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DETERMINATION OF THE LEADING COEFFICIENT $a(x)$ IN THE HEAT EQUATION $u_t = a(x)\Delta u$

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Abstract. This note deals with the parabolic inverse problem of determination of the leading coefficient in the heat equation with an extra condition at the terminal. After introducing a new variable, we reformulate the problem as a nonclassical parabolic equation along with the initial and boundary conditions. The existence of a solution is established by means of the Schauder fixed-point theorem.

1. Introduction. Recently, considerable effort was made in dealing with inverse problems in partial differential equations. These inverse problems not only have the intrinsic mathematical interests, but also have a variety of applications in industry and engineering sciences. It is known that an inverse problem is not well-posed in general. An important task is to formulate the problem properly and to find the conditions that ensure its well-posedness. In the present work, we study the inverse problem of finding $a(x) > 0$ and $u(x, t)$, which satisfy:

$$u_t = a(x)\Delta u, \qquad (x, t) \in Q_T; \tag{1.1}$$

$$u(x, t) = g(x, t), \qquad (x, t) \in S_T = \partial\Omega \times [0, T]; \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

along with an extra condition

$$u(x, T) = u_1(x), \qquad x \in \Omega, \tag{1.4}$$

where $T > 0$ is fixed and $Q_T = \Omega \times (0, T]$, where Ω is a bounded domain in $\mathbf{R}^n$.

When an unknown coefficient appears in the lower-order terms, various results are obtained in [1, 2, 4] (also see [7] and the references therein). The uniqueness of solution of the problem (1.1)-(1.4) was studied in [8]. In the present work we shall follow the idea of [4] to establish the existence for the problem (1.1)-(1.4). After

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introducing a new variable, we have a nonlinear parabolic equation with the involvement of a trace-type functional as the leading coefficient. To avoid the degeneracy of the equation, we construct appropriate auxiliary functions and deduce some a priori estimates. The Schauder fixed point is used to prove the existence.

2. The assumptions and the main result. Let $v(x, t) = u_t(x, t)$. Then the extra condition (1.4) implies that

$$a(x) = \frac{v(x, T)}{\Delta u_1(x)},$$

provided $\Delta u_1(x) \neq 0$.

Now we differentiate Eq. (1.1) with respect to t; then $v(x, t)$ satisfies:

$$v_t = \frac{v(x, T)}{\Delta u_1(x)} \Delta v, \qquad (x, t) \in Q_T; \tag{2.1}$$

$$v(x, t) = g_t(x, t), \qquad (x, t) \in S_T; \tag{2.2}$$

$$v(x, 0) = k(x) v(x, T), \qquad x \in \Omega, \tag{2.3}$$

where

$$k(x) = \frac{\Delta u_0(x)}{\Delta u_1(x)}.$$

Because of the nonlocal term $v(x, T)$, Eq. (2.1) is nonclassical. Moreover, the initial condition (2.3) is not known. In the sequel a solution of problem (1.1)–(1.4) or (2.1)–(2.3) is always understood in the classical sense.

PROPERTY. The problems (1.1)–(1.4) and (2.1)–(2.3) are equivalent if $\Delta u_1 \neq 0$ in $\overline{\Omega}$.

Indeed, we have seen that if $u(x, t)$, $a(x)$ is a solution of problem (1.1)–(1.4), then $v(x, t)$ is a solution of problem (2.1)–(2.3). Conversely, assuming that $v(x, t)$ is a solution of problem (2.1)–(2.3), we easily verify that

$$u(x, t) = u_0(x) + \int_0^t v(x, \tau)\, d\tau, \qquad a(x) = \frac{v(x, T)}{\Delta u_1}$$

is a solution of the inverse problem (1.1)–(1.4). Therefore, we shall investigate the problem (2.1)–(2.3).

Throughout this paper the following conditions are assumed:

H(1) The functions $u_0(x)$, $u_1(x) \in C^{4+\alpha}(\overline{\Omega})$,

$$\Delta u_0(x) \geq 0, \quad 0 < \Delta u_1(x) \leq M_0 \quad \text{in } \overline{\Omega}.$$

H(2) The function $g(x, t) \in C^{4+\alpha, 2+\alpha/2}(S_T)$, and

$$0 < g_0 \leq g_t(x, t) \leq G_0, \quad g_t(x, 0) = k(x) g_t(x, T) \quad \text{for } x \in \partial\Omega$$

$$\text{and} \quad \frac{d^2}{T} \leq \frac{2 g_0}{M_0 e^{3/2}},$$

where $d = \mathrm{MD}(\Omega)$ is the *minimum diameter* of Ω, i.e., the infimum of distances between pairs Π_1, Π_2 of parallel planes such that Ω is contained in the strip determined by Π_1 and Π_2.

H(3) The function $k(x)$ satisfies

$$0 \le k(x) \le \exp\left(\frac{g_0 T}{e^{3/2} M_0} e^{-2d}\right).$$

The essential difficulty lies in that Eq. (2.1) may be degenerate, i.e., $v(x, T)$ may become zero at some points in Ω. This would easily be avoided by using the maximum principle if the initial and the boundary data were uniformly positive; however, our initial condition is given by a relation between the initial and final states. The condition H(2) is physically reasonable since we require that $a(x)$ is positive, which is equivalent to saying that $v(x, T) > 0$ on Ω. This is the case if the surrounding temperature is high enough.

The main result is

THEOREM. Under the conditions H(1)–H(3), the problem (1.1)–(1.4) admits a solution.

3. Proof. We shall use the Schauder fixed-point theorem to prove the result.

Proof of Theorem. Without loss of generality, we may assume that $0 \in \partial\Omega$ and that Ω lies in the strip $0 \le x_1 \le d$. Let

$$K = \Big\{ w(x) \in C^\alpha(\overline{\Omega}) : k_0 \le w(x) \le G_0(e^{2d} - e^{x_1}) \text{ for } x \in \Omega,$$
$$w(x) = \frac{g_t(x, 0)}{k(x)} \text{ for } x \in \partial\Omega \text{ and } \|w\|_{C^\alpha(\overline{\Omega})} \le k_1 \Big\},$$

where the positive constants k_0 and k_1 will be specified later.

Obviously, each $w(x)$ in K is bounded from above. For each $w(x) \in K$, we consider the problem:

$$v_t = \frac{w(x)}{\Delta u_1(x)} \Delta v, \qquad (x, t) \in Q_T; \tag{3.1}$$

$$v(x, t) = g_t(x, t), \qquad (x, t) \in S_T, \tag{3.2}$$

$$v(x, 0) = k(x) w(x), \qquad x \in \Omega. \tag{3.3}$$

The standard theory of parabolic equations (cf. [6]) implies the problem admits a unique classical solution

$$v(x, t; w) \in C(\overline{Q}_T) \cap C^{2+\alpha, 1+\alpha/2}(Q_T).$$

Moreover, that for any $t_0 > 0$ $(t_0 < T)$,

$$v(x, t; w) \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_{Tt_0}),$$

where $Q_{Tt_0} = Q_T \cap \{(x, t) : x \in \overline{\Omega},\ t_0 \le t \le T\}$ and

$$\|v\|_{C^{2+\alpha, 1+\alpha/2}(\overline{Q}_{Tt_0})} \le C_0, \tag{3.4}$$

where C_0 depends only on k_0, k_1 and known data. Also, by Krylov-Safanov's C^α-estimate, $\|v\|_{C^{\alpha, \alpha/2}(\overline{Q}_{Tt_0})} \le C_1$, where C_1 depends only on k_0 and known data and is independent of k_1. Hence we can take $k_1 = C_1$.

Now we define a mapping M from K into $C^{2+\alpha}(\overline{\Omega})$ as follows:

$$M: w \in K \to v(x, T; w) \in C^{2+\alpha}(\overline{\Omega}) \subset C^{\alpha}(\overline{\Omega}),$$

where $v(x, t; w)$ is the solution of problem (3.1)–(3.3).

We first show that M is a continuous mapping from $C^{\alpha}(\overline{\Omega})$ to $C^{\alpha}(\overline{\Omega})$. Let $\{w_n(x)\} \subset K$ with $w_n(x) \to w(x)$ in $C^{\alpha}(\overline{\Omega})$ as n tends to infinity. Let $v_n(x, t)$ and $v(x, t)$ be the corresponding solutions of Eqs. (3.1)–(3.3), respectively. Then the function $U(x, t) = v(x, t) - v_n(x, t)$ satisfies

$$LU = U_t - \frac{w(x)}{\Delta u_1(x)}\Delta U = \frac{\Delta v_n}{\Delta u_1}(w - w_n), \qquad (x, t) \in Q_T;$$
$$U(x, t) = 0 \quad \text{on } S_T; \qquad U(x, 0) = k(x)[w(x) - w_n(x)] \quad \text{on } \Omega.$$

By Green's representation, we have

$$U(x, t) = \int_\Omega G(x, y; t, 0)k(y)[w(y) - w_n(y)]\,dy + \int_0^t \int_\Omega G\frac{\Delta v_n}{\Delta u_1}[w - w_n]\,dy\,d\tau,$$

where $G(x, y; t, \tau)$ is the Green's function corresponding to the operator L. It follows by Eq. (3.4) that

$$\max_{\overline{Q}_T} |U(x, t)| \le C_1\|w - w_n\|_0 + C_1\|w - w_n\|_0 \to 0$$

as $n \to \infty$. Using the Schauder estimate on Q_{Tt_0}, we have

$$\|U\|_{C^{2+\alpha, 1+\alpha/2}(\overline{Q}_{Tt_0})} \le C[\|U\|_0 + \|w - w_n\|_{C^{\alpha}(\overline{\Omega})}] \to 0$$

as $n \to \infty$. In particular, as $n \to \infty$,

$$\|v(x, T) - v_n(x, T)\|_{C^{\alpha}(\Omega)} \to 0.$$

The compactness of M is clear since the embedding operator from $C^{2+\alpha}(\overline{\Omega})$ into $C^{\alpha}(\overline{\Omega})$ is compact. In order to apply the Schauder fixed-point theorem, it remains to prove that the mapping M maps K into itself.

Note that for $x \in \partial\Omega$, $v(x, 0) = k(x)w(x) = g_t(x, 0)$, it follows by H(2) that for $x \in \partial\Omega$

$$v(x, T) = g_t(x, T) = \frac{g_t(x, 0)}{k(x)}.$$

We shall next construct a subsolution for $v(x, t) \equiv v(x, t; w)$. Let $\lambda = M_0/k_0$. For $x = (x_1, x_2, \dots, x_n) \in \overline{\Omega}$, we introduce an auxiliary function

$$\psi(x, t) = \frac{C^*}{\sqrt{t}} \exp\left(-\frac{\lambda(x_1 - \xi_1)^2}{4t}\right),$$

where $\xi = (\xi_1, 0, \dots, 0)$ is a fixed point, which lies in the outside of $R^n\backslash\overline{\Omega}$ and C^* is a positive constant to be determined later. Then

$$\psi_t(x, t) = C^*\left[-\frac{1}{2} + \frac{\lambda(x_1 - \xi_1)^2}{4t}\right]\frac{1}{\sqrt{t^3}} \exp\left(-\frac{\lambda(x_1 - \xi_1)^2}{4t}\right).$$

We choose ξ such that for all $x = (x_1, \dots, x_n) \in \Omega$,

$$D \le |x_1 - \xi_1| \le 2D;$$

this is possible if we choose D such that $D \ge d$. With the above choice of ξ, we have $\psi_t(x, t) \ge 0$ on Q_T if we choose D such that $D^2\lambda = 2T$. By a direct calculation, we see

$$\Delta\psi = \lambda C^* \left[-\frac{1}{2} + \frac{\lambda(x_1 - \xi_1)^2}{4t} \right] \frac{1}{\sqrt{t^3}} \exp\left(-\frac{\lambda(x_1 - \xi_1)^2}{4t} \right).$$

It follows that

$$\Delta\psi = \frac{1}{\lambda}\psi_t \ge 0.$$

Hence

$$\begin{aligned} \psi_t - \frac{w(x)}{\Delta u_1(x)}\Delta\psi &\le \psi_t - \frac{k_0}{M_0}\Delta\psi \\ &= \psi_t - \frac{1}{\lambda}\Delta\psi = 0. \end{aligned}$$

Moreover, since $\xi \notin \overline{\Omega}$, we have, for all $x \in \Omega$,

$$\psi(x, 0) = \lim_{t \to 0} \psi(x, t) = 0.$$

Furthermore, on S_T, as $|x_1 - \xi_1| \ge D$, we have

$$\begin{aligned} \psi(x, t) &\le \frac{C^*}{\sqrt{t}} \exp\left(-\frac{\lambda D^2}{4t} \right) \\ &\le \frac{C^*}{\sqrt{T}} \sqrt{\frac{T}{t}} \exp\left(-\frac{\lambda D^2}{4T}\frac{T}{t} \right) \\ &\le \frac{C^*}{\sqrt{T}} \sup_{0<s<\infty} \left[\sqrt{s} \exp\left(-\frac{s}{2} \right) \right] \\ &= \frac{C^*}{\sqrt{T}} \frac{1}{\sqrt{e}} \\ &\le g_0, \end{aligned}$$

if we choose $C^* = \sqrt{e}\sqrt{T} g_0$. It follows that

$$\psi(x, t) \le g_t(x, t) \quad \text{on } S_T.$$

By the comparison principle, one obtains

$$v(x, t) \ge \psi(x, t), \qquad (x, t) \in \overline{Q}_T.$$

In particular, on $\overline{\Omega}$,

$$\begin{aligned} v(x, T) &\ge \psi(x, T) \\ &\ge \sqrt{e} g_0 \exp\left(-\frac{\lambda D^2}{T} \right) \\ &\ge \frac{1}{\sqrt{e^3}} g_0. \end{aligned}$$

Therefore, if we take $k_0 = e^{-3/2} g_0$, then we have, on $\overline{\Omega}$,

$$Mw = v(x, T; w) \geq k_0 \quad \text{for all } x \in \overline{\Omega}.$$

With our choice of the constants D, λ, and k_0 and also using assumption H(2), we get

$$D = \sqrt{\frac{2T}{\lambda}} = \sqrt{\frac{2Tk_0}{M_0}} = \sqrt{\frac{2Tg_0}{M_0 e^{3/2}}} \geq d,$$

which is exactly what we assumed in the proof.

To show $v(x, T) \leq G_0(e^{2d} - e^{x_1})$, we introduce another auxiliary function:

$$\varphi(x, t) = G_0 e^{\gamma(T-t)}[e^{2d} - e^{x_1}].$$

Then

$$\begin{aligned}
\varphi_t - \frac{w(x)}{\Delta u_1(x)}\Delta\varphi &= G_0 e^{\gamma(T-t)}\left[-\gamma(e^{2d} - e^{x_1}) + \frac{w(x)}{\Delta u_1(x)} e^{x_1}\right] \\
&\geq G_0 e^{\gamma(T-t)}\left(-\gamma e^{2d} + \frac{k_0}{M_0} e^{x_1}\right) \\
&\geq G_0 e^{\gamma(T-t)}\left(-\gamma e^{2d} + \frac{k_0}{M_0}\right) \\
&= 0,
\end{aligned}$$

if we choose $\gamma = (k_0/M_0)e^{-2d}$. Recalling the definition for k_0, we conclude that $e^{\gamma T} = \exp((g_0 T/e^{3/2} M_0)e^{-2d})$. Thus, for $x \in \Omega$,

$$\begin{aligned}
\varphi(x, 0) &= e^{\gamma T} G_0(e^{2d} - e^{x_1}) \\
&\geq e^{\gamma T} w(x) \\
&\geq k(x) w(x) \quad \text{[by assumption H(3)]} \\
&= v(x, 0).
\end{aligned}$$

On the boundary S_T,

$$\begin{aligned}
\varphi(x, t) &\geq G_0(e^{2D} - e^{x_1}) \\
&\geq G_0 = \max|g_t| \\
&\geq v(x, t).
\end{aligned}$$

Again by the comparison principle, we have

$$v(x, t) \leq \varphi(x, t) \quad \text{on } \overline{Q}_T.$$

It follows that

$$v(x, T) \leq G_0(e^{2D} - e^{x_1}).$$

Thus, the mapping M is from K into itself. By the Schauder fixed-point theorem, the mapping M admits a fixed point, which is a solution of the problem (2.1)–(2.3). This completes our proof.

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