QUARTERLY OF APPLIED MATHEMATICS
VOLUME LI, NUMBER 3
SEPTEMBER 1993, PAGES 519–534

NONLINEAR FIELD INSTABILITY AND CHAOS IN MAGNETIC FLUIDS

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Abstract. We study the nonlinear dynamics of normal field instability in a ferrofluid under the influence of a uniform magnetic field. In addition, a small normal sinusoidal magnetic field is superimposed on the system. An equation governing the evolution of small but finite amplitude is obtained. Applying the Melnikov method, it is shown that there exist transverse homoclinic orbits leading to chaotic motions.

1. Introduction. The normal field instability in magnetic fluids has received considerable attention of various scientists [1–9]. Cowley and Rosensweig [2] investigated the linear stability of two superposed magnetic fluids in the presence of an externally applied magnetic field acting normal to the interface. They reported that an instability sets in when the applied magnetic field H exceeds the critical magnetic field H_c. Their pioneer experiment demonstrated that the flat interface deforms to form a regular hexagonal pattern, and the subsequent finite amplitude state is time independent. Inspired by this experimentation, Gailītis [4], Brancher [6], Malik and Singh [9], Twombly and Thomas [5], and Silber and Knoblock [8] formulated nonlinear theoretical models for normal field instability. With the use of the global energy method, Gailītis [4] studied the stability behaviour of the surface peak amplitude against the magnetic field. His analysis reveals the existence of hard excitation of a steady wave. Furthermore, the hexagonal cell is replaced by a square cell with a possible hysteresis behaviour when the magnetic field is subcritical. By integrating numerically the magnetic fluid equations, Boudouvis et al [7] not only confirmed results of Gailītis [4] theoretically but experimentally as well. Such an experiment demonstrates first-order excitation with hysteresis as predicted by nonlinear stability and bifurcation theory. In physical reality, the bifurcation is imperfect if the magnetic fluid containers are of finite size or if the field nonuniformities and wall wetting are taken into account.

The external modulations of a nonlinear system can dramatically alter its behaviour and can induce novel dramatic states, particularly near a point of instability H_c. In this paper, we undertake to investigate how a nonlinear magnetic fluid system displays instability when the applied magnetic field is modulated with respect to time.

Received September 17, 1991.
1991 *Mathematics Subject Classification.* Primary 76E25.

In this presentation, we examine the time dependence of the magnetic fluid surface pattern brought about by an additional magnetic field of frequency Ω_0 for small values of the control parameter $\delta = m_0/M_c$. Here M_c is the critical magnetization and $M_c + m_0 \cos(\Omega_0 t)$ represents the total applied magnetization. Under static conditions, the system undergoes a simple bifurcation such as a transition from a homogeneous state to a partially structured state.

Recently, Bacri, d'Ortona, and Salin [10] reported interesting experimental observations of nonlinear period doubling in the normal field instability problem. These authors observe that for low values of δ, the peak oscillates at the resonance frequency Ω_0; however, for large enough values of δ, the oscillation is subharmonic. In two dimensions, this transition occurs with symmetry breaking from a triangular to a square pattern.

In Sec. 2 we employ the method of multiple scales and derive an evolution equation governing the amplitude for H in the close vicinity of H_c. When the driving terms are absent and the dissipation effects are neglected, the evolution equation reduces to a nonlinear Klein-Gordon equation as reported earlier [3, 9]. When the spatial effects are ignored, the equation for the amplitude leads to exponential decays, subcritical and explosive instabilities, all of which depend sensitively on the sign of the magnetic field parameter $\varepsilon = (H^2/H_c^2 - 1)^{1/2}$, the nonlinear interaction parameter Q, and the magnetic permeabilities of the media involved. In Sec. 3, we formulate the Melnikov function [11–14] to show that if the ratio of forcing to dissipation is sufficiently small, then there exist transverse homoclinic orbits resulting in chaotic behaviour.

2. Formulation. We consider an incompressible magnetic fluid of density ρ_1, magnetic permeability μ_1, and kinematic viscosity ν in the region $z \leq 0$. The other half space $z > 0$ is occupied by an inviscid fluid of density ρ_2 and magnetic permeability μ_2. The motion of the fluids under gravity $\mathbf{g}(0, 0, -1)$ is influenced by an externally driven magnetic field acting normal to the interface. The magnetic field parameter $\varepsilon = (H^2/H_c^2 - 1)^{1/2}$ defines the departure of the system from the bifurcation value H_c which is assumed to be small. The equation of motion for the magnetic fluid is

$$\rho_l \frac{dv_i^{(l)}}{dt} = -(\Pi_{ij}^{(l)})_{,j}, \qquad l = 1, 2, \quad i = 1, 2, 3, \tag{1}$$

where

$$\Pi_{ij}^{(l)} = p^{(l)}\delta_{ij} - \rho_l \nu_l (v_{i,j}^{(l)} + v_{j,i}^{(l)}) + m_{ij}^{(l)} + \rho_l g z \delta_{ij} \tag{2}$$

and

$$m_{ij}^{(l)} = -\frac{\delta_{ij}}{4\pi} \int_0^{H^{(l)}} \left[\mu_l - \rho_l \left(\frac{\partial \mu_l}{\partial \rho_l} \right)_{H, \theta_0} \right] H^{(l)} \cdot dH^{(l)} + \mu_l \frac{H_i^{(l)} H_j^{(l)}}{4\pi}. \tag{3}$$

Here μ_l is a function of H, θ_0, and ρ, where θ_0 represents the temperature, and p is the hydrostatic pressure. The equations governing the perturbed magnetic potentials ψ $(H = -\nabla\psi)$ are

$$\nabla^2 \psi^{(1)} = 0, \qquad -\infty < z < \eta(x, t), \tag{4}$$

$$\nabla^2 \psi^{(2)} = 0, \qquad \eta(x, t) < z < \infty, \tag{5}$$

where $\eta(x, t)$ is the free surface elevation at time t. The vanishing of the motion away from the interface requires

$$\begin{aligned} |\nabla\psi^{(1)}| &\to 0 \quad \text{as } z \to -\infty, \\ |\nabla\psi^{(2)}| &\to 0 \quad \text{as } z \to \infty. \end{aligned} \tag{6}$$

The boundary conditions at the free surface $z = \eta(x, t)$ are given by

$$\frac{\partial\eta}{\partial t} + \mathbf{v}^{(l)} \cdot \nabla\eta = w^{(l)}, \tag{7}$$

$$(\mathbf{B}^{(1)} - \mathbf{B}^{(2)}) \cdot \mathbf{n} = 0, \tag{8}$$

$$(\mathbf{H}^{(1)} - \mathbf{H}^{(2)}) \times \mathbf{n} = 0, \tag{9}$$

$$(\rho_1 p^{(1)}\delta_{ij} + m_{ij}^{(1)})n_j + (\rho_1 - \rho_2)g\eta n_i = T\frac{\partial^2\eta}{\partial x^2}\left[1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]^{-3/2} n_i + (\rho_2 p^{(2)}\delta_{ij} + m_{ij}^{(2)})n_j, \tag{10}$$

and

$$\frac{\partial u^{(l)}}{\partial z} + \frac{\partial w^{(l)}}{\partial x} = 0.$$

The superscripts 1, 2 denote the media 1 and 2, respectively. Here, T is the coefficient of the surface tension, $\mathbf{v} = (u, 0, w)$, and the quantities $\mathbf{n}$ and $\mathbf{B}$ denote the outward drawn normal and the magnetic induction, respectively. We shall assume flows to be isothermal and the magnetic fluids to be linearly magnetizable to the lowest order. It should be noted that in magnetic fluids there is a significant distinction between the magnetic induction $\mathbf{B}$ and the magnetic field $\mathbf{H}$. These vector fields are related as

$$\mathbf{B}^{(l)} = \mu_0(\mathbf{H}^{(l)} + \mathbf{M}^{(l)}) = \mu_l\mathbf{H}^{(l)}, \qquad l = 1, 2, \tag{11}$$

where

$$\mu_l = \mu_0(1 + \chi_l), \qquad \chi_l\mathbf{H}^{(l)} = \mathbf{M}^{(l)}.$$

Here μ_0, μ_l, and χ_l represent the magnetic permeability of the vacuum, the magnetic permeabilities of the magnetic fluids, and the susceptibilities of the fluids, respectively. To describe the significant nonlinear interactions in a slightly unstable region, we introduce a small parameter $\varepsilon = (H^2/H_c^2 - 1)^{1/2}$. This parameter defines the departure of the system from the bifurcation value H_c and is assumed to be small. We are interested in studying the effect on pattern formation when the externally applied magnetic field is near the value of the critical magnetization M_c of the fluid. The magnetic field is also subject to an additional vertical alternating magnetic field of driving frequency Ω_0 corresponding to magnetization $m_0\cos(\Omega_0 t)$. For small values of the control parameter $\delta = m_0/M_c$, the surface oscillates with

frequency Ω_0. In this paper, we confine ourselves to the principal resonance only. The effect of viscosity, assumed to be small, is $O(\varepsilon^2)$. We want to show that if the driving terms are strong enough compared to the viscous term, then there exists a possibility of chaotic motion in the system. To derive asymptotic solutions from Eqs. (1)–(11) for small but finite amplitude, we use the method of multiple scales. Introducing spatial and temporal scales

$$x_n = \varepsilon^n x, \quad t_n = \varepsilon^n t, \qquad n = 0, 1, 2, \tag{12}$$

we expand the various physical quantities

$$\Phi(x, z, t) = \sum_{n=1}^{N} \varepsilon^n \phi_n(z, x_0, x_1, x_2; t_0, t_1, t_2) + O(\varepsilon^{N+1}). \tag{13}$$

Here Φ can be any of the physical quantities $\mathbf{v}$, ψ, p, and η. To apply the boundary conditions (7)–(11), we express various physical quantities in terms of the Maclaurin Series expansion at $z = 0$. The system of equations (1) to (11) can now be solved by using Eqs. (12) and (13) and by equating the coefficients of equal powers in ε. We obtain the linear as well as the successive higher-order perturbation equations. The hierarchy of equations for each order can be solved with the knowledge of the solutions of the previous orders. Furthermore, it is sufficient to consider only the solutions up to $O(\varepsilon^3)$ as far as the lowest significant approximation is concerned.

We introduce the following linear operators:

$$L_1[u^{(l)}, \Pi^{(l)}] = \rho_l \frac{\partial u^{(l)}}{\partial t_0} + \frac{\partial \Pi^{(l)}}{\partial x_0}, \tag{14}$$

$$L_2[w^{(l)}, \Pi^{(l)}] = \rho_l \frac{\partial w^{(l)}}{\partial t_0} + \frac{\partial \Pi^{(l)}}{\partial z}, \tag{15}$$

$$L_3[\psi^{(l)}] = \frac{\partial^2 \psi^{(l)}}{\partial x_0^2} + \frac{\partial^2 \psi^{(l)}}{\partial z^2}, \tag{16}$$

$$L_4[\eta, w^{(l)}] = \frac{\partial \eta}{\partial t_0} - w^{(l)}, \tag{17}$$

$$L_5[\psi^{(1)}, \psi^{(2)}] = \mu \frac{\partial \psi^{(1)}}{\partial z} - \frac{\partial \psi^{(2)}}{\partial z}, \qquad \mu = \frac{\mu_1}{\mu_2}, \tag{18}$$

$$L_6[\psi^{(1)}, \psi^{(2)}, \eta] = \frac{\partial \psi^{(1)}}{\partial x_0} - \frac{\partial \psi^{(2)}}{\partial x_0} + \frac{\mu - 1}{\mu} H \frac{\partial \eta}{\partial x_0}, \tag{19}$$

$$\begin{aligned} &L_7[p^{(1)}, p^{(2)}, \eta, \psi^{(1)}] \\ &\quad = p^{(2)} - p^{(1)} + (\rho_1 - \rho_2) g \eta - T \frac{\partial^2 \eta}{\partial x_0^2} + \frac{\mu_2(\mu - 1)}{4\pi} H \frac{\partial \psi^{(1)}}{\partial z}, \end{aligned} \tag{20}$$

where

$$\Pi^{(l)} = p^{(l)} + g \rho_l z - \mu_0 \int_0^{H^{(l)}} M \, dH. \tag{21}$$

A. *The linear problem.* The first-order problem is generated by the equations

$$L_1[u_1^{(l)}, \Pi_1^{(l)}] = 0, \quad L_2[w_1^{(l)}, \Pi_1^{(l)}] = 0, \qquad l = 1, 2, \tag{22}$$

$$L_3[\psi_1^{(l)}] = 0, \tag{23}$$

with the boundary conditions at $z = 0$ being

$$L_4[\eta_1, w_1^{(l)}] = 0, \qquad L_5[\psi_1^{(1)}, \psi_1^{(2)}] = 0, \tag{24}$$

$$L_6[\psi_1^{(1)}, \psi_1^{(2)}, \eta_1] = 0, \tag{25}$$

$$L_7[p_1^{(1)}, p_1^{(2)}, \eta, \psi_1^{(1)}] = 0. \tag{26}$$

The boundary-value problem posed by Eqs. (22)–(26) admits a solution of the type

$$u_1^{(1)} = \omega[A \exp i\theta + \text{c. c.}] \exp(kz), \tag{27}$$

$$u_1^{(2)} = \omega[A \exp i\theta + \text{c. c.}] \exp(-kz), \tag{28}$$

$$w_1^{(1)} = -i\omega[A \exp i\theta + \text{c. c.}] \exp(kz), \tag{29}$$

$$w_1^{(2)} = -i\omega[A \exp i\theta + \text{c. c.}] \exp(-kz), \tag{30}$$

$$\psi_1^{(1)} = \frac{B}{\mu}[A \exp i\theta + \text{c. c.}] \exp(kz), \tag{31}$$

$$\psi_1^{(2)} = -B[A \exp i\theta + \text{c. c.}] \exp(-kz), \tag{32}$$

$$p_1^{(1)} = \frac{\rho_1 \omega^2}{k}[A \exp i\theta + \text{c. c.}] \exp(kz), \tag{33}$$

$$p_1^{(2)} = -\frac{\rho_2 \omega^2}{k}[A \exp i\theta + \text{c. c.}] \exp(-kz), \tag{34}$$

where

$$\theta = kx_0 - \omega t_0 \quad \text{and} \quad B = H(1-\mu)/(1+\mu). \tag{35}$$

For the first-order problem $O(\varepsilon)$, the frequencey ω and the wavenumber k satisfy the dispersion relation

$$D(\omega, k) = -\omega^2(\rho_1 + \rho_2) + (\rho_1 - \rho_2)gk + Tk^3 - \frac{\mu_2(\mu-1)^2 H^2 k^2}{4\pi\mu(\mu+1)} = 0. \tag{36}$$

Setting $\mu_2 = \mu_0$ and $\rho_2 = 0$ and recognizing that $H^{(1)} - H^{(2)} = M_0$, we recover Rosensweig's result [1].

The condition for criticality, i.e., the transition between a stable and an unstable mode is given by $\text{Real}(\omega) = 0$. The neutral stability curve furnishes the critical value of the magnetic field:

$$H_c^2 = \left[\frac{\mu_2(\mu-1)^2}{8\pi\mu(\mu+1)}\right]^{-1} [(\rho_1 - \rho_2)gT]^{1/2}, \tag{37}$$

when

$$k_c = \left[\frac{(\rho_1 - \rho_2)g}{T}\right]^{1/2}. \tag{38}$$

The instability sets in when $H \geq H_c$. The system bifurcates into a new steady state for the post critical values of the magnetic field. Cowley and Rosensweig [2] have experimentally verified the existence of such a critical field.

B. *Second-order problem.* The equations that govern the second-order solution are

$$\begin{aligned} L_1[u_2^{(l)}, \Pi_2^{(l)}] = \nu_l \rho_l \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z^2}\right) u_1^{(l)} - \rho_l \left(u_1^{(l)} \frac{\partial}{\partial x_0} + w_1^{(l)} \frac{\partial}{\partial z}\right) u_1^{(l)} \\ - \frac{\partial \Pi_1^{(l)}}{\partial x_1} - \rho_l \frac{\partial}{\partial t_1} u_1^{(l)}, \end{aligned} \tag{39}$$

$$L_2[w_2^{(l)}, \Pi_2^{(l)}] = \nu_l \rho_l \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z^2}\right) w_1^{(l)} - \rho_l \left(u_1^{(l)} \frac{\partial}{\partial x_0} + w_1^{(l)} \frac{\partial}{\partial z}\right) w_1^{(l)} - \rho_l \frac{\partial}{\partial t_1} w_1^{(l)}, \tag{40}$$

$$L_3[\psi_2^{(l)}] = -2 \frac{\partial^2 \psi_1^{(l)}}{\partial x_0 \partial x_1}, \tag{41}$$

and the boundary conditions at $z = 0$:

$$L_4[\eta_2, w_2^{(l)}] = -\frac{\partial \eta_1}{\partial t_1} - u_1^{(l)} \frac{\partial \eta_1}{\partial x_0} + \eta_1 \frac{\partial}{\partial z} w_1^{(l)}, \tag{42}$$

$$L_5[\psi_2^{(1)}, \psi_2^{(2)}] = -\eta_1 \left(\mu \frac{\partial^2 \psi_1^{(1)}}{\partial z^2} - \frac{\partial^2 \psi_1^{(2)}}{\partial z^2}\right) + \left(\frac{\partial \eta_1}{\partial x_0}\right) \left(\mu \frac{\partial \psi_1^{(1)}}{\partial x_0} - \frac{\partial \psi_1^{(2)}}{\partial x_0}\right), \tag{43}$$

$$\begin{aligned} L_6[\psi_2^{(1)}, \psi_2^{(2)}, \eta_2] = & -\eta_1 \left(\frac{\partial^2 \psi_1^{(1)}}{\partial x_0 \partial z} - \frac{\partial^2 \psi_1^{(2)}}{\partial x_0 \partial z}\right) - \left(\frac{\partial \eta_1}{\partial x_0}\right) \left(\frac{\partial \psi_1^{(1)}}{\partial z} - \frac{\partial \psi_1^{(2)}}{\partial z}\right) \\ & + \left(\frac{1-\mu}{\mu}\right) H \left(\frac{\partial \eta_1}{\partial x_1}\right) - \left(\frac{\partial \psi_1^{(1)}}{\partial x_1} - \frac{\partial \psi_1^{(2)}}{\partial x_1}\right), \end{aligned} \tag{44}$$

$$\begin{aligned} & L_7[p_2^{(1)}, p_2^{(2)}, \eta_2, \psi_2^{(2)}] \\ & = -\eta_1 \frac{\partial}{\partial z}\left(p_1^{(1)} - p_1^{(2)}\right) - 2\rho_1 \nu_1 \frac{\partial w_1^{(1)}}{\partial z} + 2T \frac{\partial^2 \eta_1}{\partial x_0 \partial x_1} \\ & + \frac{\mu_2(\mu - 1)}{8\pi} \left[\left(\frac{\partial \psi_1^{(1)}}{\partial x_0}\right)^2 + \mu \left(\frac{\partial \psi_1^{(2)}}{\partial z}\right)^2 + \left(\frac{\partial \eta_1}{\partial x_0}\right)^2 \frac{H^2}{\mu^2}(1-\mu) \right. \\ & \left. - 2H\eta_1 \frac{\partial^2 \psi_1^{(1)}}{\partial z^2} + 2H \frac{\partial \eta_1}{\partial x_0} \frac{\partial \psi_1^{(1)}}{\partial x_0} \left(\frac{\mu - 1}{\mu}\right)\right]. \end{aligned} \tag{45}$$

We shall now obtain the solution of Eqs. (39)–(45) on the neutral stability curve by letting $H = H_c$ and $k = k_c$. On substituting the solution of the first-order problem

from Eqs. (27)–(35) into Eqs. (39)–(45), we get

$$L_1[u_2^{(l)}, \Pi_2^{(l)}] = 0, \qquad L_2[w_2^{(l)}, \Pi_2^{(l)}] = 0, \tag{46}$$

$$L_3[\psi_2^{(1)}] = \frac{2iB}{\mu} k_c \frac{\partial A}{\partial x_1} \exp(i\theta_c + k_c z) + \text{c.c.}, \tag{47}$$

$$L_3[\psi_2^{(2)}] = -2iBk_c \frac{\partial A}{\partial x_1} \exp(i\theta_c - k_c z) + \text{c.c.}, \tag{48}$$

where

$$\theta_c = k_c x_0. \tag{49}$$

The boundary conditions at $z = 0$ reduce to

$$L_4[\eta_2, w_2^{(l)}] = -\frac{\partial A}{\partial t_1} \exp(i\theta_c) + \text{c.c.}, \tag{50}$$

$$L_5[\psi_2^{(1)}, \psi_2^{(2)}] = -4k_c^2 \mu B A^2 \exp(2i\theta_c) + \text{c.c.}, \tag{51}$$

$$L_6[\psi_2^{(1)}, \psi_2^{(2)}, \eta_2] = \frac{\partial A}{\partial x_1} B \exp(i\theta_c) - 2ik_c^2(1-\mu)BA^2 \exp(2i\theta_c) + \text{c.c.}, \tag{52}$$

$$\begin{aligned} L_7[p_2^{(1)}, p_2^{(2)}, \eta_2, \psi_2^{(2)}] = {} & 2ik_c T \frac{\partial A}{\partial x_1} \exp(i\theta_c) \\ & + \frac{3\mu_2 \mu(\mu+1)}{4\pi} k_c^2 B^2 A^2 \exp(2i\theta_c). \end{aligned} \tag{53}$$

Since the homogeneous part of the second-order problem has a nontrivial solution, which is the same as that of the first-order problem, the inhomogeneous problem has a solution if and only if the inhomogeneous part is orthogonal to every solution of the adjoint homogeneous problem. Following Malik and Singh [3], this solvability condition yields

$$\left(2Tk - \frac{\mu_2(\mu-1)^2 H^2}{4\pi\mu(\mu+1)}\right) \frac{\partial A}{\partial x_1} = 0 \tag{54}$$

on the marginally neutral curve, where $H = H_c$ and $k = k_c$. Equation (54) states that $\partial A/\partial x_1$ is nonvanishing; hence, the particular solution of the second-order problem defined by Eqs. (46)–(53) is

$$u_2^{(1)} = i\frac{\partial A}{\partial t_1} \exp(i\theta_c + k_c z) + \text{c.c.}, \tag{55}$$

$$u_2^{(2)} = -i\frac{\partial A}{\partial t_1} \exp(i\theta_c - k_c z) + \text{c.c.}, \tag{56}$$

$$w_2^{(1)} = \frac{\partial A}{\partial t_1} \exp(i\theta_c + k_c z) + \text{c.c.}, \tag{57}$$

$$w_2^{(2)} = \frac{\partial A}{\partial t_1} \exp(i\theta_c - k_c z) + \text{c.c.}, \tag{58}$$

$$\eta_2 = \lambda A^2 \exp(2i\theta_c) + \text{c.c.}, \tag{59}$$

$$\psi_2^{(1)} = \frac{iB}{\mu k_c}(1 - zk_c)\frac{\partial A}{\partial x_1}\exp(i\theta_c + k_c z) + (\lambda - k_c)\frac{B}{\mu}A^2\exp(2i\theta_c + 2k_c z) + \text{c.c.}, \tag{60}$$

$$\psi_2^{(2)} = -\frac{iB}{k_c}(1 + zk_c)\frac{\partial A}{\partial x_1}\exp(i\theta_c - k_c z) - (\lambda + k_c)BA^2\exp(2i\theta_c - 2k_c z) + \text{c.c.}, \tag{61}$$

where

$$\lambda = \frac{k_c^2 V_A^2(\mu - 1)/(\mu + 1)}{(\rho_1 - \rho_2)g + 4k_c^2 T - 2V_A^2 k_c^2} \tag{62}$$

and

$$V_A^2 = \frac{\mu_2 H^2}{4\pi}. \tag{63}$$

It is interesting to note here that there is no contribution to the pressure terms in the second-order problem, because when $H = H_c$ and $k = k_c$, there is no mean and no harmonic forcing. Equations (59) and (62) indicate singularity when $k_c^2 = \frac{1}{2}[(\rho_1 - \rho_2)g/T]$. Physically, this represents the phenomenon of second harmonic resonance. In magnetic fluids, this phenomenon has been studied earlier by Malik and Singh [15].

C. *Third-order problem.* The field equations describing the third-order problem are

$$\begin{aligned} L_1[u_3^{(l)}, \Pi_3^{(l)}] = &-\rho_l\frac{\partial u_2^{(l)}}{\partial t_1} - \rho_l\frac{\partial u_1^{(l)}}{\partial t_2} - \frac{\partial \Pi_2^{(l)}}{\partial x_1} - \frac{\partial \Pi_1^{(l)}}{\partial x_2} \\ &- \left(u_1^{(l)}\frac{\partial}{\partial x_0} + w_1^{(l)}\frac{\partial}{\partial z}\right)u_2^{(l)} - \left(u_2^{(l)}\frac{\partial}{\partial x_0} + w_2^{(l)}\frac{\partial}{\partial z}\right)u_1^{(l)} \\ &+ \rho_l\nu_l\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z^2}\right)u_2^{(l)} + 2\rho_l\nu_l\frac{\partial^2}{\partial x_0\partial x_1}u_1^{(l)} - u_1^{(l)}\frac{\partial u_1^{(l)}}{\partial x_1}, \end{aligned} \tag{64}$$

$$\begin{aligned} L_2[w_3^{(l)}, \Pi_3^{(l)}] = &-\rho_l\frac{\partial w_2^{(l)}}{\partial t_1} - \rho_l\frac{\partial w_1^{(l)}}{\partial t_2} - \left(u_1^{(l)}\frac{\partial}{\partial x_0} + w_1^{(l)}\frac{\partial}{\partial z}\right)w_2^{(l)} \\ &- \left(u_2^{(l)}\frac{\partial}{\partial x_0} + w_2^{(l)}\frac{\partial}{\partial z}\right)w_1^{(l)} + \rho_l\nu_l\left(\frac{\partial}{\partial x_0^2} + \frac{\partial^2}{\partial z^2}\right)w_2^{(l)} \\ &+ 2\rho_l\nu_l\frac{\partial^2}{\partial x_0\partial x_1}w_1^{(l)} - u_1^{(l)}\frac{\partial w_1^{(l)}}{\partial x_1}, \end{aligned} \tag{65}$$

$$L_3[\psi_3^{(l)}] = -2\frac{\partial \psi_1^{(l)}}{\partial x_0\partial x_2} - 2\frac{\partial^2\psi_2^{(l)}}{\partial x_0\partial x_1} - \frac{\partial^2\psi_1^{(l)}}{\partial x_1^2}, \tag{66}$$

and the boundary conditions at $z = 0$:

$$L_4[\eta_3, w_3^{(l)}] = -\frac{\partial \eta_1}{\partial t_2} - \frac{\partial \eta_1}{\partial t_1} + \eta_1 \frac{\partial w_2^{(l)}}{\partial z} + \eta_2 \frac{\partial w_1^{(l)}}{\partial z} + \frac{1}{2}\eta_1^2 \frac{\partial^2 w_1^{(l)}}{\partial z^2} - u_1^{(l)} \frac{\partial \eta_2}{\partial x_0} - u_2^{(l)} \frac{\partial \eta_1}{\partial x_1} - \eta_1 \frac{\partial u_1^{(l)}}{\partial z} \frac{\partial \eta_1}{\partial x_0} - u_1^{(l)} \frac{\partial \eta_1}{\partial x_1}, \tag{67}$$

$$\begin{aligned} L_5[\psi_3^{(1)}, \psi_3^{(2)}] = & -\eta_1 \left[\mu \frac{\partial^2 \psi_2^{(1)}}{\partial z^2} - \frac{\partial^2 \psi_2^{(2)}}{\partial z^2}\right] - \eta_2 \left[\mu \frac{\partial^2 \psi_1^{(1)}}{\partial z^2} - \frac{\partial^2 \psi_1^{(2)}}{\partial z^2}\right] \\ & - \frac{\eta_1^2}{2}\left(\mu \frac{\partial^3 \psi_1^{(1)}}{\partial z^3} - \frac{\partial^3 \psi_1^{(2)}}{\partial z^3}\right) + \left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\mu \frac{\partial \psi_2^{(1)}}{\partial x_0} - \frac{\partial \psi_2^{(2)}}{\partial x_0}\right) \\ & + \left(\frac{\partial \eta_2}{\partial x_0}\right)\left(\mu \frac{\partial \psi_1^{(1)}}{\partial x_0} - \frac{\partial \psi_1^{(2)}}{\partial x_0}\right) + \eta_1 \frac{\partial \eta_1}{\partial x_0}\left(\mu \frac{\partial^2 \psi_1^{(1)}}{\partial x_0 \partial z} - \frac{\partial^2 \psi_1^{(2)}}{\partial x_0 \partial z}\right) \\ & + \left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\mu \frac{\partial \psi_1^{(1)}}{\partial x_0} - \frac{\partial \psi_1^{(2)}}{\partial x_0}\right) + \left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\mu \frac{\partial \psi_1^{(1)}}{\partial x_1} - \frac{\partial \psi_1^{(2)}}{\partial x_1}\right), \end{aligned} \tag{68}$$

$$\begin{aligned} L_6[\psi_3^{(1)}, \psi_3^{(2)}, \eta_3] = & -\eta_1 \left(\frac{\partial^2 \psi_2^{(1)}}{\partial x_0 \partial z} - \frac{\partial^2 \psi_2^{(2)}}{\partial x_0 \partial z}\right) - \eta_2 \left(\frac{\partial^2 \psi_1^{(1)}}{\partial x_0 \partial z} - \frac{\partial^2 \psi_1^{(2)}}{\partial x_0 \partial z}\right) \\ & - \frac{\eta_1^2}{2}\left(\frac{\partial^3 \psi_1^{(1)}}{\partial x_0 \partial z^2} - \frac{\partial^3 \psi_1^{(2)}}{\partial x_0 \partial z^2}\right) - \left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\frac{\partial \psi_2^{(1)}}{\partial z} - \frac{\partial \psi_2^{(2)}}{\partial z}\right) \\ & - \left(\frac{\partial \eta_2}{\partial x_0}\right)\left(\frac{\partial \psi_1^{(1)}}{\partial z} - \frac{\partial \psi_1^{(2)}}{\partial z}\right) \\ & - \eta_1 \left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\frac{\partial^2 \psi_1^{(1)}}{\partial z^2} - \frac{\partial^2 \psi_1^{(2)}}{\partial z^2}\right) - \left(\frac{\partial \psi_2^{(1)}}{\partial x_1} - \frac{\partial \psi_2^{(2)}}{\partial x_1}\right) \\ & - \left(\frac{\partial \eta_1}{\partial x_1}\right)\left(\frac{\partial \psi_1^{(1)}}{\partial z} - \frac{\partial \psi_1^{(2)}}{\partial z}\right) - \left(\frac{\partial \psi_1^{(1)}}{\partial x_2} - \frac{\partial \psi_1^{(2)}}{\partial x_2}\right) \\ & + \left(\frac{1-\mu}{\mu}\right) H \left(\frac{\partial \eta_2}{\partial x_1} + \frac{\partial \eta_1}{\partial x_2}\right), \end{aligned} \tag{69}$$

$$
\begin{aligned}
&L_7[p_3^{(1)}, p_3^{(2)}, w_3^{(1)}, \eta_3, \psi_3^{(1)}] \\
&= -\eta_1\left(\frac{\partial p_2^{(2)}}{\partial z} - \frac{\partial p_2^{(1)}}{\partial z}\right) - \eta_2\left(\frac{\partial p_1^{(2)}}{\partial z} - \frac{\partial p_1^{(1)}}{\partial z}\right) \\
&\quad - \frac{1}{2}\eta_1^2\left(\frac{\partial^2 p_1^{(2)}}{\partial z^2} - \frac{\partial^2 p_1^{(1)}}{\partial z^2}\right) + 2T\frac{\partial^2 \eta_1}{\partial x_0 \partial x_2} + 2T\frac{\partial^2 \eta_2}{\partial x_0 \partial x_1} + T\frac{\partial^2 \eta_1}{\partial x_1^2} \\
&\quad - \frac{3}{2}T\frac{\partial^2 \eta_1}{\partial x_0^2}\left(\frac{\partial \eta_1}{\partial x_0}\right)^2 - 2\rho_1\nu\frac{\partial w_2^{(1)}}{\partial z} - 2\rho_1\nu\eta\frac{\partial w_1^{(1)}}{\partial z} \\
&\quad + \frac{\mu_2(\mu-1)}{8\pi}\left[2\frac{\partial \psi_1^{(1)}}{\partial x_0}\frac{\partial \psi_2^{(1)}}{\partial x_0} + 2\eta_1\frac{\partial \psi_1^{(1)}}{\partial x_0}\frac{\partial^2 \psi_1^{(1)}}{\partial x_0 \partial z}\right. \\
&\qquad + 2\mu\left(\frac{\partial \psi_1^{(1)}}{\partial z}\frac{\partial \psi_2^{(1)}}{\partial z} + \eta_1\frac{\partial \psi_1^{(1)}}{\partial z}\frac{\partial^2 \psi_1^{(1)}}{\partial z^2}\right) \\
&\qquad - 2(\mu-1)\frac{\partial \eta_1}{\partial x_0}\frac{\partial \psi_1^{(1)}}{\partial x_0}\frac{\partial \psi_1^{(1)}}{\partial z} + \frac{2H^2}{\mu^2}(1-\mu)\frac{\partial \eta_1}{\partial x_0}\frac{\partial \eta_2}{\partial x_0} \\
&\qquad + \frac{2H}{\mu}(\mu-1)\left(\frac{\partial \eta_1}{\partial x_0}\frac{\partial \psi_2^{(1)}}{\partial x_0} + \frac{\partial \eta_2}{\partial x_0}\frac{\partial \psi_1^{(1)}}{\partial x_0} + 2\eta_1\frac{\partial \eta_1}{\partial x_0}\frac{\partial^2 \psi_1^{(1)}}{\partial x_0 \partial z}\right) \\
&\qquad - 2H\left(\eta_1\frac{\partial^2 \psi_2^{(1)}}{\partial z^2} + \frac{\eta_1^3}{2}\frac{\partial^3 \psi_1^{(1)}}{\partial z^3} + \eta_2\frac{\partial^2 \psi_1^{(1)}}{\partial z^2}\right) \\
&\qquad + \frac{2H}{\mu}\left(\frac{\partial \eta_1}{\partial x_0}\right)^2\left(\frac{\partial \psi_1^{(1)}}{\partial z}\right)(\mu-1) + 2\frac{\partial \psi_1}{\partial x_1}\frac{\partial \psi_2}{\partial x_0} \\
&\qquad \left. + \frac{2H^2}{\mu^2}(1-\mu)\frac{\partial \eta_1}{\partial x_0}\frac{\partial \eta_1}{\partial x_1} + \frac{2H(\mu-1)}{\mu}\left(\frac{\partial \eta_1}{\partial x_0}\frac{\partial \psi_1^{(1)}}{\partial x_1} + \frac{\partial \eta_1}{\partial x_1}\frac{\partial \psi_1}{\partial x_0}\right)\right].
\end{aligned}
\tag{70}
$$

On substituting the first- and the second-order solutions given by Eqs. (27)–(35) and (55)–(63) into Eqs. (64)–(70), we obtain

$$L_1[u_3^{(1)}, \Pi_3^{(1)}] = -i\rho_1\frac{\partial^2 A}{\partial t_1^2}\exp(i\theta_c + k_c z) + \text{c.c.}, \tag{71}$$

$$L_1[u_3^{(2)}, \Pi_3^{(2)}] = i\rho_2\frac{\partial^2 A}{\partial t_1^2}\exp(i\theta_c - k_c z) + \text{c.c.}, \tag{72}$$

$$L_2[w_3^{(1)}, \Pi_3^{(1)}] = -\rho_1\frac{\partial^2 A}{\partial t_1^2}\exp(i\theta_c + k_c z) + \text{c.c.}, \tag{73}$$

$$L_2[w_3^{(2)}, \Pi_3^{(2)}] = -\rho_2\frac{\partial^2 A}{\partial t_1^2}\exp(i\theta_c - k_c z) + \text{c.c.}, \tag{74}$$

$$L_3[\psi_3^{(1)}] = -\frac{iB}{\mu}\left[2k_c\frac{\partial A}{\partial x_2} + (2k_c z - 1)\frac{\partial^2 A}{\partial x_1^2}\right]\exp(i\theta_c + k_c z) + \text{c.c.}, \tag{75}$$

$$L_3[\psi_3^{(2)}] = iB\left[2k_c\frac{\partial A}{\partial x_2} + (2k_c z + 1)\frac{\partial^2 A}{\partial x_1^2}\right]\exp(i\theta_c - k_c z) + \text{c.c.}, \tag{76}$$

$$L_4[w_3^{(1)}, \eta_3] = -2\frac{\partial A}{\partial t_2}\exp(i\theta_c) + \text{c.c.}, \tag{77}$$

$$L_5[\psi_3^{(1)}, \psi_3^{(2)}] = -2k_c^2\mu\lambda B A^2\overline{A} + \text{c.c.}, \tag{78}$$

$$L_6[\psi_3^{(1)}, \psi_3^{(2)}, \eta_3] = -\frac{(1-\mu)B}{k_c}\frac{\partial^2 A}{\partial x_1^2} - \frac{k_c^2 B A^2\overline{A}}{2}[6\lambda(1-\mu) - k_c(l+\mu)], \tag{79}$$

and

$$\begin{aligned} &L_7[p_3^{(1)}, p_3^{(2)}, w_3^{(1)}, \psi_3^{(1)}] \\ &= -\frac{1}{k_c}(\rho_1+\rho_2)\frac{\partial^2 A}{\partial t_1^2} + 2ik_c T\frac{\partial A}{\partial x_2} + \frac{3}{2}Tk_c^4 A^2\overline{A} \\ &\quad + T\frac{\partial^2 A}{\partial x_1^2} - 2\rho_1\nu k_c\frac{\partial A}{\partial t_1} + \frac{\mu_2(\mu-1)}{8\pi}H_c k_c^2 B(-2\lambda + 5k_c)A^2\overline{A}. \end{aligned} \tag{80}$$

Since the homogeneous part of the third-order problem defined by Eqs. (71)–(80) has a nontrivial solution, which is identical to that of the first-order problem, the inhomogeneous problem admits a nontrivial solution if and only if the inhomogeneous part is orthogonal to every solution of the adjoint problem. This yields the nonsecularity condition for η_3 when $k = k_c$ and $H = H_c$:

$$\frac{\partial^2 A}{\partial t_1^2} - \Omega_R^2\frac{\partial^2 A}{\partial x_1^2} + \frac{2\nu k_c^2\rho_1}{(\rho_1+\rho_2)}\frac{\partial A}{\partial t_1} = \sigma(\varepsilon^2 + \delta\cos(\Omega_0 t_1))A + Q|A|^2 A, \tag{81}$$

where

$$\Omega_R^2 = \frac{(\rho_1-\rho_2)g}{k_c(\rho_1+\rho_2)}, \qquad \sigma = \frac{2(\rho_1-\rho_2)gk_c}{(\rho_1+\rho_2)}, \tag{82}$$

$$Q = \frac{(\rho_1-\rho_2)gk_c^3}{2(\mu+1)^2(\rho_1+\rho_2)}(11(\mu^2+1) - 42\mu). \tag{83}$$

Equation (81) is the damped nonlinear Klein-Gordon equation. The undamped equation appears in various physical problems such as the buckling problem (see Lang and Newell [16]), the baroclinic instability (see Pedlosky [17]), the Kelvin-Helmholtz instability (see Weissman [18]), and the normal field instability in magnetic fluids (see Malik and Singh [9]). When A is real and the spatial modulation and dissipation are neglected, Eq. (81) admits a solution in terms of the Jacobi elliptic function. The

nature of such a solution depends sensitively on the coefficient ε and the nonlinear interaction parameter Q. The parameter Q is positive when $\mu > \mu_c$ $(= 3.54)$ and negative for $\mu < \mu_c$. This leads to hard and soft modes of excitations. We shall now examine Eq. (81), when $(H^2/H_c^2 - 1)$ is positive while Q is negative. The phase diagram is shown in Fig. 1. The origin and the points $A = \pm A_e$ happen to be the equilibrium points where

$$A_e = \frac{(2)^{1/2}}{k_c}\left|\frac{H^2}{H_c^2} - 1\right|^{1/2} |42\mu - 11(\mu^2 + 1)|^{-1/2}. \tag{84}$$

Introducing a dimensionless variable

$$A = \left|\frac{Q}{\sigma}\right|^{1/2} A' \tag{85}$$

and dropping the prime over A in subsequent discussions, Eqs. (81)–(85) furnish

$$m\frac{d^2A}{dt_1^2} + m\gamma\frac{dA}{dt_1} - [\varepsilon^2 + \delta\cos(\Omega_0 t_1)]A + A^3 + F\sin(\Omega_0 t_1) = 0, \tag{86}$$

where

$$m = \frac{\rho_1 + \rho_2}{2k_c(\rho_1 - \rho_2)g}, \qquad \gamma = \frac{2\nu k_c^2 \rho_1}{\rho_1 + \rho_2}, \tag{87}$$

and

$$F = f|Q|^{1/2}\Omega_0 \delta m. \tag{88}$$

Equation (86) is a damped parametrically modulated anharmonic oscillator equation. The inhomogeneous forcing term in Eq. (86) accounts for the inclusion of the finite width of the tank, imperfections, and field nonuniformities as observed in the experiments by Boudouvis et al [7]. Such a forcing term is responsible for imperfect bifurcation. The quantity f is merely a convenient parameterization. The forcing term in the amplitude equation (86) is taken to be $f d\varepsilon/dt_1$ in a phenomenological way to take into account rate of change of magnetization. Equation (86) has a reflection symmetry so that each solution is doubly degenerate. This degeneracy is removed by introducing a forcing term.

3. *Melnikov function and chaos.* We now discuss the mechanism of chaos that occurs when we follow the time dependence of regular peak pattern forced by an additional small normal alternating magnetic field of frequency Ω_0. The interesting case is that of the static normal field H when H is supercritical. The control parameter δ is assumed to be nonzero, and we wish to find the appropriate conditions that force a homoclinic bifurcation to occur. We will employ the Melnikov method [11] and the subsequent modifications [12–14]. By introducing the stretched variables

$$\tau = t_1\left(\frac{m^{1/2}}{\varepsilon}\right), \qquad X = \varepsilon A, \tag{89}$$

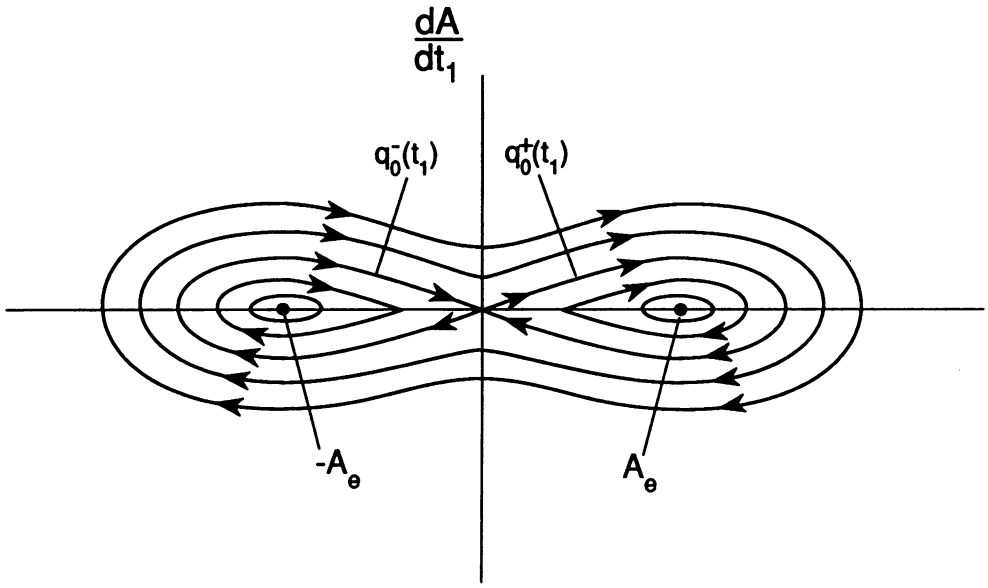

FIG. 1.

in Eq. (86), we obtain

$$\frac{d^2X}{d\tau^2} - X + X^3 = \Delta[-\beta_1 \frac{dX}{d\tau} + \beta_2 \cos(\Omega\tau)X - \beta_3 \sin(\Omega\tau)], \tag{90}$$

where

$$\begin{aligned} \beta_1 &= \left(\frac{m^{1/2}}{\varepsilon}\right)\gamma, \qquad \beta_2 = \left(\frac{\delta}{\varepsilon^2}\right), \\ \beta_3 &= \frac{f|Q|^{1/2}\Omega_0\delta}{\sigma\varepsilon^3}, \qquad \Omega = \Omega_0\left(\frac{m^{1/2}}{\varepsilon}\right). \end{aligned} \tag{91}$$

Here Δ is a parameter. It may be observed that when $\Delta = 0$, Eq. (90) is integrable. This corresponds to the case when the external magnetic field modulation and the viscous dissipations are absent. Rewriting Eq. (90) as a first-order system, we obtain

$$\begin{aligned} \frac{dX}{d\tau} &= V, \\ \frac{dV}{d\tau} &= X - X^3 + \Delta[-\beta_1 V + \beta_2 X\cos\phi - \beta_3 \sin\phi], \\ \frac{d\phi}{d\tau} &= \Omega. \end{aligned} \tag{92}$$

When $\Delta = 0$, the system has centres at $(\pm 1, 0)$ and a hyperbolic saddle at $(0, 0)$. Furthermore, it has a pair of homoclinic orbits given by

$$q_0^{\pm} = (X_0^{\pm}(\tau), V_0^{\pm}(\tau)) = (\pm\sqrt{2}\,\mathrm{sech}\,\tau, \mp\sqrt{2}\,\mathrm{sech}\,\tau\tanh\tau). \tag{93}$$

These separatrix orbits $q_0^{\pm}$ are exhibited in Fig. 1. When $\Delta \neq 0$ but is assumed to be a small quantity, the perturbed phase space is extended to three dimensions

(X, V, ϕ) and the motion can be seen in the Poincaré section

$$\Sigma^{\phi_0} = \{(X, V, \phi) \in \mathbb{R} \times \mathbb{R} \times S^1 | \phi = \phi_0\}. \tag{94}$$

The perturbed stable and unstable manifolds $w^s(\tau)$, $w^u(\tau)$ can still be identified in the Poincaré section and are made up of a sequence of distinct points. If the ratio of viscous dissipation to the external forcing is sufficiently small, the stable and unstable manifolds will intersect transversally, creating a homoclinic bifurcation point. If there exist such intersections, then chaotic motion of the Smale horseshoe may occur. The Melnikov function provides a measure of separation between $w^s(\tau)$ and $w^u(\tau)$ as functions of the phase ϕ of the Poincaré map [12]. This function is represented by the integral

$$M(\tau_0) = -\int_{-\infty}^{\infty} (\mathbf{f}_0 \wedge \mathbf{f}_1)\, dt, \tag{95}$$

where

$$\begin{aligned} f_{01} &= V, & f_{11} &= 0, \\ f_{02} &= X - X^3, & f_{12} &= -\beta_1 V + \beta_2 X \cos\phi - \beta_3 \sin\phi. \end{aligned} \tag{96}$$

Therefore,

$$\begin{aligned} M(\tau_0) = -\int_{-\infty}^{\infty} dt[&-\beta_1 V_0^2(t-\tau_0) + \beta_2 \cos(\Omega t) \\ &\times V_0(t-\tau_0) X_0(t-\tau_0) - \beta_3 V_0(t-\tau_0)\sin(\Omega t)]. \end{aligned} \tag{97}$$

The function $M(\tau_0)$ above measures the distance between the perturbed stable and unstable manifolds in the Poincaré section Σ^{ϕ_0}. Substituting Eq. (94) into Eq. (97), making the change of variables $\tau' = t - \tau_0$, and using the method of residues, we get

$$M(\tau_0) = \frac{4}{3}\beta_1 - \beta_2 \pi \Omega^2 \sin(\Omega\tau_0)\operatorname{csch}\left(\frac{\pi\Omega}{2}\right) - \sqrt{2}\beta_3 \pi\Omega \cos(\Omega\tau_0)\operatorname{sech}\left(\frac{\pi\Omega}{2}\right). \tag{98}$$

The Melnikov function $M(\tau_0)$ has a simple zero when

$$\frac{1}{\delta} < \mathscr{H}(\omega, \varepsilon, f, \nu), \tag{99}$$

where

$$\begin{aligned} \mathscr{H}(\omega, \varepsilon, f, \nu) = \frac{3}{8\varepsilon^3} \cdot \frac{\pi\Omega_0^2}{\Omega_H^2} &\left(\frac{\rho_1 \nu k_c^2}{\rho_1 + \rho_2}\right)^{-1} \\ &\times \left[\Omega_H^2 \operatorname{csch}^2\left(\frac{\pi\Omega_0}{2\Omega_H \varepsilon}\right) + f^2 |Q| \operatorname{sech}^2\left(\frac{\pi\Omega_0}{2\Omega_H\varepsilon}\right)\right]^{1/2}, \end{aligned} \tag{100}$$

with

$$\Omega_H^2 = \frac{2(\rho_1 - \rho_2) g k_c}{(\rho_1 + \rho_2)}. \tag{101}$$

The condition (99) implies the existence of transverse intersections of stable and unstable manifolds. It is independent of the choice of any particular Poincaré section Σ^{ϕ_0}. This leads to a local criterion for chaos valid near the unperturbed separatrix. According to the Smale-Birkhoff homoclinic theorem [12], the presence of such orbits

implies that there exists an invariant Cantor set Λ on which the standard Poincaré map is equivalent to the Smale horseshoe map. The set Λ contains a countable set of periodic orbits of arbitrary long periods, an uncountable set of nonperiodic orbits, and a dense orbit (cf. Guckenheimer and Holmes [12]). Thus, the condition (99) yields the regions on the parametric space where the chaotic dynamics may occur.

4. Discussion. The analysis in the previous section was carried out for small values of Δ. For the perturbation approach to be valid, we require that $\Delta\beta_n$ $(n = 1, 2, 3)$ be small. This puts restrictions on the parameters δ, f, and dissipation due to viscosity. For fixed values of ω and ε, the condition $\Delta\beta_n \ll 1$ requires

$$\left(\frac{\rho_1 \nu k_c^2}{(\rho_1+\rho_2)\varepsilon\Omega_H}\right), \qquad \delta\left(=\frac{m_0}{M_c}\right), \quad \text{and} \quad \frac{f|Q|^{1/2}\Omega_0\delta}{\Omega_H^3\varepsilon^3}$$

to be all $\ll 1$. For a given fluid and a fixed magnetic field parameter, we obtain $\Delta \ll \varepsilon$. It should be noted that the perturbation approach given in this paper will break down for $\Delta \simeq 1$. The function $\mathscr{H}$ defined in Eq. (100) approaches zero as $\Omega_0 \to 0$. For fixed values of the parameters ε, ν, and f, the function $\mathscr{H}$ attains a maximum value when

$$\frac{\partial \mathscr{H}}{\partial \Omega_0} = 0. \tag{102}$$

For small values of ε,

$$\Omega_{\max} = \frac{4}{\pi}\left[\frac{2(\rho_1-\rho_2)gk_c}{(\rho_1+\rho_2)}\left|\frac{H^2}{H_c^2}-1\right|\right]^{1/2}. \tag{103}$$

It is interesting to observe that as the modulating frequency Ω_0 is decreased, a window of chaos will appear in ferrofluids when $\Omega_1 < \Omega_0 < \Omega_2$. Newton [19] made similar predictions for the Bénard problem. The values of Ω_1 and Ω_2, which are given by Eq. (100), are sketched in Fig. 2. Equation (90) also supports subharmonic

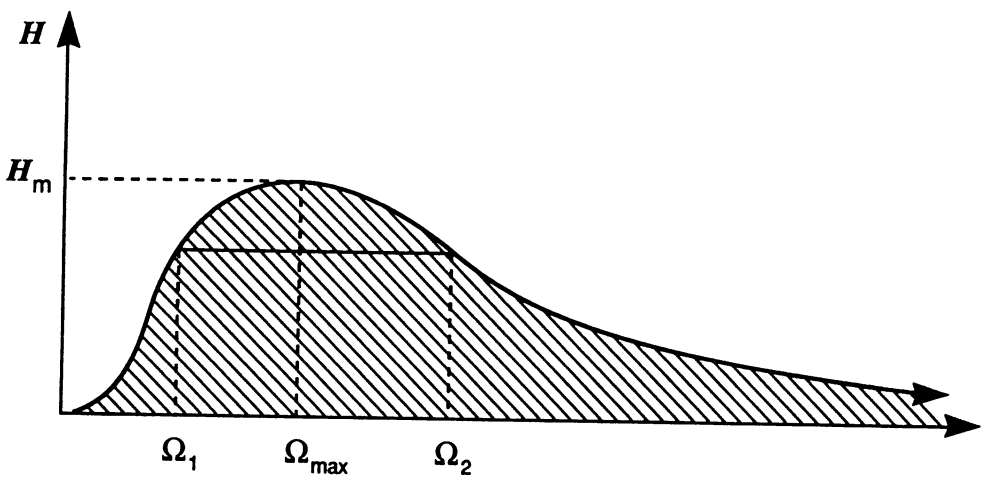

FIG. 2.

(period doubling) resonance. Such a nonlinear period doubling has been observed experimentally by Bacri et al [10]. By employing the averaging method and Melnikov's method, we have demonstrated the existence of subharmonic resonance resulting in chaotic motion [20]. Such a chaotic motion is a direct consequence of the existence of cascade of period doubling and saddle node bifurcation.

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