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UNIMODALITY AND VISCOELASTIC PULSE PROPAGATION

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Abstract. Sufficient conditions on the stress relaxation modulus in a viscoelastic material are given in order for initially short mechanical pulses in the material to remain unimodal.

Introduction. The purpose of this paper is to study the shape of an initially short mechanical pulse propagating in a viscoelastic material, in particular, to try to clarify under what conditions the pulse will be unimodal, i.e., have only one local maximum.

Consider, for example, a fluid filling a half space resting on a plate. At time $t \leq 0$ everything is at rest, but then the plate is suddenly moved for a short time. The velocity $v(t, x)$ of the fluid will then only depend on time $t \geq 0$ and distance $x \geq 0$ from the plate, and if the fluid is linearly viscoelastic with unit density, then the velocity v satisfies the equation

$$v_t(t, x) = a_0 v_{xx}(t, x) + \int_0^t a(t-s) v_{xx}(s, x)\, ds, \tag{1}$$
$$v(0, x) = 0, \quad t > 0, \quad x > 0.$$

Here the kernel $G(dt) = a_0\delta(dt) + a(t)dt$ is the stress relaxation modulus. For details, see [6] or [8].

In equation (1) the boundary value $v(t, 0)$ must be specified, and if it is chosen to be the delta functional, then one gets the fundamental solution $\nu(dt, x)$ that has the property that the solution of equation (1) is given by

$$v(t, x) = \int_{[0,t]} v(t-s, 0)\nu(ds, x), \qquad t \geq 0, \quad x \geq 0.$$

In [7] it is shown that if $a_0 \geq 0$ and $a \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R}^+)$ is such that $\log(a(t))$ is nonincreasing and convex (which will be assumed throughout this paper), then $\nu(\cdot, x)$ is a probability measure (i.e., a nonnegative measure with total mass 1) supported on $\mathbb{R}^+$ for each $x \geq 0$.

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The question to be studied in this paper is under what conditions on the function a does it follows that $\nu(\cdot, x)$ is unimodal. A probability measure κ is said to be unimodal if there exists a point t_0 such that $\kappa((-\infty, t])$ is convex on $(-\infty, t_0]$ and is concave on $[t_0, \infty)$. Thus $\kappa(dt) = k(t)\,dt + k_0\delta_{t_0}(dt)$ where k is nondecreasing on $(-\infty, t_0)$ and nonincreasing on (t_0, ∞) and $k_0 \geq 0$. Here δ_τ denotes the unit point mass at τ, i.e., the measure defined by $\delta_\tau(E) = 1$ if $\tau \in E$ and $\delta_\tau(E) = 0$ if $\tau \notin E$ for every Borel set $E \subset \mathbb{R}$. It is clear that the restriction to probability measures in the definition of unimodality is not essential.

A probability measure is said to be strongly unimodal if it is unimodal and the convolution of this measure and an arbitrary unimodal probability measure is again unimodal. In [3] it is shown that κ is a strongly unimodal probability distribution with support not contained in a single point if and only if $\kappa(dt) = k(t)\,dt$ where $\log(k)$ is concave. Unfortunately there is no hope that $\nu(\cdot, x)$ would be strongly unimodal in general, because in the case where $a \equiv 0$ and $a_0 > 0$ we have

$$\nu(dt, x) = \frac{x}{2\sqrt{a_0 \pi t^3}} e^{-x^2/(4a_0 t)}\,dt,$$

and the logarithm of this function is not concave. If $a_0 = 0$ and $a(t) \equiv a_\infty$, then $\nu(dt, x) = \delta_{x/\sqrt{a_\infty}}(dt)$, and this measure is, of course, strongly unimodal.

In [4] experimental results on pulse propagation along rods of different polymers are presented. It is observed that the pulses, although decreasing in amplitude and broadening in width, otherwise preserve their unimodal shapes and also that the shapes for different materials can be brought into congruence by a simple scaling operation. This suggests, as further elaborated in [5], that the stress relaxation modulus is of so-called power law form, i.e., $a(t) = ct^{-\alpha}$ with $\alpha \in [0, 1)$ and $a_0 = 0$ (unless $\alpha = 1$).

In [1] equation (1) with kernel $a(t) = ct^{-\alpha}$ and $x \in \mathbb{R}$ is studied, but there the emphasis is on the behaviour with respect to the x-variable for each fixed t and it is proved that the fundamental solution, which is not identical with ν defined above, is unimodal with respect to the x-variable on each half-axis.

Statement of Results.

THEOREM 1. Assume that $a_0 \geq 0$ and that

$$a(t) = \int_{[0,1)} \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \mu(d\alpha), \qquad t > 0,$$

where μ is a finite positive measure on $[0, 1)$ with $a_0 + \mu([0, 1)) > 0$, and let ν be the fundamental solution of equation (1). Then there exists a constant $\gamma \geq 0.823$ such that if the support of the measure $\mu + a_0\delta_1$ is contained in an interval of length at most γ, then $\nu(\cdot, x)$ is a unimodal probability measure for each $x \geq 0$.

The (quite limited) numerical evidence available to me suggests that the assertion of Theorem 1 is true in the case where $\gamma = 1$ as well, but I have not been able to prove this result. It is, however, relatively easy to show that if the support of $\mu + a_0\delta_1$ is contained in at most two points in $[0, 1]$, then the claim holds.

It is not clear what are necessary conditions on the kernel a for $\nu(\cdot, x)$ to be unimodal. One can, however, show that it does not suffice to assume that a is completely monotone, i.e., $(-1)^j a^{(j)}(t) \geq 0$ for all $j = 0, 1, 2, \ldots$ and $t > 0$.

Proposition 2. The assumptions that $a_0 \geq 0$ and that $a \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R})$ is completely monotone on $(0, \infty)$ do not imply that the fundamental solution ν of equation (1) is such that $\nu(\cdot, x)$ is unimodal for each $x \geq 0$.

Proof of Theorem 1. Let $G(dt) = a_0\delta(dt) + a(t)\,dt$. It is clear from the assumptions that if we let $\mu(1) = a_0$, then ("$\hat{\ }$" denotes Laplace transform)

$$\widehat{G}(z) = \int_{[0,1]} z^{\alpha-1}\mu(d\alpha), \qquad \Re z > 0 \quad \text{or} \quad \Im z \neq 0.$$

Taking Laplace transforms of both sides of equation (1) and solving the resulting differential equation, we easily see that

$$\hat{\nu}(z, x) = e^{-xz\varphi(z)}, \quad x \geq 0, \qquad \text{where} \quad \varphi(z) = \sqrt{\frac{1}{z\hat{G}(z)}}.$$

It follows from Theorem 5.2.6 in [2] that $\varphi(z) = \hat{k}(z)$ where k is a locally integrable, completely monotone function. Since all probability distributions that are self-decomposable, or equivalently, belong to the class L, are unimodal (see [9] and [10]), it suffices to show that $\hat{\nu}(z, x)$ is the Laplace transform of such a distribution, and this, in turn, is equivalent to the fact that

$$\text{the function } t \mapsto t|k'(t)| \text{ is nonincreasing on } (0, \infty). \tag{2}$$

(We shall, in fact, prove a stronger result, namely that the function $t \mapsto t|k'(t)|$ is completely monotone.)

Since k is locally integrable and completely monotone, it can be written in the form

$$k(t) = \int_{(0,\infty)} e^{-t\lambda}\eta(d\lambda), \qquad t > 0, \tag{3}$$

where η is a nonnegative measure such that $\int_{(0,\infty)} \frac{1}{1+\lambda}\eta(d\lambda) < \infty$. (The observation that $\eta(\{0\}) = 0$ follows from the fact that $\lim_{t\to\infty} k(t) = \lim_{z\to 0} z\hat{k}(z) = 0$.) Recall that $\hat{k}(z) = \varphi(z)$ and use (3) to get

$$\Im\varphi(-\sigma + i\epsilon) = -\int_{(0,\infty)} \frac{\epsilon}{(\sigma-\lambda)^2 + \epsilon^2}\eta(d\lambda). \tag{4}$$

We let

$$\int_{[0,1]} e^{i\pi\alpha}\sigma^{\alpha}\mu(d\alpha) = r(\sigma)e^{i\phi(\sigma)}, \qquad \sigma > 0,$$

and since we can assume that the support of μ is not contained in $\{0, 1\}$ (in this case it is easy to see that (1) holds), we deduce that

$$\Im\varphi(-\sigma + i\epsilon) \to -\frac{\sin(\phi(\sigma)/2)}{\sqrt{r(\sigma)}}\,d\sigma \quad \text{as } \epsilon \downarrow 0,$$

uniformly on compact subsets of $(0, \infty)$. Thus we conclude from (3) and (4) that

$$k(t) = \frac{1}{\pi}\int_0^\infty e^{-t\sigma}\frac{\sin(\phi(\sigma)/2)}{\sqrt{r(\sigma)}}\,d\sigma.$$

Now an integration by parts shows that a sufficient (but not a necessary condition) for (2) to hold is that

$$\text{the function } \sigma \mapsto \frac{\sigma}{\sqrt{r(\sigma)}}\sin\left(\frac{\phi(\sigma)}{2}\right) \text{ is nondecreasing on } (0, \infty). \tag{5}$$

In order to prove that (5) holds, we shall first prove that the function ϕ is nondecreasing. If $t > 0$, then we get from an integration by parts that

$$\int_{[0,1]} t^\alpha e^{i\pi\alpha}\sigma^\alpha \mu(d\alpha) = \int_{[0,1]} e^{i\pi\alpha}\sigma^\alpha\mu(d\alpha) + \ln(t)\int_0^1 t^\alpha \int_{[\alpha,1]} e^{i\pi\beta}\sigma^\beta\mu(d\beta)\,d\alpha. \tag{6}$$

Now we recall that if $t_j > 0$ and

$$\arg(z_j) \in [0, \pi] \quad \text{for } j = 1, \dots, n,$$

then $\arg(\sum_{j=1}^n t_j z_j)$ belongs to the interval $[\min_{1\le j\le n}\{\arg(z_j)\}, \max_{1\le j\le n}\{\arg(z_j)\}]$. In particular, this implies that $\arg(\int_{[\alpha_0,\alpha_1]} e^{i\pi\alpha}\sigma^\alpha\mu(d\alpha)) \in [\alpha_0, \alpha_1]$. It follows from this fact that $\arg(\int_{[\alpha,1]} e^{i\pi\beta}\sigma^\beta\mu(d\beta))$ is a nondecreasing function of α with values in $[0, \pi]$. If now $t > 1$, then we get

$$\arg\left(\ln(t)\int_0^1 t^\alpha\int_{[\alpha,1]} e^{i\pi\beta}\sigma^\beta\mu(d\beta)\,d\alpha\right) \ge \arg\left(\int_{[0,1]} e^{i\pi\alpha}\sigma^\alpha\mu(d\alpha)\right)$$

and we conclude from (6) that the function ϕ is nondecreasing.

Since the range of the function ϕ is contained in $[0, \pi]$, it follows that the function $\sin(\phi(\sigma)/2)$ is nondecreasing as well. Hence it is sufficient, in order to establish (5), to show that the function $\sigma \mapsto \sigma/\sqrt{r(\sigma)}$ is nondecreasing.

Let us denote by $c(\sigma)$ and $s(\sigma)$ the functions

$$c(\sigma) = \int_{[0,1]} \cos(\pi\alpha)\sigma^\alpha\mu(d\alpha), \qquad \sigma > 0,$$

$$s(\sigma) = \int_{[0,1]} \sin(\pi\alpha)\sigma^\alpha\mu(d\alpha), \qquad \sigma > 0,$$

so that $r(\sigma)^2 = c(\sigma)^2 + s(\sigma)^2$. It is clear that $\sigma/\sqrt{r(\sigma)}$ is nondecreasing if and only if

$$g(\sigma) \stackrel{\text{def}}{=} 2c(\sigma)^2 + 2s(\sigma)^2 - \sigma c'(\sigma)c(\sigma) - \sigma s'(\sigma)s(\sigma) \ge 0, \qquad \sigma > 0.$$

Now $\sigma c'(\sigma) = \int_{[0,1]} \alpha\cos(\pi\alpha)\sigma^\alpha\mu(d\alpha)$ and $\sigma s'(\sigma) = \int_{[0,1]} \alpha\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha)$, and it follows that $0 \le \sigma s'(\sigma) \le s(\sigma)$ and also that

$$\sigma c'(\sigma) = \frac{1}{2}c(\sigma) + \int_{[0,1]}\left(\alpha - \frac{1}{2}\right)\sigma^\alpha\cos(\pi\alpha)\mu(d\alpha) \le \frac{1}{2}c(\sigma).$$

Thus we see that if $c(\sigma) \ge 0$, then $g(\sigma) \ge \frac{3}{2}c(\sigma)^2 + s(\sigma)^2 \ge 0$.

Hence it remains to consider the case where $c(\sigma) < 0$. Suppose that the support of the measure μ is contained in the interval $[\alpha_0, \alpha_1]$. Then we get the following inequalities:

$$\begin{aligned} g(\sigma) &= s(\sigma)\int_{[0,1]}(2-\alpha)\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha) \\ &\qquad + (2-\alpha_1)c(\sigma)^2 - c(\sigma)\int_{[0,1]}(\alpha-\alpha_1)\cos(\pi\alpha)\sigma^\alpha\mu(d\alpha) \\ &\geq s(\sigma)\int_{[0,1]}(2-\alpha)\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha) \\ &\qquad + (2-\alpha_1)c(\sigma)^2 - |c(\sigma)|\int_{[\alpha_0,1/2]}(\alpha_1-\alpha)\cos(\pi\alpha)\sigma^\alpha\mu(d\alpha). \end{aligned} \tag{7}$$

If α_0 and α_1 are such that

$$(\alpha_1-\alpha)\cos(\pi\alpha) \leq 2\sqrt{2-\alpha_1}\sqrt{2-\alpha}\sin(\pi\alpha), \qquad \alpha\in\left[\alpha_0, 1/2\right], \tag{8}$$

then it follows from an application of Hölder's inequality that

$$\begin{aligned} &\int_{[\alpha_0,1/2]}(\alpha_1-\alpha)\cos(\pi\alpha)\sigma^\alpha\mu(d\alpha) \\ &\qquad \leq 2\sqrt{2-\alpha_1}\sqrt{\int_{[0,1]}\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha)}\sqrt{\int_{[0,1]}(2-\alpha)\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha)}. \end{aligned}$$

If we insert this result into (7), then we see that $g(\sigma)\geq 0$, which is what we need.

By the same argument used above, we have

$$\begin{aligned} g(\sigma) &= s(\sigma)\int_{[0,1]}(2-\alpha)\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha) \\ &\qquad + (2-\alpha_0)c(\sigma)^2 - c(\sigma)\int_{[0,1]}(\alpha-\alpha_0)\cos(\pi\alpha)\sigma^\alpha\mu(d\alpha) \\ &\geq s(\sigma)\int_{[0,1]}(2-\alpha)\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha) \\ &\qquad + (2-\alpha_0)c(\sigma)^2 - |c(\sigma)|\int_{[1/2,\alpha_1]}(\alpha-\alpha_0)|\cos(\pi\alpha)|\sigma^\alpha\mu(d\alpha). \end{aligned} \tag{9}$$

Now we assume that α_0 and α_1 are such that

$$(\alpha-\alpha_0)|\cos(\pi\alpha)| \leq 2\sqrt{2-\alpha_0}\sqrt{2-\alpha}\sin(\pi\alpha), \qquad \alpha\in[1/2, \alpha_1], \tag{10}$$

and we use Hölder's inequality to get

$$\begin{aligned} &\int_{[1/2,\alpha_1]}(\alpha-\alpha_0)|\cos(\pi\alpha)|\sigma^\alpha\mu(d\alpha) \\ &\qquad \leq 2\sqrt{2-\alpha_0}\sqrt{\int_{[0,1]}\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha)}\sqrt{\int_{[0,1]}(2-\alpha)\sin(\pi\alpha)\sigma^\alpha\mu(d\alpha)}. \end{aligned}$$

If we combine this inequality with (9), then we get the desired inequality $g(\sigma)\geq 0$.

It is easy to see that if (8) holds with $\alpha = \alpha_0$ or (10) holds with $\alpha = \alpha_1$, then at least one of these inequalities holds for all values of α in $[\alpha_0, \alpha_1]$. In the extreme cases, one of these inequalities becomes an equality. Thus we see that we can take γ to be at least the distance from the line $\alpha_0 = \alpha_1$ in the (α_0, α_1)-plane to the curve given by

$$(\alpha_1 - \alpha_0) = 2\sqrt{2-\alpha_0}\sqrt{2-\alpha_1}\max\{\tan(\pi\alpha_0), |\tan(\pi\alpha_1)|\} \tag{11}$$

in the region where $0 \le \alpha_0 < 1/2 < \alpha_1 \le 1$.

Using the implicit function theorem one sees that (11) determines α_1 as a function of α_0, and when $\alpha_0 < 1 - \alpha_1$ we have

$$\frac{\partial \alpha_1}{\partial \alpha_0} = \frac{1 + \frac{\sqrt{2-\alpha_1}}{\sqrt{2-\alpha_0}}\tan(\pi\alpha_1)}{1 - \frac{\sqrt{2-\alpha_0}}{\sqrt{2-\alpha_1}}\tan(\pi\alpha_1) + \frac{2\pi\sqrt{2-\alpha_0}\sqrt{2-\alpha_1}}{\cos^2(\pi\alpha_1)}} < 1$$

because $\tan(\pi\alpha_1) < 0$. Similarly, when $\alpha_0 > 1 - \alpha_1$ we get from (11) that

$$\begin{aligned}\frac{\partial \alpha_1}{\partial \alpha_0} &= \frac{1 - \frac{\sqrt{2-\alpha_1}}{\sqrt{2-\alpha_0}}\tan(\pi\alpha_0) + \frac{2\pi\sqrt{2-\alpha_0}\sqrt{2-\alpha_1}}{\cos^2(\pi\alpha_0)}}{1 + \frac{\sqrt{2-\alpha_0}}{\sqrt{2-\alpha_1}}\tan(\pi\alpha_0)} \\ &= \frac{1 - \frac{\alpha_1-\alpha_0}{2(2-\alpha_0)} + \frac{\pi(\alpha_1-\alpha_0)}{\sin(\pi\alpha_0)\cos(\pi\alpha_0)}}{1 + \frac{\alpha_1-\alpha_0}{2(2-\alpha_1)}} > 1\end{aligned}$$

because

$$\frac{1}{2-\alpha_0} + \frac{1}{2-\alpha_1} \le \frac{2\pi}{\sin(\pi\alpha_0)\cos(\pi\alpha_0)}.$$

Thus we see that the shortest distance from the curve determined by equation (11) to the line $\alpha_0 = \alpha_1$ is obtained at the point where $\alpha_0 = 1 - \alpha_1$. Since $\alpha_1 - \alpha_0 = 1 - 2\alpha_0$ at this point we conclude that $\gamma \ge \lambda$ where λ satisfies $\lambda - \sqrt{9-\lambda^2}\cot(\pi\lambda/2) = 0$. It is easy to see that this equation has a unique solution in the interval $(0, 1)$ that is at least 0.823.

This completes the proof of Theorem 1.

Proof of Proposition 2. Let $a(t) = (1 + 8e^{-9t})/9$, $t \ge 0$, and $a_0 = 0$, and suppose that the fundamental solution ν of equation (1) is unimodal, at least when, e.g., $x = 1$. A straightforward calculation shows that the Laplace transform of $\nu(\cdot, 1)$ is $\exp(-z\sqrt{1 + 8/(z+1)})$. Thus we see that $\nu(\cdot, 1) = \delta_1 * \eta$ where $\hat{\eta}(z) = \exp(-z\sqrt{1+8/(z+1)} + z)$ (and where " $*$ " denotes convolution). If $\nu(\cdot, 1)$ is unimodal it follows that η must be unimodal. Since η is supported on $\mathbb{R}^+$ and $\lim_{z\to\infty}\hat{\eta}(z) = e^{-4}$, it follows from the unimodality of η that η can be written as $\eta(dt) = e^{-4}\delta_0(dt) + w(t)\,dt$ where w is nonincreasing. Thus it follows that

$$w(0) = \lim_{z\to\infty} z\hat{w}(z) = 12e^{-4}.$$

Since w is nonincreasing we must have $\liminf_{z\to\infty} z(z\hat{w}(z) - w(0)) \le 0$ but we have in fact

$$\lim_{z\to\infty} z(z\hat{w}(z) - w(0)) = 20e^{-4}.$$

This gives a contradiction and shows that $\nu(\cdot, 1)$ cannot be unimodal.

Note also that since the set of unimodal distributions is closed under weak convergence, and since a sequence of probability measures converges weakly to another probability measure if the Fourier transforms converge pointwise, it follows that the counterexample given above does not depend on a being continuous, or on having any number of continuous derivatives at zero. It does not depend on $\lim_{t\to\infty} a(t)$ being positive either.

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