction of the circles remain along circumferential direction but with inverse bending direction.

We assume that the deformation is also symmetric, i.e., (2.4)–(2.6) hold. In this case, r decreases as R increases and θ decreases as Θ increases. So $r' < 0$ and $\theta' < 0$. Then the equations in (2.7) should be replaced by

$$\lambda_R = -r', \qquad \lambda_\Theta = -\theta' r/R. \tag{4.1}$$

The equations in (2.8)–(2.11) are still valid here, but (2.12) should be written as

$$\lambda_\Theta = -Cr/R, \tag{4.2}$$

since $\lambda_\Theta > 0$ while $C = \theta' < 0$. Introducing a new constant $C^* = -C > 0$, we have

$$\lambda_\Theta = C^* r/R. \tag{4.3}$$

With the aid of (4.1), (4.3), and (2.14), we derive, from the second equation of (2.8), the equation for $r(R)$

$$r'' - C^{*2}k^2 r/R = -C^* k^2, \tag{4.4}$$

which is of the same form as (2.15). So the general solution is given by (2.16) with C replaced by C^*. The other alternatives are along the boundary. If after the deformation r_2 and r_1 are the radii of the circumferential cords which are initially along the internal and external circles, then $r_2 > r_1$.

5. Symmetric deformations of circular arc networks. Now we turn to the discussion of the deformation of an elastic network whose cords lie along the radial directions and chords of circles. The radial cords are joined with the chord cords at their intersection. The intersection points of a chord cord with radial cords are on the same circle. We assume that the cords are evenly distributed: the adjacent chord cords are apart by a distance H and adjacent radial cords are apart by an angle α. Part of the network with five radial cords is shown in Fig. 1(a), where A is radius of the cord zero and L_i $(i = 0, 1, \dots)$ is length of the chord.

The deformation is such that the intersection points on a circle are still on the circle deformed from it. Then after the deformation the radial cords are still evenly distributed, apart by an angle β, but the distance between the chord cords are different, being h_i $(i = 0, 1, \dots)$. Assume that A, L_i become a, l_i respectively. Here we have relations

$$L_0 = 2A\sin(\alpha/2), \qquad l_0 = 2a\sin(\beta/2).$$

The tensions and stretch ratios do not vary along the chord cord. The deformed configuration of the part of the network shown in Fig. 1(a) is illustrated in Fig. 1(b), where T_i and S_i $(i = 0, 1, 2, \dots)$ are the tensions carried by radial cords and chord cords respectively.

From the symmetric geometry, we have

$$\begin{aligned} L_i &= L_{i-1} + 2H\sin(\alpha/2) = L_0 + 2iH\sin(\alpha/2), \\ l_i &= l_{i-1} + 2h_{i-1}\sin(\beta/2) = l_0 + 2\sin(\beta/2)\sum_{k=0}^{i-1} h_k, \qquad (i = 1, 2, \dots). \end{aligned} \tag{5.1}$$

The stretch ratios of each segment of radial and chord cords are given by

$$\lambda_i = h_i/H, \quad \lambda_i^* = l_i/L_i \qquad (i = 0, 1, 2, \dots). \tag{5.2}$$

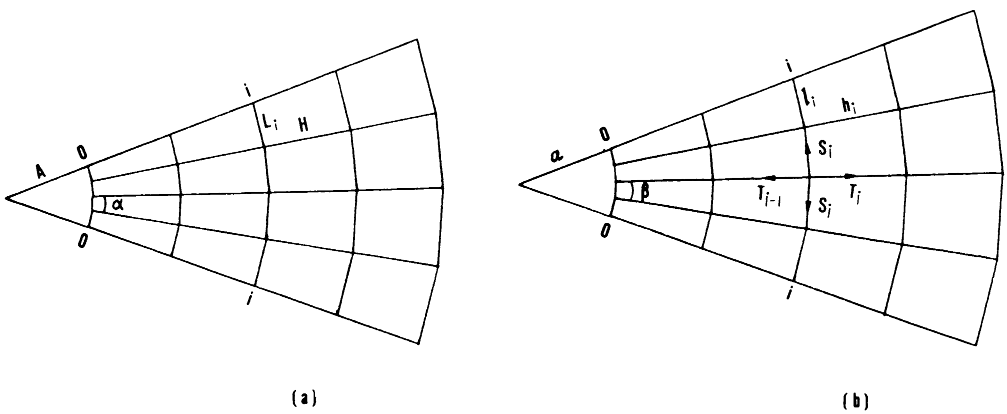

FIG. 1. Network with cords along radial and chord directions of circles, (a) before the deformation, (b) after the deformation

Substituting (5.1) into the second equation of (5.2), we have

$$\begin{aligned}\lambda_i^* &= \frac{l_{i-1} + 2h_{i-1}\sin(\beta/2)}{L_0 + 2iH\sin(\alpha/2)} \\ &= \frac{\lambda_{i-1}^*[\mu + (i-1)\alpha_0] + \lambda_{i-1}\beta_0}{\mu + i\alpha_0}, \qquad (i = 1, 2, \ldots),\end{aligned} \tag{5.3}$$

where

$$\mu = L_0/H, \qquad \beta_0 = 2\sin(\beta/2), \qquad \alpha_0 = 2\sin(\alpha/2).$$

Therefore, it follows that

$$\beta_0\lambda_{i-1} = [\mu + i\alpha_0]\lambda_i^* - [\mu + (i-1)\alpha_0]\lambda_{i-1}^* \qquad (i = 1, 2, \ldots). \tag{5.4}$$

The balance of the force at the ith chord cord yields

$$T_i = T_{i-1} + S_i\beta_0 \qquad (i = 0, 1, 2, \ldots). \tag{5.5}$$

The relations between the tensions and stretch ratios are assumed to be

$$\begin{aligned} T_i &= K(\lambda_i - 1), \\ S_i &= K^*(\lambda_i^* - 1), \end{aligned} \qquad (i = 0, 1, 2, \ldots). \tag{5.6}$$

Here K and K^* are elastic constants of the radial and chord cords respectively. Then substituting (5.6) into (5.5) and using (5.4) we derive the equations for λ_i^*

$$\begin{aligned} &[\mu + (i+1)\alpha_0]\lambda_{i+1}^* - [2(\mu + i\alpha_0) + \beta_0^2 k_0^2]\lambda_i^* \\ &\quad + [\mu + (i-1)\alpha_0]\lambda_{i-1}^* + \beta_0^2 k_0^2 = 0, \qquad (i = 1, 2, \ldots), \end{aligned} \tag{5.7}$$

where $k_0^2 = K^*/K$.

Assume that there are $N+1$ points (or chord cords), numbered as $0, 1, 2, \ldots, N$. The equation (5.7) holds at the interior points $1, 2, \ldots, N-1$ along radial cords. At the two end points of the radial cord, we can derive two additional equations according to the specified boundary conditions. So it forms a system of $N+1$ linear algebraic equations for $N+1$ variables λ_i^* $(i = 0, 1, 2, \ldots, N)$. Therefore the number of equations is equal to that of variables and the problem is well posed. After the stretch ratios λ_i^* $(i = 0, 1, 2, \ldots, N)$ of the chord cords are determined from (5.7) with the boundary conditions, the stretch ratios λ_i $(i = 0, 1, 2, \ldots, N-1)$ of the radial cords can be found from (5.4), then the tensions from (5.6). When the chord cords form whole circles, $\beta = \alpha$, i.e., $\beta_0 = \alpha_0$.

In passing, instead of (5.7) as the governing equation, we can derive the equations for λ_i $(i = 1, 2, \ldots, N-1)$, which are

$$\begin{aligned} &[\mu + (i+1)\alpha_0]\lambda_{i+1} - [2(\mu + i\alpha_0) + \alpha_0 + \beta_0^2 k_0^2]\lambda_i + \\ &\quad [\mu + i\alpha_0]\lambda_{i-1} + \beta_0\alpha_0 k_0^2 = 0 \qquad (i = 1, 2, \ldots, N-1). \end{aligned} \tag{5.8}$$

The stretch ratios of the chord cords are given by

$$\lambda_i^* = 1 + (\lambda_i - \lambda_{i-1})/k_0\beta_0. \tag{5.9}$$

The equation (2.15) is derived from the continuum model-membrane, while the equation (5.7) is derived from the discrete model-network. Now we try to derive

(2.15) from (5.7) by taking the limit: H, α, K, and $K^* \to 0$ in such a way that K/α and K^*/H are finite, and $\beta \to 0$. We can see that, by this procedure of limiting,

$$\lim_{K^*, H \to 0} \frac{K^*}{H} = E_2, \qquad \lim_{K, \alpha \to 0} \frac{K}{\alpha} = E_1,$$

and

$$\lim_{\beta, \alpha \to 0} \frac{\beta_0}{\alpha_0} = \lim_{\beta, \alpha \to 0} \frac{\beta}{\alpha} = C.$$

Rearranging (5.7), we have

$$[\mu + i\alpha_0](\lambda^*_{i+1} - 2\lambda^*_i + \lambda^*_{i-1}) + \alpha_0(\lambda^*_{i+1} - \lambda^*_{i-1}) - \beta_0^2 k_0^2 \lambda^*_i + \beta_0^2 k_0^2 = 0,$$

or

$$[\mu + i\alpha_0]\frac{H}{\alpha_0}\frac{\lambda^*_{i+1} - 2\lambda^*_i + \lambda^*_{i-1}}{H^2} + 2\frac{\lambda^*_{i+1} - \lambda^*_{i-1}}{2H} - \frac{\beta_0^2 K^* \alpha_0}{\alpha_0^2 K H}\lambda^*_i + \frac{\beta_0^2 K^* \alpha_0}{\alpha_0^2 K H} = 0.$$

From the geometry, it follows that

$$(\mu + i\alpha_0)H/\alpha_0 = L_0/\alpha_0 + iH = R.$$

Now taking the limit, we obtain

$$R\lambda^{*\prime\prime} + 2\lambda^{*\prime} - k^2 C^2 \lambda^* + k^2 C^2 = 0.$$

This is the same as that derived by introducing (2.12) into (2.15) with $\lambda_\Theta = \lambda^*$.

6. Straightening of a circular arc network. Now we consider a special deformation of the network described in Sec. 5, with general relationships between the tensions and stretch ratios, in which the chord cords are staightened such that the radial cords are parallel to each other and perpendicular to the staightened chord cords. Therefore after the deformation, every segment of the chord cords has the same length l and

$$\lambda^*_i = \frac{l}{L_i} = \frac{\lambda^*_0 \mu}{\mu + i\alpha_0}, \qquad \lambda_i = \frac{h_i}{H}. \tag{6.1}$$

Equilibrium of the points along a radial cord gives rise to

$$T_i = T_{i-1} = T_{i-2} = \cdots = T_0 = C_1. \tag{6.2}$$

Then from any tension-stretch ratio relation, we have

$$\lambda_i = \lambda_{i-1} = \lambda_{i-2} = \cdots = \lambda_0 = \text{constant}, \tag{6.3}$$

and

$$h_i = \text{constant}.$$

The constant C_1 is related to λ_0 by the relation between the tension and stretch ratios. When λ^*_0 is given, we can determine λ^*_i from the first of (6.1), and then the tension of the chord cord from the constitutive relation $S_i = S(\lambda^*_i)$.

This deformation is again a universal deformation and should be maintained by the tractions calculated from the above equations.

Rearranging (6.1), we have

$$\lambda_i^* = \frac{\lambda_0^* A}{R}, \tag{6.4}$$

where $A = \mu H/\alpha_0$ is the radius of the 0th chord cord. This is the same as that in (3.8) with $C_2 = \lambda_0^* A$. We again derive the result for the continuum model by using the discrete model.

7. Inverse bending of circular arc networks. As in Sec. 4, we investigate in the present section the deformation in which the straightened arc network, discussed in Sec. 6, is further bent inversely. In this case the deformation is symmetric, then corresponding to (5.1) we have

$$\begin{aligned} L_i &= L_{i-1} + 2H\sin(\alpha/2) = L_0 + 2iH\sin(\alpha/2), \\ l_i &= l_{i-1} - 2h_{i-1}\sin(\beta/2) = l_0 - 2\sin(\beta/2)\sum_{k=0}^{i-1} h_k, \qquad (i = 1, 2, \ldots). \end{aligned} \tag{7.1}$$

With the definition in (5.2), we have, instead of (5.3) and (5.4),

$$\begin{aligned} \lambda_i^* &= \frac{l_{i-1} - 2h_{i-1}\sin(\beta/2)}{L_0 + 2iH\sin(\alpha/2)} \\ &= \frac{\lambda_{i-1}^*[\mu + (i-1)\alpha_0] - \lambda_{i-1}\beta_0}{\mu + i\alpha_0} \qquad (i = 1, 2, \ldots), \end{aligned} \tag{7.2}$$

$$\beta_0\lambda_{i-1} = [\mu + (i-1)\alpha_0]\lambda_{i-1}^* - [\mu + i\alpha_0]\lambda_i^* \qquad (i = 1, 2, \ldots). \tag{7.3}$$

The balance of the force at the ith chord cord yields

$$T_i = T_{i-1} - S_i\beta_0 \qquad (i = 0, 1, 2, \ldots). \tag{7.4}$$

Then substituting the constitutive relations (5.6) into (7.4) and using (7.3) we derive the equations for λ_i^* $(i = 1, 2, \ldots)$.

$$\begin{aligned} &[\mu + (i+1)\alpha_0]\lambda_{i+1}^* - [2(\mu + i\alpha_0) + \beta_0^2 k_0^2]\lambda_i^* \\ &\quad + [\mu + (i-1)\alpha_0]\lambda_{i-1}^* + \beta_0^2 k_0^2 = 0 \qquad (i = 1, 2, \ldots), \end{aligned} \tag{7.5}$$

which is the same as (5.7). Therefore the problem is complete when the conditions at the points on the boundary are specified.

Acknowledgment. Part of the work presented here is from the author's Ph.D. thesis submitted to Nottingham University with the support from the Agricultural Ministry of the People's Republic of China and the ORS Awards Committee of the U.K. The remaining research was conducted at the University of Queensland when the author was appointed as a Raybould Fellow. This financial support is strongly appreciated and the author is very grateful to Dr. W. A. Green for his supervision of the thesis and to Dr. V. G. Hart for his useful suggestions and careful check.

REFERENCES

[1] S. M. Genensky and R. S. Rivlin, *Infinitesimal plane strain in a network of elastic cords*, Arch. Rational Mech. Anal. **4**, 30–44 (1959–60)

[2] W. A. Green and Jingyu Shi, *Plane deformations of membranes formed with elastic cords*, Quart. J. Mech. Appl. Math. **43**, 317–333 (1990)

[3] ——, *Degenerate deformations and uniqueness in highly elastic networks*, Quart. Appl. Math. **50**, 501–516 (1992)

[4] ——, *Deformations of discrete elastic networks*, IMA J. Appl. Math. **45**, 99–113 (1990)

[5] J. Shi, *Plane deformations of shear-resistant membranes formed of elastic cords*, J. Engrg. Math., to appear

[6] ——, *Elastic networks and membranes*, Ph.D. thesis, Nottingham University, U.K., 1988

[7] A. J. M. Spencer, *Continuum Mechanics*, Longman, New York, 1980

[8] N. W. Mclachlan, *Ordinary Non-Linear Differential Equations in Engineering and Physical Sciences*, Oxford Press, Oxford, 1950

[9] ——, *Bessel Functions for Engineers*, Oxford Press, Oxford, 1946