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ON PHASE-LOCKED MOTIONS ASSOCIATED WITH STRONG RESONANCE

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Abstract. This paper is concerned with the stability and bifurcation behaviour of a nonlinear autonomous system in the vicinity of a compound critical point characterized by two pairs of pure imaginary eigenvalues of the Jacobian. Attention is focused on the local dynamics of the system near-to-resonance. The methodology developed earlier for the bifurcation analysis into periodic and quasi-periodic motions (unification technique coupled with the intrinsic harmonic balancing) is extended to consider the stability and bifurcations of resonant cases. A set of simplified rate equations characterizing the local dynamics of the system is derived. These equations differ from those associated with nonresonant cases in that they are phase-coupled. Furthermore, the stability conditions of the phase-locked periodic bifurcation solutions are presented. All the results are expressed in explicit forms.

1. Introduction. It is well known that a nonautonomous system may quite often exhibit periodic as well as quasi-periodic bifurcation solutions. Nonlinear systems with periodic driving forces have been considered in detail by, for example, Nayfeh and Mook [1], Iooss and Joseph [2], who used perturbation techniques and operator function theory to obtain the first-order approximations of bifurcation solutions and stability conditions. In a nonlinear autonomous system, interactions of static and dynamic bifurcation modes in the vicinity of a *compound* critical point may lead to *invariant tori*, which has been investigated by many authors (e.g., see [3-7]). Recently, such phenomena are studied by using the *unification technique* [7-11] coupled with the *intrinsic harmonic balancing* [12] and its generalization, *multiple-scale intrinsic harmonic balancing* [11]. The approach enables one to obtain systematically a set of simplified *rate equations* which govern the local dynamics of the system. The solutions for static bifurcations, Hopf bifurcations, and quasi-periodic motions lying on two- and higher-dimensional invariant tori, as well as the associated stability conditions, are expressed in terms of the system coefficients.

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An interesting problem associated with nonlinear autonomous systems is concerned with a Jacobian that has two pairs of pure imaginary eigenvalues. Mainly, two cases may arise. The case without resonance (i.e., when the ratio of the two frequencies at the critical point is irrational) has been studied by a number of authors [5, 11, 13, 14].

The second case is concerned with resonance which can occur in several ways: the ratio of frequencies ω_1/ω_2 can be equal to 1:1, 1:2, 1:3, etc. and also has been studied by some authors (e.g., see [15–18]). In this paper, the *unification technique* (coupled with the *intrinsic harmonic balancing*) is extended to the analysis of such resonance problems, with particular attention to the case of $\omega_1/\omega_2 = \frac{1}{2}$. Based on the assumption of *near-to-resonance* (that is, the ratio of the two frequencies of the bifurcation solutions is near $\frac{1}{2}$), a set of rate equations is derived by using the techniques which are then used to explore *phase-locked* periodic bifurcation solutions. Furthermore, the bifurcation critical lines and associated stability conditions are presented. All the results are expressed in terms of the system coefficients. The methods used in this paper can be applied to investigate other special and even more degenerate resonant cases.

2. Formulation of the problem. Consider an n-dimensional nonlinear autonomous system described by a set of first-order differential equations. Suppose that the Jacobian matrix of the linearized system evaluated at a critical point c has two pairs of pure imaginary eigenvalues, while the rest of the eigenvalues have negative real parts. With the aid of the *Center Manifold Theory*, one may reduce the system to a new system in terms of four critical state variables as follows:

$$\frac{dz^i}{dt} = Z_i(z^j\,;\,\eta^\beta) \qquad (i, j = 1, 2, 3, 4;\ \beta = 1, 2), \tag{1}$$

where the z^i denote the state variables and the η^β are two certain independent parameters. It is assumed that the functions Z_i are analytic, at least in the region of interest, and an equilibrium path $z^i = f_i(\eta^\beta)$ in this region exhibits a critical point c with two pairs of pure imaginary eigenvalues. Introduce a nonsingular transformation,

$$z^i = f_i(\eta^\beta) + T_{ij}w^j, \tag{2}$$

into Eq. (1) to obtain the system

$$\frac{dw^i}{dt} = W_i(w^j\,;\,\eta^\beta) \qquad (i, j = 1, 2, 3, 4;\ \beta = 1, 2) \tag{3}$$

such that its Jacobian matrix evaluated at the critical point c is in the canonical form

$$\left[W_{ij}\right]_c = \begin{bmatrix} 0 & \omega_{1c} & 0 & 0 \\ -\omega_{1c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{2c} \\ 0 & 0 & -\omega_{2c} & 0 \end{bmatrix} \qquad (\omega_{1c} > 0,\ \omega_{2c} > 0), \tag{4}$$

where the ratio ω_{1c}/ω_{2c} is assumed to be rational, and will be assigned the value of

$\frac{1}{2}$. It follows from the transformation (2) that system (3) has the property

$$W_i(0;\eta^\xi) = W_{i\beta}(0,\eta^\xi) = W_{i\beta\gamma}(0;\eta^\xi) = \cdots = 0, \tag{5}$$

where the subscripts on W_i denote differentiations with respect to the corresponding parameters.

Suppose that the eigenvalues of the Jacobian matrix $\left[W_{ij}(\mu^\beta)\right]$, where $\mu^\beta = \eta^\beta - \eta_c^\beta$, consist of two distinct complex conjugate pairs given by $\alpha_1(\mu^\beta) \pm \omega_1(\mu^\beta)i$ and $\alpha_2(\mu^\beta) \pm \omega_2(\mu^\beta)i$. Then

$$\alpha_1(0) = \alpha_2(0) = 0, \quad \omega_1(0) = \omega_{1c}, \quad \text{and} \quad \omega_2(0) = \omega_{2c}. \tag{6}$$

It is further assumed that a *transversality condition*, given by

$$\det\begin{bmatrix} \partial\alpha_1/\partial\mu^1 & \partial\alpha_1/\partial\mu^2 \\ \partial\alpha_2/\partial\mu^1 & \partial\alpha_2/\partial\mu^2 \end{bmatrix} \neq 0, \tag{7}$$

is satisfied, which implies that both pairs of the eigenvalues cross the imaginary axis in the complex plane with nonzero velocity. It can be shown (see [11, Appendix A]) that

$$\frac{\partial\alpha_1}{\partial\mu^\beta} \equiv \alpha_1^\beta = \frac{1}{2}(W_{11\beta} + W_{22\beta}), \quad \frac{\partial\alpha_2}{\partial\mu^\beta} \equiv \alpha_2^\beta = \frac{1}{2}(W_{33\beta} + W_{44\beta}) \qquad (\beta = 1, 2) \tag{8}$$

and

$$\frac{\partial\omega_1}{\partial\mu^\beta} \equiv \omega_1^\beta = \frac{1}{2}(W_{12\beta} - W_{21\beta}), \quad \frac{\partial\omega_2}{\partial\mu^\beta} \equiv \omega_2^\beta = \frac{1}{2}(W_{34\beta} - W_{43\beta}) \qquad (\beta = 1, 2). \tag{9}$$

In order to apply the *multiple-scale intrinsic harmonic balancing* procedure and the *unification technique* here, suppose that the steady-state solutions of Eq. (3) in the vinicity of c are in the parametric form

$$w^i = w^i(\tau_k;\sigma^a), \quad \eta^\beta = \eta^\beta(\sigma^a), \quad \omega_k = \omega_k(\sigma^a) \qquad (i = 1, 2, 3, 4;\ k, a, \beta = 1, 2), \tag{10}$$

where $\tau_k = \omega_k t$ and σ^a are certain unidentified small perturbation parameters [12]. Assume further that these solutions can be expressed as a *multiple time scale Fourier series* [11]

$$w^j(\tau_k;\sigma^a) = \sum_{\substack{m=0 \\ m_1+m_2=m}}^{M} \Big[p_{j,m_1m_2}(\sigma^a)\cos(m_1\tau_1 + m_2\tau_2) + r_{j,m_1m_2}(\sigma^a)\sin(m_1\tau_1 + m_2\tau_2)\Big]. \tag{11}$$

This expression can be introduced for both resonance and nonresonance cases. In the latter situation, the governing equations can be constructed readily from the perturbation equations, which will be generated next. This paper is concerned with resonant cases and attention will be focused on phase-locked periodic bifurcations near-to-resonance. Here, only the case of $\omega_1/\omega_2 = \frac{1}{2}$ is considered in detail. Setting $\tau_1 = 2\tau_1 \equiv 2\tau$ and $\omega = 2\omega_1 \equiv 2\omega$, Eq. (10) takes the form

$$w^i = w^i(\tau;\sigma^a), \quad \eta^\beta = \eta^\beta(\sigma^a), \quad \omega = \omega(\sigma^a) \qquad (i = 1, 2, 3, 4;\ a, \beta = 1, 2), \tag{12}$$

and Eq. (11) therefore, is in the form of an ordinary Fourier series

$$w^j(\tau;\sigma^a) = \sum_{m=0}^{M}(p_{jm}\cos m\tau + r_{jm}\sin m\tau). \tag{13}$$

Next, a sequence of perturbation equations is generated by substituting Eq. (12) into Eq. (13) and differentiating the resulting identity with respect to the σ^a successively,

$$\omega^{,a}w_\tau^i + \omega_\tau^{i,a} = W_{ij}w^{j,a} + W_{i\beta}\mu^{\beta,a}, \tag{14}$$

$$\begin{aligned}\omega^{,ab}w_\tau^i + \omega^{,a}w_\tau^{i,b} + \omega^{,b}w_\tau^{i,a} + \omega w_\tau^{i,ab} &= W_{ijk}w^{j,a}w^{k,b} + W_{ij\beta}(w^{j,a}\mu^{\beta,b} + w^{j,b}\mu^{\beta,a}) \\ &\quad + W_{ij}w^{j,ab} + W_{i\beta\gamma}\mu^{\beta,a}\mu^{\gamma,b} + W_{i\beta}\mu^{\beta,ab},\end{aligned} \tag{15}$$

etc., where $i, j, k = 1, 2, 3, 4$; $a, b = 1, 2$; $\beta, \gamma = 1, 2$, the subscripts on the functions W_i denote differentiations with respect to the corresponding variables, and summation convention applies. For clarity, differentiations of the variables with respect to the σ^a are indicated by the superscripts after a comma.

Now, evaluating the first-order perturbation equation (14) at the critical point c with the aid of Eqs. (5) and (13) yields

$$\omega_c\sum_{m=1}^{M} m(r_{im}^{;a}\cos m\tau - p_{im}^{;a}\sin m\tau) = W_{ij}\sum_{m=0}^{M}(p_{jm}^{;a}\cos m\tau + r_{jm}^{;a}\sin m\tau), \tag{16}$$

where W_{ij} is given by Eq. (4). Comparing the coefficients of $\cos m\tau$ and $\sin m\tau$ for each m gives the nontrivial solutions

$$\begin{matrix} p_{21}^{;a} = r_{11}^{;a} \\ r_{21}^{;a} = -p_{11}^{;a} \end{matrix} \quad \text{and} \quad \begin{matrix} p_{42}^{;a} = r_{32}^{;a} \\ r_{42}^{;a} = -p_{32}^{;a}, \end{matrix} \qquad (a = 1.2). \tag{17}$$

Next, evaluating the second-order perturbation equation (15) at the critical point c, introducing Eqs. (5), (13), and (17), and then comparing the coefficients of $\cos m\tau$ and $\sin m\tau$ results in

$$\frac{1}{2}(W_{i11}+W_{i22})(p_{11}^{;a}p_{11}^{;b}+r_{11}^{;a}r_{11}^{;b})+\frac{1}{2}(W_{i33}+W_{i44})(p_{33}^{;a}p_{44}^{;b}+r_{32}^{;a}r_{32}^{;b})+W_{ij}p_{j0}^{;ab} = 0, \tag{18}$$

for $m = 0$;

$$\begin{aligned}&(\omega_1^{;a}r_{i1}^{;b} + \omega_1^{;b}r_{i1}^{;a}) + \omega_{1c}r_{i1}^{;ab} \\ &= \frac{1}{2}\Big[(W_{i13} + W_{i24})(p_{11}^{;a}p_{32}^{;b} + p_{11}^{;b}p_{32}^{;a} + r_{11}^{;a}p_{32}^{;b} + r_{11}^{;b}r_{32}^{;a}) \\ &\quad +(W_{i14} - W_{i23})(p_{11}^{;a}r_{32}^{;b} + p_{11}^{;b}r_{32}^{;a} - r_{11}^{;a}p_{32}^{;b} - r_{11}^{;b}p_{32}^{;a})\Big] \\ &\quad + \Big[W_{i1\beta}(p_{11}^{;a}\mu^{\beta,b} + p_{11}^{;b}\mu^{\beta,a}) + W_{i2\beta}(r_{11}^{;a}\mu^{\beta,b} + r_{11}^{;b}\mu^{\beta,a})\Big] + W_{ij}p_{j1}^{;ab},\end{aligned} \tag{19a}$$

$$(\omega_1^{,a} p_{i1}^{,b} + \omega_1^{,b} p_{i1}^{,a}) + \omega_{1c} p_{i1}^{,ab}$$
$$= \frac{1}{2}\Big[(W_{i14} - W_{i23})(p_{11}^{,a} p_{32}^{,b} + p_{11}^{,b} p_{32}^{,a} + r_{11}^{,a} r_{32}^{,b} + r_{11}^{,b} r_{32}^{,a}) \tag{19b}$$
$$- (W_{i13} + W_{i24})(p_{11}^{,a} r_{32}^{,b} + p_{11}^{,b} r_{32}^{,a} - r_{11}^{,a} p_{32}^{,b} - r_{11}^{,b} p_{32}^{,a})\Big]$$
$$+ \Big[W_{i2\beta}(p_{11}^{,a}\mu^{\beta,b} + p_{11}^{,b}\mu^{\beta,a}) - W_{i1\beta}(r_{11}^{,a}\mu^{\beta,b} + r_{11}^{,b}\mu^{\beta,a})\Big] - W_{ij} r_{j1}^{,ab},$$

for $m = 1$ (where the first two terms on the left-hand side vanish for $i = 3, 4$);

$$(\omega_2^{,a} r_{i2}^{,b} + \omega_2^{,b} r_{i2}^{,a}) + \omega_{2c} r_{i2}^{,ab}$$
$$= \frac{1}{2}\Big[(W_{i11} - W_{i22})(p_{11}^{,a} p_{11}^{,b} - r_{11}^{,a} r_{11}^{,b}) + 2W_{i12}(p_{11}^{,a} r_{11}^{,b} + p_{11}^{,b} r_{11}^{,a})\Big]$$
$$+ \Big[W_{i3\beta}(p_{32}^{,a}\mu^{\beta,b} + p_{32}^{,b}\mu^{\beta,a}) + W_{i4\beta}(r_{32}^{,a}\mu^{\beta,b} + r_{32}^{,b}\mu^{\beta,a})\Big] + W_{ij} p_{j2}^{,ab}, \tag{20a}$$

$$(\omega_2^{,a} p_{i2}^{,b} + \omega_2^{,b} p_{i2}^{,a}) + \omega_{2c} p_{i2}^{,ab}$$
$$= \frac{1}{2}\Big[2W_{i12}(p_{11}^{,a} p_{11}^{,b} - r_{11}^{,b} r_{11}^{,b}) - (W_{i11} - W_{i22})(p_{11}^{,a} r_{11}^{,b} + p_{11}^{,b} r_{11}^{,a})\Big]$$
$$+ \Big[W_{i4\beta}(p_{32}^{,a}\mu^{\beta,b} + p_{32}^{,b}\mu^{\beta,a}) - W_{i3\beta}(r_{32}^{,a}\mu^{\beta,b} + r_{32}^{,b}\mu^{\beta,a})\Big] - W_{ij} r_{j2}^{,ab}, \tag{20b}$$

for $m = 2$ (where the first two terms on the left-hand side vanish for $i = 1, 2$);

$$(\omega_{1c} + \omega_{2c}) r_{i3}^{,ab} = \frac{1}{2}\Big[(W_{i13} - W_{i24})(p_{11}^{,a} p_{32}^{,b} + p_{11}^{,b} p_{32}^{,a} - r_{11}^{,a} r_{32}^{,b} - r_{11}^{,b} r_{32}^{,a})$$
$$+ (W_{i14} + W_{i23})(p_{11}^{,a} r_{32}^{,b} + p_{11}^{,b} r_{32}^{,a} + r_{11}^{,a} p_{32}^{,b} + r_{11}^{,b} p_{32}^{,a})\Big] + W_{ij} p_{j3}^{,ab}, \tag{21a}$$

$$(\omega_{1c} + \omega_{2c}) p_{i3}^{,ab} = \frac{1}{2}\Big[(W_{i14} + W_{i23})(p_{11}^{,a} p_{32}^{,b} + p_{11}^{,b} p_{32}^{,a} - r_{11}^{,a} r_{32}^{,b} + r_{11}^{,b} r_{32}^{,a})$$
$$- (W_{i14} - W_{i23})(p_{11}^{,a} r_{32}^{,b} + p_{11}^{,b} r_{32}^{,a} + r_{11}^{,a} p_{32}^{,b} + r_{11}^{,b} p_{32}^{,a})\Big] - W_{ij} r_{j3}^{,ab}, \tag{21b}$$

for $m = 3$; and

$$2\omega_{2c} r_{i4}^{,ab} = \frac{1}{2}\Big[(W_{i33} - W_{i44})(p_{32}^{,a} p_{32}^{,b} - r_{32}^{,a} r_{32}^{,b}) + 2W_{i34}(p_{32}^{,a} r_{32}^{,b} + p_{32}^{,b} r_{32}^{,a})\Big] + W_{ij} p_{j4}^{,ab},$$
$$2\omega_{2c} p_{i4}^{,ab} = \frac{1}{2}\Big[2W_{i34}(p_{32}^{,a} p_{32}^{,b} - r_{32}^{,a} r_{32}^{,b}) - (W_{i33} - W_{i44})(p_{32}^{,a} r_{32}^{,b} + p_{32}^{,b} r_{32}^{,a})\Big] - W_{ij} r_{j4}^{,ab}, \tag{22}$$

for $m = 4$.

A close inspection of these equations suggests that Eqs. (19) and (20) may yield important relationships, and are rewritten more explicitly

$$\omega_1^{,a} r_{i1}^{,b} + \omega_1^{,b} r_{i1}^{,a} = A(p_{11}^{,a} p_{32}^{,b} + p_{11}^{,b} p_{32}^{,a} + r_{11}^{,a} p_{32}^{,b} + r_{11}^{,b} r_{32}^{,a})$$
$$+ B(p_{11}^{,a} r_{32}^{,b} + p_{11}^{,b} r_{32}^{,a} - r_{11}^{,a} p_{32}^{,b} - r_{11}^{,b} p_{32}^{,a})$$
$$+ \alpha_1^{\beta}(p_{11}^{,a}\mu^{\beta,b} + p_{11}^{,b}\mu^{\beta,a}) + \omega_1^{\beta}(r_{11}^{,a}\mu^{\beta,b} + r_{11}^{,b}\mu^{\beta,a}), \tag{23a}$$

$$\begin{aligned}\omega_1^{;a}p_{i1}^{;b}+\omega_1^{;b}p_{i1}^{;a} &= B(p_{11}^{;a}p_{32}^{;b}+p_{11}^{;b}p_{32}^{;a}+r_{11}^{;a}p_{32}^{;b}+r_{11}^{;b}r_{32}^{;a})\\ &\quad - A(p_{11}^{;a}r_{32}^{;b}+p_{11}^{;b}r_{32}^{;a}-r_{11}^{;a}p_{32}^{;b}-r_{11}^{;b}p_{32}^{;a})\\ &\quad + \omega_1^{\beta}(p_{11}^{;a}\mu^{\beta,b}+p_{11}^{;b}\mu^{\beta,b})-\alpha_1^{\beta}(r_{11}^{;a}\mu^{\beta,b}+r_{11}^{;b}\mu^{\beta,a}) \qquad (23b)\end{aligned}$$

and

$$\begin{aligned}\omega_2^{;a}p_{i2}^{;b}+\omega_2^{;b}p_{i2}^{;a} &= C(p_{11}^{;a}p_{11}^{;b}-r_{11}^{;a}r_{11}^{;b})-D(p_{11}^{;a}r_{11}^{;b}+p_{11}^{;b}r_{11}^{;a})\\ &\quad +\alpha_2^{\beta}(p_{32}^{;a}\mu^{\beta,b}+p_{32}^{;b}\mu^{\beta,a})+\omega_2^{\beta}(r_{32}^{;a}\mu^{\beta,b}+r_{32}^{;b}\mu^{\beta,a}), \qquad (24a)\end{aligned}$$

$$\begin{aligned}\omega_2^{;a}r_{i2}^{;b}+\omega_2^{;b}r_{i2}^{;a} &= -D(p_{11}^{;a}p_{11}^{;b}-r_{11}^{;b}r_{11}^{;b})-C(p_{11}^{;a}r_{11}^{;b}+p_{11}^{;b}r_{11}^{;a})\\ &\quad +\omega_2^{\beta}(p_{32}^{;a}\mu^{\beta,b}+p_{32}^{;b}\mu^{\beta,a})-\alpha_2^{\beta}(r_{32}^{;a}\mu^{\beta,b}+r_{32}^{;b}\mu^{\beta,a}), \qquad (24b)\end{aligned}$$

where α_k^{β} and ω_k^{β} $(k=1,2)$ are given by Eqs. (8) and (9), respectively, and

$$\begin{aligned}A &= \frac{1}{4}\left[(W_{113}-W_{223})+(W_{124}+W_{214})\right],\\ B &= \frac{1}{4}\left[(W_{114}-W_{224})-(W_{123}+W_{213})\right],\\ C &= \frac{1}{4}\left[(W_{311}-W_{322})+(W_{412}+W_{421})\right],\\ D &= \frac{1}{4}\left[(W_{411}-W_{422})-(W_{312}+W_{321})\right].\end{aligned} \qquad (25)$$

These equations, which are in terms of the derivatives of p_{11}, p_{32}, ω, and μ^{β}, can be combined through the unification technique to yield the *governing bifurcation equations*. Consider, for example, the first equation (23) which involves three equations (for $a, b = 1, 2$). Multiplying the first equation for $a = b = 1$ by $(\sigma^1)^2/2$, the equation for $a = 1$, $b = 2$ by $\sigma^1\sigma^2$, and the equation for $a = b = 2$ by $(\sigma^2)^2/2$ and adding them together yields

$$\begin{aligned}\Omega_1 r_{11} &= A(p_{11}p_{32}+r_{11}r_{32})+B(p_{11}r_{32}-r_{11}p_{32})+p_{11}\alpha_1^{\beta}\mu^{\beta}+r_{11}\omega_1^{\beta}\mu^{\beta},\\ \Omega_1 p_{11} &= B(p_{11}p_{32}+r_{11}r_{32})-A(p_{11}r_{32}-r_{11}p_{32})-r_{11}\alpha_1^{\beta}\mu^{\beta}+p_{11}\omega_1^{\beta}\mu^{\beta}.\end{aligned} \qquad (26)$$

Similarly, the first equation of (24) gives

$$\begin{aligned}\Omega_2 r_{32} &= C(p_{11}^2-r_{11}^2)-D(2p_{11}r_{11})+p_{32}\alpha_2^{\beta}\mu^{\beta}+r_{32}\omega_2^{\beta}\mu^{\beta},\\ \Omega_2 p_{32} &= -D(p_{11}^2-r_{11}^2)-C(2p_{11}r_{11})-r_{32}\alpha_2^{\beta}\mu^{\beta}+p_{32}\omega_2^{\beta}\mu^{\beta},\end{aligned} \qquad (27)$$

where $\Omega_k = \omega_k - \omega_{kc}$ $(k = 1, 2)$. Similarly, after some algebra one obtains

$$\begin{aligned}&(p_{11}^2+r_{11}^2)\alpha_1^{\beta}\mu^{\beta}+A\left[(p_{11}^2-r_{11}^2)p_{32}+2p_{11}r_{11}r_{32}\right]\\ &\qquad +B\left[(p_{11}^2-r_{11}^2)r_{32}-2p_{11}r_{11}p_{32}\right]=0,\\ &(p_{32}^2+r_{32}^2)\alpha_2^{\beta}\mu^{\beta}+C\left[(p_{11}^2-r_{11}^2)p_{32}+2p_{11}r_{11}r_{32}\right]\\ &\qquad +D\left[(p_{11}^2-r_{11}^2)r_{32}-2p_{11}r_{11}p_{32}\right]=0 \qquad (28)\end{aligned}$$

and

$$
\begin{aligned}
(p_{11}^2 + r_{11}^2)\Omega_1 &= (p_{11}^2 + r_{11}^2)\omega_1^\beta \mu^\beta + B\left[(p_{11}^2 - r_{11}^2)p_{32} + 2p_{11}r_{11}r_{32}\right], \\
&\quad - A\left[(p_{11}^2 - r_{11}^2)r_{32} - 2p_{11}r_{11}p_{32}\right], \\
(p_{32}^2 + r_{32}^2)\Omega_2 &= (p_{32}^2 + r_{32}^2)\omega_2^\beta \mu^\beta - D\left[(p_{11}^2 - r_{11}^2)r_{32} + 2p_{11}r_{11}r_{32}\right] \\
&\quad + C\left[(p_{11}^2 - r_{11}^2)r_{32} - 2p_{11}r_{11}p_{32}\right].
\end{aligned}
\tag{29}
$$

In order to obtain the dynamical equations in the vicinity of c, let p_{11}, r_{11}, p_{32}, and r_{32} be replaced by $\rho_1 \cos\phi_1$, $-\rho_1 \sin\phi_1$, $\rho_2 \cos\phi_2$, $-\rho_2 \sin\phi_2$, respectively. It can be shown (following the approach described in [11, Appendix B]) that the *rate equations* up to second-order terms are given by

$$
\begin{aligned}
\frac{d\rho_1}{dt} &= \rho_1 \left\{\alpha_1^\beta \mu^\beta + \rho_2 \left[A\cos(2\phi_1 - \phi_2) + B\sin(2\phi_1 - \phi_2)\right]\right\}, \\
\frac{d\rho_2}{dt} &= \rho_2 \alpha_2^\beta \mu^\beta + \rho_1^2 \left[C\cos(2\phi_1 - \phi_2) + D\sin(2\phi_1 - \phi_2)\right]
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
\rho_1 \frac{d\phi_1}{dt} &= \rho_1(-\Omega_1 + \omega_1^\beta \mu^\beta) + \rho_1\rho_2 \left[B\cos(2\phi_1 - \phi_2) - A\sin(2\phi_1 - \phi_1)\right], \\
\rho_2 \frac{d\phi_2}{dt} &= \rho_2(-\Omega_2 + \omega_2^\beta \mu^\beta) + \rho_1^2 \left[-D\cos(2\phi_1 - \phi_2) + C\sin(2\phi_1 - \phi_2)\right].
\end{aligned}
\tag{31}
$$

Next it is noted that the first-order approximation of the periodic solutions is expressed by

$$
\begin{aligned}
p_{11}\cos\tau + r_{11}\sin\tau &= \rho_1 \cos\phi_1 \cos\tau - \rho_1 \sin\phi_1 \sin\tau \\
&= \rho_1 \cos(\tau + \phi_1) = \rho_1 \cos(\tau_1 + \phi_1), \\
p_{32}\cos 2\tau + r_{32}\sin 2\tau &= \rho_2 \cos\phi_2 \cos 2\tau - \rho_2 \sin\phi_2 \sin 2\tau \\
&= \rho_2 \cos(2\tau + \phi_2) = \rho_2 \cos(\tau_2 + \phi_2).
\end{aligned}
\tag{32}
$$

Let

$$
\theta_1 = \tau_1 + \phi_1 \quad \text{and} \quad \theta_2 = \tau_2 + \phi_2, \tag{33}
$$

and define the phase difference

$$
\psi = 2\phi_1 - \phi_2 \equiv 2\theta_1 - \theta_2. \tag{34}
$$

Thus, Eqs. (30) and (31) can be written as

$$
\begin{aligned}
\frac{d\rho_1}{dt} &= \rho_1 \alpha_1^\beta \mu^\beta + \rho_1\rho_2(A\cos\psi + B\sin\psi), \\
\frac{d\rho_2}{dt} &= \rho_2 \alpha_2^\beta \mu^\beta + \rho_1^2(C\cos\psi + D\sin\psi)
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
\rho_1 \frac{d\theta_1}{dt} &= \rho_1(\omega_{1c} + \omega_1^\beta \mu^\beta) + \rho_1\rho_2(B\cos\psi - A\sin\psi), \\
\rho_2 \frac{d\theta_2}{dt} &= \rho_2(\omega_{2c} + \omega_2^\beta \mu^\beta) + \rho_1^2(-D\cos\psi + C\sin\psi).
\end{aligned}
\tag{36}
$$

It may be noted that setting $\mu^\beta = 0$ in Eqs. (35) and (36) leads to the *normal form*, which is equivalent to that given in [5]. Moreover, for the nontrivial solution $(\rho_1 \neq 0,\ \rho_2 \neq 0)$, one may reduce Eq. (36) to

$$\frac{d\psi}{dt} = (2\omega_1^\beta - \omega_2^\beta)\mu^\beta + \left(2B\rho_2 + D\frac{\rho_1^2}{\rho_2}\right)\cos\psi - \left(2A\rho_2 + C\frac{\rho_1^2}{\rho_2}\right)\sin\psi. \tag{37}$$

3. Bifurcation and stability analysis. Based on Eqs. (35) and (37), we shall analyze the bifurcation and stability properties of the system. The steady-state solutions can be obtained by setting $d\rho_1/dt = d\rho_2/dt = d\psi/dt = 0$ as follows:

(I) initial equilibrium solution $\rho_1 = \rho_2 = 0$; and

(II) phase-locked periodic solution

$$\left.\begin{aligned}
\rho_2^2 &= \frac{(D\alpha_1^\beta\mu^\beta - B\rho_{21}\alpha_2^\beta\mu^\beta)^2 + (C\alpha_1^\beta\mu^\beta - A\rho_{21}\alpha_2^\beta\mu^\beta)^2}{(AD-BC)^2}, \\
\rho_1^2 &= \rho_{12}\rho_2^2, \\
\sin\psi &= \frac{C(\alpha_1^\beta\mu^\beta)\rho_1^2 - A(\alpha_2^\beta\mu^\beta)\rho_2^2}{(AD-BC)\rho_1^2\rho_2}, \\
\cos\psi &= -\frac{D(\alpha_1^\beta\mu^\beta)\rho_1^2 - B(\alpha_2^\beta\mu^\beta)\rho_2^2}{(AD-BC)\rho_1^2\rho_2},
\end{aligned}\right\} \quad (AD - BC \neq 0), \tag{38}$$

where

$$\begin{aligned}
\rho_{12} &= \frac{-P \pm \sqrt{P^2+Q}}{2(C^2+D^2)\alpha_1^\beta\mu^\beta}, \qquad \rho_{21} = \frac{P \pm \sqrt{P^2+Q}}{4(A^2+B^2)\alpha_2^\beta\mu^\beta}, \\
P &= \left[(AC+BD)(2\alpha_1^\beta - \alpha_2^\beta) - (AD-BC)(2\omega_1^\beta - \omega_2^\beta)\right]\mu^\beta, \\
Q &= 8(A^2+B^2)(C^2+D^2)(\alpha_1^\beta\mu^\beta)(\alpha_2^\beta\mu^\beta).
\end{aligned} \tag{39}$$

Here, it can be proved that $\rho_{12} = 1/\rho_{21}$ and $(\rho_1/\rho_2)^2 = \rho_{12}$.

The stability of the initial equilibrium solution is determined by the Jacobian of Eq. (35). It gives the conditions

$$\alpha_1^\beta\mu^\beta < 0 \quad \text{and} \quad \alpha_2^\beta\mu^\beta < 0, \tag{40}$$

which are identical to those obtained for the nonresonance case [11]. These stability conditions imply two critical bifurcation lines:

$$L_1\colon \alpha_1^\beta\mu^\beta = 0 \qquad (\alpha_2^\beta\mu^\beta < 0) \tag{41}$$

and

$$L_2\colon \alpha_2^\beta\mu^\beta = 0 \qquad (\alpha_1^\beta\mu^\beta < 0), \tag{42}$$

along which a family of phase-locked periodic solutions lying on a 2-D torus bifurcates from the initial equilibrium solution. This family of periodic solutions is represented by solution (38).

Before discussing the stability of solution (38), consider the existence conditions for the family of phase-locked periodic solution (38). It is clear from Eq. (38) that the existence conditions are given by $\rho_1^2 > 0$ and $\rho_2^2 > 0$, and so $\rho_{12} > 0$ (or $\rho_{21} > 0$). Therefore, the existence of the periodic solution is equivalent to the existence of the positive solution ρ_{12}. One may conclude from Eq. (39) that there are three cases:

(i) $\alpha_1^\beta \mu^\beta > 0$ but $\alpha_2^\beta \mu^\beta < 0$, which implies $Q < 0$; then if $P > 0$, there is no solution; if $P < 0$, there exist two solutions when $P^2 + Q \geq 0$.

(ii) $\alpha_1^\beta \mu^\beta < 0$ but $\alpha_2^\beta \mu^\beta > 0$, which also implies $Q < 0$; then if $P < 0$, there is no solution; if $P > 0$, there exist two solutions when $P^2 + Q \geq 0$.

(iii) $\alpha_1^\beta \mu^\beta > 0$ and $\alpha_2^\beta \mu^\beta > 0$, which implies $Q > 0$; then there exists only one solution

$$\rho_{12} = \frac{-P + \sqrt{P^2 + Q}}{2(C^2 + D^2)\alpha_1^\beta \mu^\beta}.$$

Now, the stability of the phase-locked periodic solution (38) can be determined by evaluating the Jacobian of Eqs. (35) and (37) on Eq. (38), which is in the form

$$J = \begin{bmatrix} 0 & -\left(\frac{\rho_1}{\rho_2}\right)(\alpha_1^\beta \mu^\beta) & \rho_1\rho_2(B\cos\psi - A\sin\psi) \\ -2\left(\frac{\rho_2}{\rho_1}\right)(\alpha_2^\beta \mu^\beta) & \alpha_2^\beta \mu^\beta & \rho_1^2(D\cos\psi - C\sin\psi) \\ 2\left(\frac{\rho_1}{\rho_2}\right)(D\cos\psi - C\sin\psi) & 2(B\cos\psi - A\sin\psi) & (2\alpha_1^\beta - \alpha_2^\beta)\mu^\beta \\ & -\left(\frac{\rho_1}{\rho_2}\right)^2(D\cos\psi - C\sin\psi) & \end{bmatrix}. \tag{43}$$

The characteristic polynomial of Eq. (43) can be written as

$$G(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c \tag{44}$$

where

$$\begin{aligned} a &= -2(\alpha_1^\beta + \alpha_2^\beta)\mu^\beta, \\ b &= \rho_1^2\left[\rho_{12}(C^2 + D^2) - 4(AC + BD)\right] + 4(\alpha_1^\beta \mu^\beta)(\alpha_2^\beta \mu^\beta), \\ c &= \pm 2\rho_1^2\sqrt{P^2 + Q}. \end{aligned} \tag{45}$$

Thus the stability conditions for the family of the phase-locked periodic solution are given by

$$a > 0, \quad c > 0, \quad \text{and} \quad ab - c > 0. \tag{46}$$

$a > 0$ gives $(\alpha_1^\beta + \alpha_2^\beta)\mu^\beta < 0$, hence, the solutions belonging to (iii) in which $\alpha_1^\beta \mu^\beta > 0$ and $\alpha_2^\beta \mu^\beta > 0$ are unstable. Moreover, $c > 0$ requires that the positive sign of the square root $(P^2 + Q)$ has to be chosen. So only one possible solution corresponding to

$$\rho_{12} = \frac{-P + \sqrt{P^2 + Q}}{2(C^2 + D^2)\alpha_1^\beta \mu^\beta}$$

in (i) or in (ii) is stable, in particular (i) $\alpha_1^\beta \mu^\beta > 0$, $\alpha_2^\beta \mu^\beta < 0$, $P < 0$ and $P^2 + Q \geq 0$; (ii) $\alpha_1^\beta \mu^\beta < 0$, $\alpha_2^\beta \mu^\beta > 0$, $P > 0$ and $P^2 + Q \geq 0$, if, in addition, satisfying $ab - c > 0$.

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