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SOME NONSTANDARD PROBLEMS FOR THE POISSON EQUATION

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1. Introduction. The motivation for this work comes from two different sources. The first is some closely related work of Backus [1] and Magnanini [7, 8] who consider a class of nonstandard boundary value problems for the Laplace equation—problems which arise in gravity and geomagnetic studies. The second comes from the seminal papers of Serrin [13] and Weinberger [18] which deal primarily with a special class of overdetermined problems for the Poisson equation.

In the Backus and Magnanini papers, one is interested in studying the boundary value problem

$$\Delta u = 0 \quad \text{in } S, \qquad |\operatorname{grad} u| = g > 0 \quad \text{on } \partial S, \tag{1.1}$$

where Δ is the Laplace operator, S is the unit ball in R^N with boundary ∂S, and g is a prescribed function. Serrin and Weinberger ask the question: if Ω is a simply connected domain in R^N with smooth boundary $\partial\Omega$ and if u is a solution of

$$\Delta u = -2 \quad \text{in } \Omega, \qquad u = 0, \quad |\operatorname{grad} u| = \text{const} \quad \text{on } \partial\Omega, \tag{1.2}$$

must Ω be a ball? The authors answered this question in the affirmative.

In this paper we study two related classes of problems. In the first, we investigate uniqueness and comparison results for solutions (if they exist) of

$$\Delta u = -2 \quad \text{in } \Omega \subset R^N, \qquad |\operatorname{grad} u| = g \geq 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

In the second, the problem is related to the problem (1.2) of Serrin and Weinberger, but now the magnitude of the gradient of the solution is prescribed to be constant not on $\partial\Omega$, but on $\partial\Omega_1$, where $\Omega_1 \subset \Omega$. Of course, Ω cannot be a ball unless Ω_1 is itself a ball, but in some cases in which Ω_1 is not a ball, we are able to characterize the allowable surface Ω. When Ω_1 is a ball, we show that if Ω_1 is appropriately situated in Ω and Ω is restricted to be convex, then Ω must be a ball and the prescribed constant c must be exactly $2/N$ times the radius of Ω_1. In any case,

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we show that either Ω is a ball or the constant value of c on $\partial\Omega_1$ must be greater than the constant that is obtained when Ω is a concentric ball.

We discuss the first problem in Sec. 2 and the second problem in Sec. 3. In Sec. 4 of this paper, we indicate how our results extend to a class of nonlinear problems such as those arising in the modeling of torsional creep [12], and in Sec. 5, we present a few applications of the foregoing results.

Throughout this paper, we assume that $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, unless otherwise specified, and use the symbol $\partial/\partial n$ to denote the outward normal derivative operator on $\partial\Omega$.

2. Problem I. We are interested in solutions of

$$\Delta u = -2 \quad \text{in } \Omega, \qquad |\operatorname{grad} u| = g \geq 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where Ω is a simply connected domain in R^N. Clearly, solutions cannot exist for arbitrary functions g, but our concern here will be primarily with the question of uniqueness of solutions (assumed to exist) and with comparing solutions corresponding to different functions g. It should be remarked that, for our problem, the uniqueness question is much easier to deal with than the corresponding question faced by Backus and Magnanini. In fact, we establish the following:

THEOREM 1. (α) If there exists a solution u of (2.1) that satisfies $\partial u/\partial n \leq 0$ on $\partial\Omega$, then this solution is unique up to an additive constant.

(β) If there exists a solution u_1 of (2.1) corresponding to a function g_1 with $\partial u_1/\partial n \leq 0$ on $\partial\Omega$, then there is no solution u_2 corresponding to g_2 if $g_2 < g_1$ on $\partial\Omega$.

(γ) If there exists a solution u_1 of (2.1) corresponding to g_1 and a solution u_2 corresponding to g_2, where $g_2 > g_1$ on $\partial\Omega$, then at some point on $\partial\Omega$,

$$\frac{\partial u_2}{\partial n} > -\frac{\partial u_1}{\partial n}. \tag{2.2}$$

Proof. Let u_1 denote the assumed solution for which $\partial u_1/\partial n \leq 0$ on $\partial\Omega$. To establish (α), we assume there is a second solution u_2 and note that $w := u_2 - u_1$ must be harmonic in Ω. Thus w takes its maximum value at some point P on $\partial\Omega$ and its minimum value at some point Q on $\partial\Omega$.

By the boundary condition in (2.1), we have

$$\nabla w \cdot \nabla(w + 2u_1) = 0 \tag{2.3}$$

on $\partial\Omega$. Since $\operatorname{grad}_S w$, the surface gradient of w, must vanish at P and Q, it follows from (2.3) that at both P and Q

$$\frac{\partial w}{\partial n}\left(\frac{\partial w}{\partial n} + 2\frac{\partial u_1}{\partial n}\right) = 0. \tag{2.4}$$

At Q, $\partial w/\partial n$ cannot be positive, and since by assumption $\partial u_1/\partial n \leq 0$ on $\partial\Omega$, neither can it be negative. Hence $\partial w(Q)/\partial n = 0$. But by Hopf's second principle [3], we conclude $w \equiv \text{const}$ in Ω and thus (α) is proved.

To establish (β), we note that by our assumptions, (2.3) now becomes

$$\nabla w \cdot \nabla(w + 2u_1) = g_2^2 - g_1^2 < 0 \tag{2.5}$$

on $\partial\Omega$. Equation (2.4) is now replaced by

$$\frac{\partial w}{\partial n}\left(\frac{\partial w}{\partial n}+2\frac{\partial u_1}{\partial n}\right)<0 \tag{2.6}$$

at the minimum point Q. But this is impossible due to the fact that at a minimum point Q, the left side of (2.6) cannot be negative. Thus no such u_2 can exist and (β) is proved.

The proof of (γ) follows along the same line of argument, but now (2.5) is replaced by

$$\nabla w\cdot\nabla(w+2u_1)>0 \tag{2.7}$$

on $\partial\Omega$ and (2.6) by

$$\frac{\partial w}{\partial n}\left(\frac{\partial w}{\partial n}+2\frac{\partial u_1}{\partial n}\right)>0 \tag{2.8}$$

at P and Q. Evaluating at P and rewriting, we have, since $\partial w/\partial n>0$,

$$\frac{\partial w}{\partial n}+2\frac{\partial u_1}{\partial n}=\frac{\partial u_2}{\partial n}+\frac{\partial u_1}{\partial n}>0$$

at P, i.e., (2.2) holds at P. This completes the proof of Theorem 1.

We note that if Ω is a ball defined by $|x|=a$ and g is the constant $2a/N$, then there exists a radial solution

$$u(x)=u(|x|)=-|x|^2/N$$

for which we have

$$\frac{\partial u}{\partial n}=-\frac{2a}{N},\qquad \operatorname{grad}_S u=0,\quad \text{on } |x|=a.$$

Thus from Theorem 1, we conclude that this solution is unique up to an additive constant.

For another example, let Ω in (2.1) be the ellipsoidal region bounded by the surface $\sum_{i=1}^N x_i^2/a_i^2=1$. Then the function

$$u(x)=-\left(\sum_{i=1}^N\frac{x_i^2}{a_i}\right)\Big/\left(\sum_{j=1}^N\frac{1}{a_j}\right)$$

satisfies the differential equation in (2.1) and the boundary condition

$$|\operatorname{grad} u|^2=4\Big/\left(\sum_{j=1}^N\frac{1}{a_j}\right)^2.$$

Clearly, $\partial u/\partial n<0$ on the ellipsoidal surface. Thus if

$$g=2\Big/\left(\sum_{j=1}^N\frac{1}{a_j}\right),$$

then any solution of (2.1) in this case must be of the form

$$u(x)=\text{const}-\left(\sum_{i=1}^N\frac{x_i^2}{a_i}\right)\Big/\left(\sum_{j=1}^N\frac{1}{a_j}\right).$$

It is clear from parts (β) and (γ) of Theorem 1 that in the case of the ball with $g = c$, if $c < 2a/N$, then there is no solution to our problem I, while if $c > 2a/N$, it follows that if a solution exists, then at some point P on $|x| = a$ we have $\partial u_2/\partial n > 2a/N$. Similar remarks can be made for the case in which Ω is an ellipsoid.

3. Problem II. We consider first the overdetermined problem

$$\begin{aligned} \Delta u + 2 &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ |\operatorname{grad} u| &= g \geq 0 && \text{on } \partial\Omega_1, \end{aligned} \tag{3.1}$$

where Ω_1 is strictly contained in Ω, i.e., $\Omega_1 \Subset \Omega$. For such problems, we do not specify the shape and size of Ω but seek to determine an Ω for which the problem has a solution. Theorem 1 tells us that if, for a prescribed g_1, there exists a solution u_1 of (2.1) in Ω_1 that satisfies $\partial u_1/\partial n \leq 0$ on $\partial\Omega_1$, then this solution is unique up to an additive constant in Ω_1. By unique continuation, it follows that Ω must be such that $u_1 + \text{const}$ is a solution of (3.1) with $g = g_1$. The theorem also tells us that if $g < g_1$, there can be no solution of (3.1).

For the remainder of this section, we restrict our attention to the case where g is a constant function. Thus we consider

$$\begin{aligned} \Delta u + 2 &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ |\operatorname{grad} u| &= c && \text{on } \partial\Omega_1. \end{aligned} \tag{3.2}$$

When $\partial\Omega_1 = \partial\Omega$, (3.2) is the problem considered by Serrin and Weinberger. However, we assume $\Omega_1 \Subset \Omega$ and examine two special cases in detail:

(1) Ω_1 is a ball, (2) Ω_1 is an ellipsoid.

THEOREM 2. Let Ω_1 be the ball $|x| < a$.

(α) If $c = 2a/N$, then Ω in (3.2) is a ball.

(β) If $c < 2a/N$, then there is no solution to (3.2).

(γ) If $c > 2a/N$, then there is no solution to (3.2) for a convex domain Ω for which the maximum value of the solution of

$$\Delta u + 2 = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.3}$$

is assumed at a point in the closed ball

$$|x| \leq a(1 + 1/N).$$

Proof. To establish (α), we observe that $u_1 = -|x|^2/N$ is the unique solution (up to an additive constant) of (2.1) in $|x| < a$. Then by unique continuation, $u = \text{const} - |x|^2/N$ must be the solution of (3.2) in Ω which, by the condition $u = 0$ on $\partial\Omega$, implies Ω is a ball.

Part (β) of Theorem 2 follows directly from part (β) of Theorem 1.

From part (γ) of Theorem 1, if $c > 2a/N$, then there is a point P on $\partial\Omega_1$ at which

$$\frac{\partial u}{\partial n} > \frac{2a}{N}, \qquad \operatorname{grad}_S u = 0. \tag{3.4}$$

But (3.4) implies that the spherical surface $|x| = a$ must be tangent to a level surface of the solution u of (3.3) at P. If Ω is restricted to be convex, then Makar-Limanov [9] (for $N = 2$) and Kennington [5] (for $N > 2$) have shown that the level surfaces of the solution of (3.3) are convex. In this regard, see also Sperb [16]. Consequently, for Ω convex, the solution u of (3.3) has a single interior maximum point. But if this maximum point lies in the closed ball $|x| \le a$, then $\partial u/\partial n$ cannot be positive at P which contradicts (3.4). Thus, if $c > 2a/N$ and Ω is convex, then no solution of (3.2) can exist when the maximum point of (3.3) lies in the closed ball $|x| \le a$.

To complete the proof of part (γ), we assume that the point of the maximum of u lies outside $|x| = a$, say at P_0. For Ω convex, the solution u of (3.3) (see Payne [10] and Sperb [17]) satisfies

$$|\operatorname{grad} u|^2 + 4u < 4u_M, \tag{3.5}$$

where u_M is the maximum value of u in Ω. From (3.5), it follows that

$$\int_u^{u_M} \frac{d\eta}{\sqrt{u_M - \eta}} < 2d, \tag{3.6}$$

where d is the distance between the point $\widetilde{P}$ at which u is evaluated and P_0, and hence

$$u_M - u(\widetilde{P}) < d^2. \tag{3.7}$$

Letting $\widetilde{P}$ be a point on $|x| = a$ and using (3.5) and (3.7), we have

$$\frac{4a^2}{N^2} < c^2 < 4d^2. \tag{3.8}$$

But (3.8) states that the minimum distance from P_0 to the surface $|x| = a$ must be greater than a/N. This completes the proof of Theorem 2.

It follows that if $c > 2a/N$, Ω is convex and a solution exists, then $P_0 \notin \Omega_2$, where $\Omega_2 := \{x\colon |x| < a(1 + 1/N)\}$. We have not proved that a solution exists in this case, but the theorem does not rule out the possibility. In fact, our theorem does not preclude the possibility of the existence of a solution for a nonconvex Ω and $c > 2a/N$ or for an elongated convex region and $c > 2a/N$. However, it does show that the solution whose gradient takes its smallest possible constant value on $|x| = a$ is achieved if and only if Ω is a ball.

We now consider the case when Ω_1 is the ellipsoid

$$\sum_{i=1}^{N} \frac{x_i^2}{a_i^2} = 1 \tag{3.9}$$

and obtain the following theorem:

THEOREM 3. Let Ω_1 be the ellipsoid (3.9).

(α) If $c = 2/\sum_{j=1}^N a_j^{-1}$, then Ω in (3.2) must be an ellipsoid of the form defined by $\sum_{i=1}^N x_i^2/a_i = \nu$, for some ν.

(β) If $c < 2/\sum_{j=1}^N a_j^{-1}$, then there is no solution to (3.2).

(γ) If $c > 2/\sum_{j=1}^{N} a_j^{-1}$, then there is no solution to (3.2) for a convex Ω for which the interior point of Ω at which the solution of (3.3) takes its maximum value lies either inside the ellipsoid (3.9) or outside at a distance less than or equal to $1/\sum_{j=1}^{N} a_j^{-1}$ from the boundary of the ellipsoid.

The proof of Theorem 3 is similar to that of Theorem 2 and is therefore not presented.

4. More general equations. The results of Secs. 2 and 3 carry over with only slight modification to equations of the form

$$[\rho(|\nabla u|^2)u_{,i}]_{,i} + k = 0 \quad \text{in } \Omega. \tag{4.1}$$

Here we use the comma notation to denote partial differentiation and the repeated index convention to indicate summation over the repeated index from 1 to N. We assume the function ρ satisfies

$$\rho(s) + 2s\rho'(s) > 0,$$

which ensures that (4.1) is uniformly elliptic, and that k is a positive constant. It is well known that equations of the form (4.1) model various physical and geometrical problems (see [15, 11]).

We first consider the analogue of (2.1), i.e., the problem

$$\begin{aligned} &[\rho(|\nabla u|^2)u_{,i}]_{,i} + k = 0 \quad \text{in } \Omega, \\ &|\nabla u| = g \geq 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

Again we are interested in uniqueness and comparison results. In fact, the only change in the argument of Theorem 1 needed for the analogous results applicable to problem (4.2) is in the equation that the difference function $w = u_2 - u_1$ satisfies. However, as previously shown by Serrin [14] (see p. 426), w satisfies an elliptic equation and takes both its maximum and minimum values on $\partial\Omega$. Thus one need only use the fact that

$$\frac{\partial w}{\partial n}\left(\frac{\partial w}{\partial n} + 2\frac{\partial u_1}{\partial n}\right) = 0$$

at the maximum and minimum points to complete the extensions to problem (4.2).

Let us now examine the special case when Ω in (4.2) is a ball and g is constant. In this case, we let $u(|x|)$ be any radial solution of (4.1). It is easy to see that

$$|\nabla u(|x|)| = |\partial u/\partial n| = \text{const} \quad \text{on } |x| = a.$$

From the fact that

$$\int_{|x|=a} \rho(|\nabla u|^2)\frac{\partial u}{\partial n}\, dS = -k\frac{|x|^N}{N}\omega_N,$$

where ω_N is the surface area of the N-dimensional unit sphere, we see that $\partial u/\partial n < 0$ on $|x| = a$. Thus the extended version of Theorem 1 tells us that if $g = \text{const}$ in (4.2) and the constant is such that $u(|x|)$ is a solution of (4.2), then this solution is unique up to an additive constant.

The analogue of Problem II is the overdetermined boundary value problem

$$\begin{aligned} [\rho(|\nabla u|^2)u_{,i}]_{,i} + k &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ |\operatorname{grad} u| &= g \geq 0 && \text{on } \partial\Omega_1, \end{aligned} \tag{4.3}$$

where we assume $\Omega_1 \Subset \Omega$. The results of Sec. 3 carry over with little change. However, since we cannot exhibit an explicit radial solution, we specify the value of the constant in the analogue of Theorem 2 as

$$c = |\operatorname{grad} u(|x|)|_{|x|=a},$$

and the determination of c involves the solution of $c\rho(c^2) = ka$. For example, if

$$\rho(|\nabla u|^2) = [1 + |\nabla u|^2]^{-1/2}, \tag{4.4}$$

we find

$$c = ka[1 - k^2 a^2]^{-1/2}$$

which is valid only if $ka < 1$.

We note that if ρ is of the form (4.4), then (4.1) is the equation of a surface of constant mean curvature. It is well known that restrictions on the size of k and/or the size of Ω are required for existence (see [14] or [15]) in this case.

The proof of the analogue of Theorem 2 follows the argument presented in Sec. 3 and will not be repeated here. Furthermore, it is possible to formulate an analogue of Theorem 3 for the overdetermined nonlinear problem (4.3), but in this case the solution u which depends only on the combination of variables $\sum_{i=1}^n x_i^2/a_i$ can seldom be exhibited explicitly.

We recall that the proof of part (γ) of Theorems 2 and 3 relied on the convexity of the level sets of the solution of (3.3) when Ω is convex. It follows from Korevaar [6] (see also Kennington [5] and Kawohl [4]) that this property holds also for solutions of (4.1) that vanish on $\partial\Omega$, a property which is used in establishing part (γ) of the modified version of Theorems 2 and 3.

5. Applications. As one application of our results, suppose we wish to construct an elastic beam of cross-sectional area A which is such that the stress is constant along the interior elliptic curve

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1. \tag{5.1}$$

The results of Theorem 3 show that a particular cross section which will achieve this goal has the equation $x_1^2/a_1 + x_2^2/a_2 = \nu$ for some ν. In fact, since the area of the cross section is prescribed, we have $A = \pi\nu\sqrt{a_1 a_2}$, i.e., the equation for the elliptic section is

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} = \frac{A}{\pi\sqrt{a_1 a_2}}. \tag{5.2}$$

Although it seems likely to be the case, Theorem 3 does not establish that this is the only solution. However, if any other solution exists, the theorem tells us that it must have a higher constant stress on the elliptic curve given by (5.1). The problem would certainly have a unique solution if we asked for the cross section of given area that leads to the smallest constant stress on the interior ellipse.

As a second illustration, let us consider a surface of constant mean curvature Λ given by $z = u(x_1, x_2)$ defined on a region $\Omega \subset R^2$. Here u is a solution of

$$[(1+|\nabla u|^2)^{-1/2}u_{,i}]_{,i} + 2\Lambda = 0 \quad \text{in } \Omega.$$

The surface is to be such that the z component of the unit normal to the surface at any point Q in Ω approaches a constant value as Q approaches the boundary. Is there a unique solution to this problem? We note that n_3, the z component of the normal to the surface, is given by $n_3 = (1+|\nabla u|^2)^{-1/2}$ so we are faced with a problem of type I, i.e.,

$$\begin{gathered}[(1+|\nabla u|^2)^{-1/2}u_{,i}]_{,i} + 2\Lambda = 0 \quad \text{in } \Omega, \\ |\operatorname{grad} u| = c \quad \text{on } \partial\Omega.\end{gathered} \tag{5.3}$$

The consequences of Theorem 1 are then applicable to (5.3).

A third possible application of our results is for the problem

$$\Delta u = -2 \quad \text{in } \Omega \subset R^2, \qquad |\operatorname{grad} u| = c > 0 \quad \text{on } \partial\Omega. \tag{5.4}$$

We assume that Ω is convex and that for some c there exists a solution with nonpositive normal derivative on $\partial\Omega$. In general, we are unable to compute u explicitly, but by Theorem 1, we know that c is determined uniquely by this solution. We can derive an upper bound for c as follows, provided we assume that $\partial\Omega$ is a $C^{2+\varepsilon}$ curve.

We introduce the P function (see Sperb [17])

$$P = |\operatorname{grad} u|^2 + 4u, \tag{5.5}$$

which is known to take its maximum value either on the boundary or at an interior point at which u takes its maximum value. Since $|\operatorname{grad} u|^2$ is constant on $\partial\Omega$, we have

$$\frac{1}{2}\frac{\partial|\operatorname{grad} u|^2}{\partial s} = \frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s\,\partial n} + \frac{\partial u}{\partial s}\frac{\partial^2 u}{\partial s^2} = 0 \tag{5.6}$$

and $\partial^2|\operatorname{grad} u|^2/\partial s^2 = 0$ on $\partial\Omega$, where $\partial/\partial s$ denotes the tangential derivative on $\partial\Omega$. Thus if P takes its maximum value at a point Q on $\partial\Omega$, then at this point

$$\frac{\partial P(Q)}{\partial n} \geq 0, \qquad \frac{\partial P(Q)}{\partial s} = 0, \qquad \frac{\partial^2 P(Q)}{\partial s^2} \leq 0$$

or, using (5.5), we have at Q

$$2\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2} + 2\frac{\partial u}{\partial s}\frac{\partial^2 u}{\partial s\,\partial n} - 2\kappa\left(\frac{\partial u}{\partial s}\right)^2 + 4\frac{\partial u}{\partial n} \geq 0, \tag{5.7}$$

$$\frac{\partial u}{\partial s} = 0, \qquad \frac{\partial^2 u}{\partial s^2} \leq 0, \tag{5.8}$$

where κ is the curvature of the boundary at Q. Using the differential equation in normal coordinates on the boundary in (5.7) (recall $\partial\Omega$ is a $C^{2+\varepsilon}$ curve), we have at Q

$$-2\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s^2}+2\frac{\partial u}{\partial s}\frac{\partial^2 u}{\partial s\,\partial n}-2\kappa c^2\geq 0. \tag{5.9}$$

Now since $\partial u(Q)/\partial s=0$ implies $|\partial u(Q)/\partial n|=c$, from (5.6), we conclude that $\partial^2 u(Q)/\partial s\,\partial n=0$.

It follows that if a maximum of the P function occurs at Q on $\partial\Omega$, then at Q

$$-\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s^2}-\kappa c^2\geq 0. \tag{5.10}$$

By assumption $\partial u/\partial n\leq 0$ on $\partial\Omega$, and since κ is nonnegative on $\partial\Omega$, it follows from (5.8) that (5.10) is impossible unless the left side is identically zero. The left side cannot be zero when Ω is strictly convex, but in any case, we are led (using Hopf's second principle [3]) to

$$|\operatorname{grad} u|^2\leq 4[u_M-u] \tag{5.11}$$

in Ω. Integrating from the boundary to an interior point $\overline{Q}$ at which u takes its maximum value as was done in Sec. 3 and noting that $\partial u/\partial n\leq 0$ on $\partial\Omega$, we find $u_M-u\leq\delta^2\leq d^2$, where δ is the distance to the boundary from $\overline{Q}$ and u is evaluated at the nearest point $\tilde{Q}$ on the boundary. The constant d is the radius of the largest inscribed circle in Ω. Reinserting into (5.11) and evaluating at $\tilde{Q}$, we conclude that if for some c and Ω convex, there exists a solution to (5.4) with nonpositive normal derivative on $\partial\Omega$, then c must satisfy

$$c\leq 2d. \tag{5.12}$$

The inequality (5.12) is an optimal inequality in the sense that the equality sign holds in the limit for a long thin ellipse as the ratio of the length of the minor axis to that of the major axis tends to zero.

A lower bound for c is obtained easily from the observation that $c\geq-\partial u/\partial n$ on $\partial\Omega$. Integrating over $\partial\Omega$ and using the differential equation, we find $cL\geq 2A$, where L is the perimeter of $\partial\Omega$ and A is the area of Ω. Thus one has $2AL^{-1}\leq c\leq 2d$ as bounds on the constant c.

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