QUARTERLY OF APPLIED MATHEMATICS
VOLUME LI, NUMBER 4
DECEMBER 1993, PAGES 633-641

SUPERIMPOSED TRAVELLING WAVE SOLUTIONS FOR NONLINEAR DIFFUSION

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Abstract. For one-dimensional nonlinear diffusion with diffusivity $D(c) = c^{-1}$, a new exact solution is noted which relates to both the well-known travelling wave solution and the source solution. The new solution can have a zero initial condition and admits essentially two distinct forms, one involving the hyperbolic tangent function and the other involving the circular tangent function. The solution involving tanh is physically meaningful and is displayed graphically while that involving the tan function is utilized together with a reciprocal Bäcklund transformation to produce a further new solution, which is physically more interesting than the tan solution, and is also displayed graphically. The basic idea used in this paper is generalized to a high-order nonlinear diffusion equation. For a third-order nonlinear diffusion-like equation, a solution reminiscent of solutions of soliton equations and involving the hyperbolic secant function is obtained and displayed graphically. The solutions investigated here are all characterized by the curious property that individually they are solutions for nonlinear diffusion and, moreover, their "sum" is also a bonafide nonlinear diffusion solution so that, for this specific solution, a nonlinear partial differential equation displays a "limited" superposition principle.

1. Introduction. The one-dimensional nonlinear diffusion equation for $c(x, t)$ with diffusivity $D(c)$ is given by

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left\{D(c)\frac{\partial c}{\partial x}\right\}, \tag{1.1}$$

and this and related equations are presently the subject of considerable research activity. In particular, for the power law diffusivities $D(c) = c^m$ a large number of exact solutions are known, many of which turn out to be surprisingly simple bearing in mind the nonlinear character of equations such as Eq. (1.1). We refer the reader to the recent papers [1-5] for an indication of current work in this area and for further references. In this paper we obtain a particularly simple exact solution applicable to

Received June 14, 1991.
1991 *Mathematics Subject Classification.* Primary 35G15.
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the diffusivity $D(c) = c^{-1}$; that is, we consider the equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{c}\frac{\partial c}{\partial x}\right). \tag{1.2}$$

This equation models the expansion of a thermalized electron cloud (see Lonngren and Hirose [6]) and amongst the power law diffusivities plays a priviledged role, in the sense that it enjoys some simple transformation properties. For example, Eq. (1.2) remains invariant under the reciprocal Bäcklund transformation (see, e.g., King [4, 5] and Rogers and Ames [7, p. 16])

$$dx' = c\,dx + \frac{1}{c}\frac{\partial c}{\partial x}dt, \qquad t' = t, \qquad c' = \frac{1}{c}, \tag{1.3}$$

as can readily be verified on calculating the quantities on both sides of (1.2), using the chain rule and making use of the relations

$$\frac{\partial x'}{\partial x} = c, \qquad \frac{\partial x'}{\partial t} = \frac{1}{c}\frac{\partial c}{\partial x} \tag{1.4}$$

to evaluate the various partial derivatives involving the primed variables. We make use of this invariance property in Sec. 3. King [5] also shows the remarkable result that if $c(x, t)$ is a solution of (1.2) then $C(r, t)$ defined by

$$C = \frac{c(\log r, t)}{r^2} \tag{1.5}$$

satisfies the two-dimensional axially symmetric nonlinear diffusion equation with the same diffusivity $D(c) = c^{-1}$; that is,

$$\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{r}{C}\frac{\partial C}{\partial r}\right\}. \tag{1.6}$$

Another curious property of Eq. (1.2) is that the traveling wave solution (see, e.g., Hill [1]), which takes the form

$$c(x, t) = \frac{1}{\alpha(x - \alpha t)}, \tag{1.7}$$

where α denotes an arbitrary constant, formally gives rise to the source solution of Eq. (1.2) as the "sum" of two such solutions of the form (1.7). That is, the expression

$$c(x, t) = \frac{1}{\alpha(x - \alpha t)} - \frac{1}{\alpha(x + \alpha t)} \tag{1.8}$$

is a solution of Eq. (1.2) and each of the individual terms are also solutions of Eq. (1.2). Moreover, Eq. (1.8) simplifies to give

$$c(x, t) = \frac{2t}{x^2 - \alpha^2 t^2}, \tag{1.9}$$

which is formally identical to the source solution (see, e.g., Hill [1]), and with $\alpha = i\beta$ the total concentration is given by

$$\int_{-\infty}^{\infty} c(x, t)\,dx = \frac{2\pi}{\beta}. \tag{1.10}$$

The form of Eq. (1.8) suggests that more generally we might investigate solutions of Eq. (1.2) of the form

$$c(x, t) = \phi(x - \alpha t) - \phi(x + \alpha t) \tag{1.11}$$

for some arbitrary function ϕ and constant α. From Eqs. (1.2) and (1.11) we find that

$$-\alpha[\phi'(\xi) + \phi'(\eta)] = \frac{\partial}{\partial x}\left\{\frac{\phi'(\xi) - \phi'(\eta)}{\phi(\xi) - \phi(\eta)}\right\}, \tag{1.12}$$

where primes denote derivatives with respect to the appropriate argument, $\xi = x - \alpha t$, and $\eta = x + \alpha t$. It is not difficult to see that, if $\phi(\xi)$ satisfies the ordinary differential equation

$$\frac{d\phi}{d\xi} = -\alpha\phi^2 + A\phi + B, \tag{1.13}$$

where A and B denote further arbitrary constants, then Eq. (1.12) is automatically satisfied. Of course, (1.13) can trivially be integrated, and the various particular forms are given in the following section. Here we merely note that if the constants A and B are both zero then Eq. (1.13) gives

$$\phi(\xi) = \frac{1}{\alpha(\xi - \xi_0)}, \tag{1.14}$$

where ξ_0 denotes an arbitrary constant which, when zero, Eq. (1.14) gives rise to the solution (1.8). We also note that the form (1.11) automatically satisfies the zero initial condition which is often a physical requirement for solutions of Eq. (1.2). However, if $\phi(\xi, C)$ denotes the solution of Eq. (1.13) with integration constant C, we could also construct solutions of Eq. (1.2) with nonzero initial condition from

$$c(x, t) = \phi(x - \alpha t, C_1) - \phi(x + \alpha t, C_2), \tag{1.15}$$

where C_1 and C_2 denote distinct values of the integration constant.

The plan of the remainder of the paper is as follows. In the following section we present specific forms for the various solutions of Eq. (1.13), and the concentration profile appropriate to the case $A^2 + 4\alpha B > 0$ is displayed graphically in Fig. 1. In the subsequent section we use a reciprocal Bäcklund transformation together with the concentration profile appropriate to the case $A^2 + 4\alpha B < 0$ to deduce a further new solution of Eq. (1.2), which is displayed graphically in Fig. 2. In the final section of the paper we extend the basic idea of the paper to the high-order nonlinear diffusion equation (4.1), and for the special case of $N = 2$ we derive the concentration profile given explicitly by Eq. (4.10) and shown in Fig. 3. We remark that in discussing Eq. (4.1) we have no particular application in mind. It merely demonstrates that the idea of Sec. 2 can be extended to certain higher-order diffusion-like equations.

2. Specific forms of the new solution. Although the integration of Eq. (1.13) is elementary and the various particular forms can be extracted from tables of integrals, in order to see how the various cases arise, we solve Eq. (1.13) as a Riccatti equation, which may seem somewhat perverse. However, the details using tables of integrals are of equal length, giving little or no insight into why different cases arise. Thus with the substitution

$$\phi(\xi) = \frac{u'(\xi)}{\alpha u(\xi)}, \tag{2.1}$$

Eq. (1.13) becomes

$$u'' - Au' - \alpha B u = 0, \tag{2.2}$$

and the various solutions arise accordingly as $A^2 + 4\alpha B$ is positive, negative, or zero.

If $A^2 + 4\alpha B > 0$ then

$$\phi(\xi) = \frac{1}{\alpha}\left\{\frac{m_1 e^{m_1\xi} + m_2 C e^{m_2\xi}}{e^{m_1\xi} + C e^{m_2\xi}}\right\}, \tag{2.3}$$

where m_1 and m_2 are constants defined by

$$m_1 = \tfrac{1}{2}\{A + (A^2 + 4\alpha B)^{1/2}\}, \qquad m_2 = \tfrac{1}{2}\{A - (A^2 + 4\alpha B)^{1/2}\}, \tag{2.4}$$

and C denotes an arbitrary constant. Now, introducing constants m and ξ_0 defined by

$$m = \frac{(A^2 + 4\alpha B)^{1/2}}{2}, \qquad C = e^{2m\xi_0}, \tag{2.5}$$

it is not difficult to show that Eq. (2.3) becomes

$$\phi(\xi) = \frac{1}{\alpha}\left\{\frac{m_1 e^{\lambda} + m_2 e^{-\lambda}}{e^{\lambda} + e^{-\lambda}}\right\} = \frac{1}{2\alpha}\{(m_1 + m_2) + (m_1 - m_2)\tanh\lambda\}, \tag{2.6}$$

where $\lambda = m(\xi - \xi_0)$. Thus altogether we have

$$\phi(\xi) = \frac{1}{2\alpha}\left\{A + (A^2 + 4\alpha B)^{1/2}\tanh\frac{(A^2 + 4\alpha B)^{1/2}}{2}(\xi - \xi_0)\right\}, \tag{2.7}$$

which is the appropriate solution of Eq. (1.13) when $A^2 + 4\alpha B > 0$. Thus the solution of Eq. (1.2) in the form of Eq. (1.11), in this case, is given explicitly by

$$c(x, t) = \tfrac{m}{\alpha}\{\tanh m(x - \alpha t - \xi_0) - \tanh m(x + \alpha t - \xi_0)\}, \tag{2.8}$$

where m is defined by Eq. $(2.5)_1$. In general for a nonzero initial condition the appropriate extension of Eq. (2.8) is simply

$$c(x, t) = \tfrac{m}{\alpha}\{\tanh m(x - \alpha t - \xi_1) - \tanh m(x + \alpha t - \xi_2)\}, \tag{2.9}$$

for arbitrary constants ξ_1 and ξ_2; this solution is illustrated graphically in Fig. 1 for $\alpha = m = \xi_2 = 1$ and $\xi_1 = -1$.

In the case $A^2 + 4\alpha B < 0$ we have

$$u(\xi) = e^{A\xi/2}\{C_1 \cos n\xi + C_2 \sin n\xi\}, \tag{2.10}$$

where the constant n is defined by

$$n = \frac{(-A^2 - 4\alpha B)^{1/2}}{2} \tag{2.11}$$

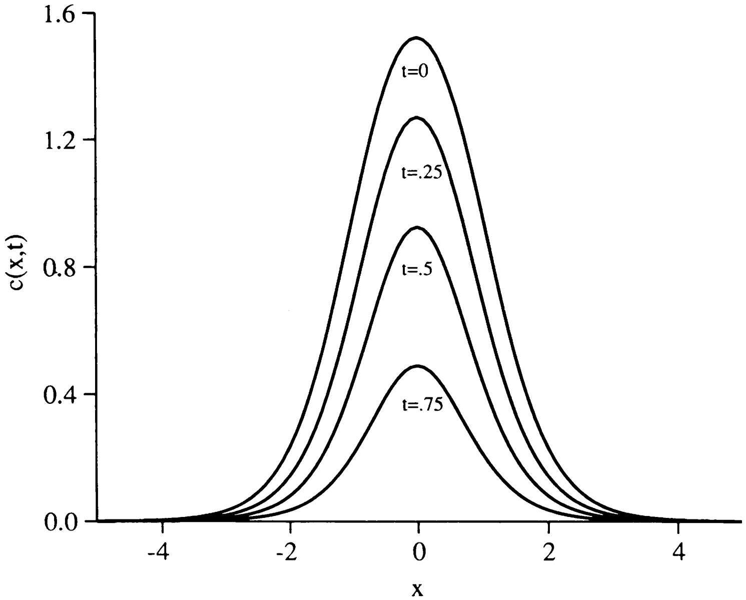

FIG. 1. Variation of the concentration (2.9) for $\alpha = m = \xi_2 = 1$ and $\xi_1 = -1$.

and C_1 and C_2 denote arbitrary constants. Thus from Eqs. (2.1) and (2.10) we have

$$\phi(\xi) = \frac{1}{\alpha}\left\{\frac{A}{2} + n\left(\frac{C - \tan n\xi}{1 + C\tan n\xi}\right)\right\}, \tag{2.12}$$

where $C = C_2/C_1$ and, setting $C = \tan n\xi_0$, we may readily deduce

$$\phi(\xi) = \frac{1}{2\alpha}\left\{A - (-A^2 - 4\alpha B)^{1/2}\tan\frac{(-A^2 - 4\alpha B)^{1/2}}{2}(\xi - \xi_0)\right\}, \tag{2.13}$$

which of course could also be deduced from Eq. (2.7) using $\tanh iz = i\tan z$. In this case the appropriate solution of Eq. (1.2) of the form (1.11) is given by

$$c(x, t) = \tfrac{n}{\alpha}\{\tan n(x + \alpha t - \xi_0) - \tan n(x - \alpha t - \xi_0)\}. \tag{2.14}$$

This solution, which appears not to be particularly physically interesting, is used in the following section to determine a further new solution from a reciprocal Bäcklund transformation. In this case $A^2 + 4\alpha B$ equals zero, Eq. (1.13) becomes

$$\frac{d\phi}{d\xi} = -\alpha\left(\phi - \frac{A}{2\alpha}\right)^2, \tag{2.15}$$

which integrates immediately to give

$$\phi(\xi) = \frac{A}{2\alpha} + \frac{1}{\alpha(\xi - \xi_0)}, \tag{2.16}$$

which evidently corresponds to the solution (1.8) or (1.9).

Finally in this section we note that if we set $\Phi(\xi) = \phi(\xi) - \phi_0$, where ϕ_0 denotes a constant, then Eq. (1.11) is unchanged with Φ in place of ϕ and that if we choose ϕ_0 such that $-\alpha\phi_0^2 + A\phi_0 + B = 0$ then Eq. (1.13) becomes

$$\frac{d\Phi}{d\xi} = -\alpha\Phi^2 + (A - 2\alpha\phi_0)\Phi; \tag{2.17}$$

and it is not difficult to show that any solution $\Phi(\xi)$ of this equation indeed generates a solution of Eq. (1.2) of the form (1.11), but, in addition, $\Phi(x-\alpha t)$ and $-\Phi(x+\alpha t)$ are themselves also solutions of Eq. (1.2). Thus for the particular nonlinear diffusion equation (1.2) we have a limited superposition principle.

3. Further new solution obtained by a reciprocal Bäcklund transformation. Now from Eq. (2.14) and the relations (1.4) it is not difficult to establish that

$$x' = \frac{1}{\alpha}\log\left\{\frac{\cos n(x - \alpha t - \xi_0)}{\cos n(x + \alpha t - \xi_0)}\right\} + x_0, \tag{3.1}$$

where x_0 denotes an arbitrary constant. Rearranging this equation, we can deduce that

$$\tanh\tfrac{\alpha}{2}(x' - x_0) = \tan n(x - \xi_0)\tan n\alpha t; \tag{3.2}$$

therefore,

$$x = \xi_0 + \frac{1}{n}\tan^{-1}\left\{\frac{\tanh\frac{\alpha}{2}(x' - x_0)}{\tan n\alpha t'}\right\}. \tag{3.3}$$

Further from Eqs. (2.14) and $(1.3)_3$ we have

$$c' = \frac{\alpha}{2n}\left\{\frac{(1 - \tan^2 n(x - \xi_0)\tan^2 n\alpha t)}{\tan n\alpha t(1 + \tan^2 n(x - \xi_0))}\right\}, \tag{3.4}$$

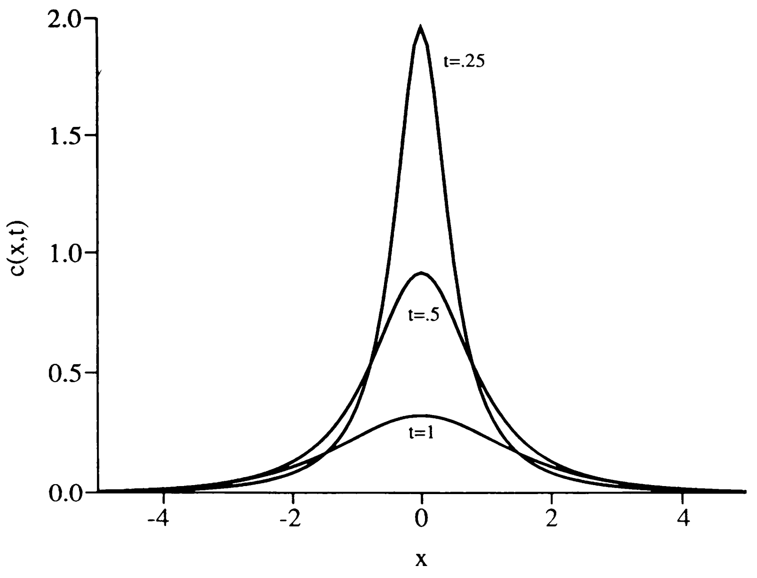

FIG. 2. Variation of the concentration (3.6) for $\alpha = n = 1$ and $x_0 = 0$.

from which using Eq. (3.2), we may deduce

$$c' = \frac{\alpha}{2n}\left\{\frac{\tan n\alpha t' \operatorname{sech}^2\frac{\alpha}{2}(x'-x_0)}{\tan^2 n\alpha t' + \tanh^2\frac{\alpha}{2}(x'-x_0)}\right\}. \tag{3.5}$$

Thus from Eqs. (1.3) and (2.14) we have deduced the additional solution of Eq. (1.2), which in terms of the original variables becomes

$$c(x, t) = \frac{\alpha}{2n}\left\{\frac{\tan n\alpha t \operatorname{sech}^2\frac{\alpha}{2}(x-x_0)}{\tan^2 n\alpha t + \tanh^2\frac{\alpha}{2}(x-x_0)}\right\}, \tag{3.6}$$

for arbitrary constants n, α, and x_0. Figure 2 shows this solution for $\alpha = n = 1$ and x_0 zero.

4. Generalization to a high-order nonlinear diffusion-like equation. In this section we extend the basic idea of the paper to the high-order nonlinear diffusion-like equation

$$\frac{\partial c}{\partial t} = (-1)^M \frac{\partial}{\partial x}\left(\frac{1}{c}\frac{\partial^N c}{\partial x^N}\right), \tag{4.1}$$

where N is a positive integer. Similar high-order equations are discussed by Smyth and Hill [8]. Based on a simple linear stability analysis about a constant solution c_0, for stability we require $M = (N-1)/2$ for N odd and $M = (N-2)/2$ for N even. If again we seek a concentration profile of the form (1.11), then it is not difficult to show that this is a bonafide solution of Eq. (4.1) provided $\phi(\xi)$ satisfies the ordinary differential equation

$$\frac{d^N\phi}{d\xi^N} = (-1)^{M+1}\alpha\phi^2 + A\phi + B, \tag{4.2}$$

where, as before, A and B designate arbitrary constants. For example, if both A and B are zero then we can readily show that one solution of Eq. (4.2) takes the form

$$\phi(\xi) = \frac{(2N)!}{2\alpha N!}\frac{(-1)^{M+1-N}}{(\xi-\xi_0)^N}, \tag{4.3}$$

which is the appropriate extension of Eq. (1.14) for $N > 1$ where as usual ξ_0 denotes an arbitrary constant. Thus for Eq. (4.1) we have the concentration profile

$$c(x, t) = \frac{(2N)!}{2\alpha N!}(-1)^{M+1-N}\left\{\frac{1}{(x-\alpha t-\xi_0)^N} - \frac{1}{(x+\alpha t-\xi_0)^N}\right\}, \tag{4.4}$$

which applies for any integer $N \geq 1$.

Alternatively for $N = 2$ we can either set B zero or translate $\phi(\xi)$ by a constant ϕ_0 which is determined as a root of the quadratic $-\alpha\phi_0^2 + A\phi_0 + B = 0$, and in either case we obtain

$$\frac{d^2\phi}{d\xi^2} = -\alpha\phi^2 + A\phi, \tag{4.5}$$

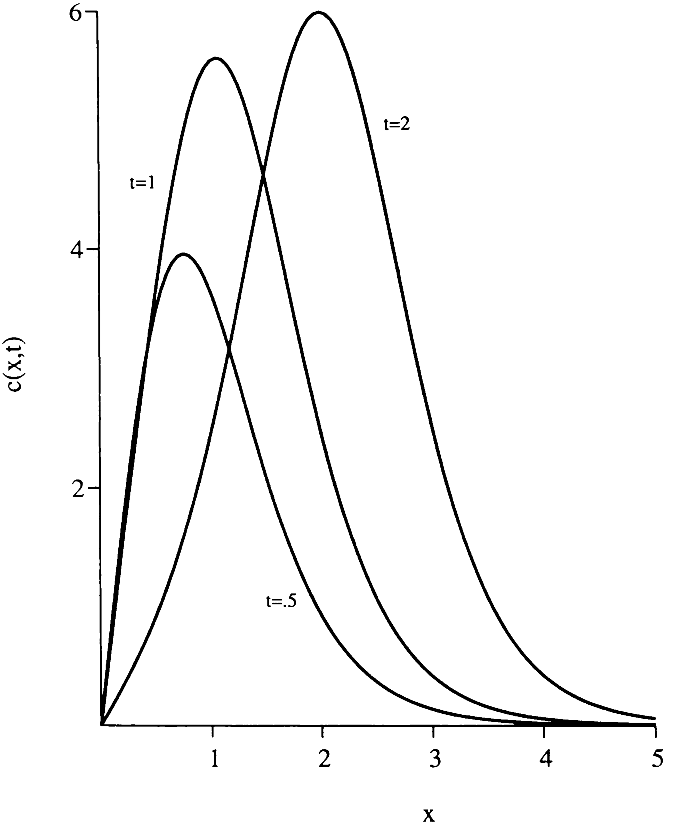

FIG. 3. Variation of the concentration (4.10) for $\alpha = 1$, $A = 4$, and $\xi_0 = 0$.

which integrates once to give

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{-2\alpha}{3}\phi^3 + A\phi^2 + C, \tag{4.6}$$

where C denotes the constant of integration. If this constant is zero, we have

$$\int \frac{d\phi}{\phi(\phi+\beta)^{1/2}} = \pm\left(\frac{-2\alpha}{3}\right)^{1/2}(\xi - \xi_0), \tag{4.7}$$

where β denotes $-3A/2\alpha$. Now for this integral we find

$$\int \frac{d\phi}{\phi(\phi+\beta)^{1/2}} = \begin{cases} \frac{1}{\beta^{1/2}} \log\left|\frac{(\phi+\beta)^{1/2}-\beta^{1/2}}{(\phi+\beta)^{1/2}+\beta^{1/2}}\right| & \text{if } \beta > 0, \\ \frac{2}{(-\beta)^{1/2}}\tan^{-1}\frac{(\phi+\beta)^{1/2}}{(-\beta)^{1/2}} & \text{if } \beta < 0. \end{cases} \tag{4.8}$$

From these expressions we may deduce solutions which are reminiscent of solutions of soliton equations. From Eqs. (4.8) we obtain

$$\phi(\xi) = \begin{cases} -\beta \operatorname{cosec}^2\left(\frac{\alpha\beta}{6}\right)^{1/2}(\xi - \xi_0) & \text{if } \beta > 0, \\ -\beta \operatorname{sech}^2\left(\frac{-\alpha\beta}{6}\right)^{1/2}(\xi - \xi_0) & \text{if } \beta < 0, \end{cases} \tag{4.9}$$

so that for example for the case $\alpha > 0$, $\beta < 0$ $(A > 0)$ we have established that

$$c(x, t) = \frac{3A}{2\alpha}\left\{\operatorname{sech}^2\left[\frac{A^{1/2}}{2}(x - \alpha t - \xi_0)\right] - \operatorname{sech}^2\left[\frac{A^{1/2}}{2}(x + \alpha t - \xi_0)\right]\right\} \tag{4.10}$$

is a solution of the nonlinear diffusion equation (4.1) with $N = 2$; namely,

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{c}\frac{\partial^2 c}{\partial x^2}\right). \tag{4.11}$$

Moreover, we again observe that individually each of the two terms are themselves also solutions of this equation. Figure 3 shows the concentration profile corresponding to Eq. (4.10) for $\alpha = 1$, $A = 4$, and ξ_0 zero.

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