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COLLAPSE OF SPHERICAL BUBBLES IN FLUIDS WITH NONLINEAR VISCOSITY

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Abstract. An analysis is given of the collapse of a spherical bubble in a large body of a viscous incompressible fluid with strain-dependent nonlinear viscosity. Two types of asymptotic behaviors are found analytically, namely the collapse over finite time and viscous damping over infinite time.

The term cavitation usually refers to the phenomenon of the growth and collapse of flow-induced voids or vapor bubbles in liquids. These effects resulting from cavitation are known to produce metal erosion, luminescence, and increases in various chemical reaction rates.

The earliest theoretical treatment is apparently that of Lord Rayleigh [1], who considered the collapse of a spherical void in an inviscid fluid. In later theoretical works, attempts have been made to account for viscous effects in both the bubble phase and in the surrounding liquid; most of the analyses have dealt with the Newtonian fluid [2, 3].

An interesting question arises as to the influence that non-Newtonian effects might have on cavitation in a viscous incompressible fluid. Actually, at the final stage of a collapse, the rate of strain tensor $\mathbf{D}$ becomes singular. We must therefore use a more thorough description than the usual Navier-Stokes one. Numerical studies based on the method of molecular dynamics show that the Navier-Stokes equations with a nonlinear viscosity coefficient, $\eta = \eta(\mathbf{D})$, provide an adequate description for various phenomena in so-called simple liquids (for a review, see, e.g., [4, 5]).

In this study we omit the influence of nonsphericity, compressibility of a fluid, surface tension, and a gas that may be present in a cavity (for a comprehensive review of these points see [6]), and concentrate on an analytical description of a collapse of spherical voids (i.e., regions containing no gas) in a simple liquid.

The flow of the liquid is governed by the Navier-Stokes equation

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \boldsymbol{\sigma} - \nabla p, \qquad \operatorname{div} \mathbf{V} = 0, \tag{1}$$

where ρ is the density of the fluid and p is the pressure. For the stress tensor $\boldsymbol{\sigma}$

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we have the usual relation

$$\boldsymbol{\sigma} = 2\eta \mathbf{D}, \tag{2}$$

with a strain-dependent viscosity coefficient, $\eta = \eta(\mathbf{D})$. Precisely, a scalar function η may depend on three independent invariants of the tensor $\mathbf{D}$, e.g., $\mathrm{Tr}(\mathbf{D})$, $\mathrm{Tr}(\mathbf{D}^2)$, and $\det(\mathbf{D})$. Note that the first of these invariants is equal to zero in an incompressible fluid, that is, $\mathrm{Tr}(\mathbf{D}) = \mathrm{div}\,\mathbf{V} = 0$.

By symmetry, the velocity has only a radial component

$$u(r, t) = r^{-2}F(t), \qquad F(t) = R^2\dot{\mathbf{R}}, \tag{3}$$

where the overdot denotes the time derivative. At the bubble surface, $r = R(t)$, the normal tension $\sigma_{rr} - p$ is zero. Using this boundary condition, one can derive the governing equation for the bubble radius [7, 8]

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_0}{\rho} = -\frac{12R^2\dot{R}}{\rho}\int_R^\infty \eta\frac{dr}{r^4}. \tag{4}$$

In Eq. (4) the viscosity coefficient η depends on $\mathrm{Tr}(\mathbf{D}^2) = 6F^2/r^6$ and $\det(\mathbf{D}) = -2F^3/r^9$, i.e., $\eta = \eta(F/r^3)$.

One cannot find a general solution of Eq. (4) for an arbitrary viscosity coefficient $\eta = \eta(F/r^3)$. To analyze the final stage of collapse we need only the behavior of $\eta(F/r^3)$ at a large value of $q = F(t)/r^3$. For further analysis it is natural to assume the power-law dependence of the viscosity coefficient, $\eta(q) \to \eta_0 q^{-\alpha}$ at $q \gg 1$. It will be shown that the behavior near the collapse point crucially depends on the exponent α.

It is difficult to extract precise values for the exponent α from numerical or experimental data, although in rheological models it is generally believed that $\alpha < 1$ (cf. [9]). Theoretically, the shear-rate dependence of shear viscosity has been derived by Ikenberry and Truesdell, and by Galkin, for a system of Maxwell molecules described by the Boltzmann equation (see [10]). For more general interaction potential, Zwanzig has obtained a closed differential equation for the shear viscosity by using the Bhatnagar-Gross-Kook (BGK) model equation [11]. Both the Ikenberry-Truesdell-Galkin solution and the solution of the BGK model kinetic equation for Maxwell molecules [11] and for hard spheres [12] exhibit the behavior $\eta(q) \sim q^{-4/3}$ for asymptotically large q. Recent computer simulations [13] also confirmed this value of the exponent α, $\alpha = \frac{4}{3}$, for hard spheres described by the Boltzmann equation. It is probably correct to say that these are the only rigorous derivations of the exponent α; however, the applicability of these results to the liquid state is quite problematic.

Now, turning to the final stage of the collapse, we assume that the ratio $\dot{R}/R$ tends to infinity. Using this assumption, one can evaluate the integral on the right-hand side of Eq. (4). At $\alpha > 1$ this integral converges as $R \to 0$, and in the vicinity of the singularity point we obtain the following equation for the bubble radius $R(t)$:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_0}{\rho} = \frac{4C}{\rho}, \tag{5a}$$

where the constant C equals the integral $\int_0^\infty dq\,\eta(q)$ and, consequently, depends on rather detailed information about the behavior of the viscosity $\eta = \eta(q)$.

In the opposite case, $\alpha < 1$, we find

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_0}{\rho} = \frac{4\eta_0}{\rho}\frac{1}{1-\alpha}\left(-\frac{\dot{R}}{R}\right)^{1-\alpha}. \tag{5b}$$

Finally, in the borderline case $\alpha = 1$ we obtain

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_0}{\rho} = \frac{4\eta_0}{\rho}\ln\left(-\frac{\dot{R}}{R}\right). \tag{5c}$$

Equation (5a) shows that in the case of strong decay of the viscosity, $\alpha > 1$, we obtain the Rayleigh equation with renormalized compressed pressure $p_0 - 4C$, so liquid becomes effectively inviscid on the final stage of collapse. Therefore, a collapse proves to be inevitable with the asymptotic decay

$$R \sim (T-t)^m \tag{6}$$

having the Rayleigh exponent $m = 2/5$. The collapse time T in this case is given by [1]

$$T = \sqrt{\frac{3\pi}{2}}\frac{\Gamma(5/6)}{\Gamma(1/3)}R_0\sqrt{\frac{\rho}{p_0 - 4C}} = 0.915R_0\sqrt{\frac{\rho}{p_0 - 4C}}. \tag{7}$$

Observe that Eq. (5a) is really the asymptotic equation with increasing accuracy in the vicinity of the singularity, $t \to T$; therefore, formula (7) provides only an approximate expression for the collapse time.

In the intermediate case, $\alpha = 1$, we again obtain the scaling behavior (6) with the Rayleigh exponent. The case of weak decay of the viscosity, $\alpha < 1$, appears to be the most interesting. Looking for the scaling solutions (6) of Eq. (5b) in the vicinity of the collapse point, we find the orders of all terms on the left-hand side ($R\ddot{R} \sim \dot{R}^2 \sim (T-t)^{2m-2}$) and on the right-hand side ($(-\dot{R}/R)^{1-\alpha} \sim (T-t)^{\alpha-1}$). If the terms on the left are more singular, we have the following most singular combined term

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 \sim (T-t)^{2m-2} m\left(\frac{5m}{2} - 1\right),$$

and, consequently, we obtain the Rayleigh exponent describing the collapse in an inviscid fluid once more.

Equation (5b) also becomes self-consistent when the singular terms in both sides are equal, i.e., $2m - 2 = \alpha - 1$; therefore, we have $m = (\alpha + 1)/2$, that is, the dynamic exponent m in this case becomes model-dependent.

For the further analysis it is convenient to introduce the nondimensional variables

$$R = R_0\overline{R}, \qquad t = R_0\sqrt{\frac{\rho}{p_0}}\,\overline{t} \tag{8}$$

and to recast Eq. (5b) in the dimensionless form

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + 1 = \frac{4}{\mathrm{Re}}\left(-\frac{\dot{R}}{R}\right)^{1-\alpha}. \tag{9}$$

Here overbars are omitted for brevity and Re denotes the Reynolds number

$$\mathrm{Re} = \sqrt{\frac{\rho p_0 R_0^2}{\eta_0^2}\left(\frac{p_0}{\rho R_0^2}\right)^{\alpha}(1-\alpha)^2}. \tag{10}$$

To proceed with a solution of Eq. (9), it is natural to reduce its order by means of the transformation $U(R) = \dot{R}(t)$. Furthermore, it is convenient to introduce new variables, $x = -\log(R)$ and $y = \log(U^2 + 2/3)$. This reduces Eq. (9) to

$$\frac{dy}{dx} = \frac{3}{2} - \frac{4}{\mathrm{Re}}\left(1 - \frac{2}{3}e^{-y}\right)^{(1-\alpha)/2} \exp\left[x(1-\alpha) - y\frac{1+\alpha}{2}\right]. \tag{11}$$

Next we carried out numerical simulations of Eq. (11) at $0 < x < 1$, subject to the initial condition $y(0) = \log(\frac{2}{3})$, by the standard fourth-order Runge-Kutta procedure. We have observed that a collapse behavior (6) with inviscid exponent exists only for the Reynolds numbers above some threshold value $\mathrm{Re}_c(\alpha)$. For $\mathrm{Re} = \mathrm{Re}_c(\alpha)$, we have found the collapse behavior with nonuniversal scaling exponent $m = (\alpha + 1)/2$. Finally, at $\mathrm{Re} < \mathrm{Re}_c(\alpha)$, a void is filled over an infinite time. Assuming an exponential asymptotic behavior

$$R \sim \exp(-\nu t) \quad \text{at } t \gg 1 \tag{12}$$

and substituting Eq. (12) into Eq. (9), one can find (nondimensional) inverse decay time ν, $\nu = (\mathrm{Re}/4)^{1/(1-\alpha)}$. In this case, however, $(-\dot{R}/R) \to \nu = \text{finite}$. Therefore, one of our assumptions fails, and we can use Eq. (5b) only for qualitative estimates; that is, the asymptotic decay (12) is valid, but the inverse decay time ν must be evaluated from the full equation (4). This last equation yields only an implicit expression for the (dimensional) inverse decay time ν, $\nu \int_0^1 dz\, \eta(\nu z) = p_0/4$, where $z = (R/r)^3$.

Figure 1 gives the plot of the critical Reynolds number, $\mathrm{Re}_c(\alpha)$, versus the model parameter α at $0 < \alpha < 1$. For the Newtonian fluid, $\alpha = 0$, the critical Reynolds number is approximately equal to 8.4.

The results of the preceding analysis indicate that non-Newtonian effects may well have a strong influence on a qualitative behavior of a void. The behavior strongly depends on the exponent α describing the asymptotic of a viscosity at large strain, $\eta(q) \sim q^{-\alpha}$ at $q \gg 1$. We have observed different dynamic behaviors including collapse over finite time and viscous damping over infinite time. For the first type of behavior, we have found the Rayleigh-like scaling at $\alpha \geq 1$ and also at $\alpha < 1$ when the Reynolds number Re lies above the threshold value $\mathrm{Re}_c(\alpha)$. When $0 < \alpha < 1$ and $\mathrm{Re} = \mathrm{Re}_c(\alpha)$, we obtain a scaling with nonuniversal model-dependent exponent. For the second type of behavior we have found the exponential viscous damping of a void over an infinite time interval. This occurs at $\alpha < 1$ and $\mathrm{Re} < \mathrm{Re}_c(\alpha)$.

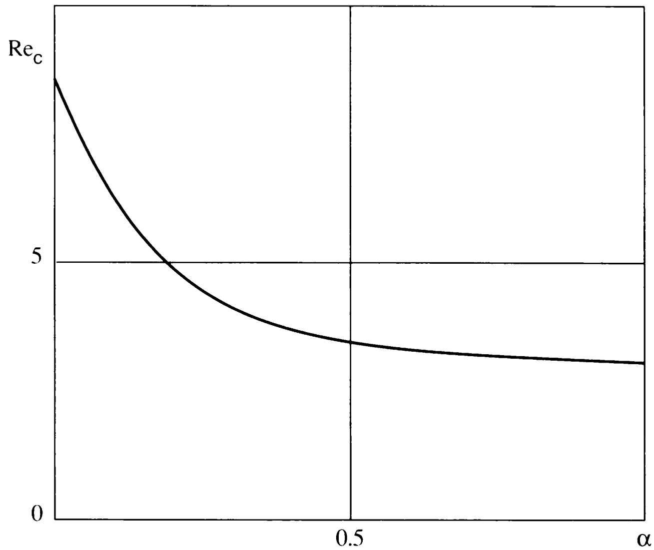

FIG. 1. The graph of the critical Reynolds number Re_c versus the model parameter α.

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REFERENCES

[1] Lord Rayleigh, *On the pressure developed in a liquid during the collapse of a spherical cavity*, Philos. Mag. **34**, 94–96 (1917)

[2] G. Birkhoff and E. H. Zarantonello, *Jets, Wakes and Cavities*, Academic Press, New York, 1957

[3] M. S. Plesset, *Cavitation in Real Liquids*, R. Davies, ed., Elsevier, Amsterdam, 1964, p. 1

[4] D. J. Evans and G. P. Morris, *Statistical Mechanics of Nonequilibrium Liquids*, Academic Press, New York, 1990

[5] M. P. Allen and D. Tildesley, *Computer Simulation of Liquids*, Clarendon, Oxford, 1987

[6] M. S. Plesset and A. Prosperetti, *Bubble dynamics and cavitation*, Ann. Rev. Fluid Mech. **9**, 145–185 (1977)

[7] H. S. Fogler and J. D. Goddard, *Collapse of spherical cavities in viscoelastic fluids*, Phys. Fluids **13**, 1135–1141 (1970)

[8] A. Prosperetti, *A generalization of the Rayleigh-Plesset equation of bubble dynamics*, Phys. Fluids **25**, 409–410 (1982)

[9] R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids, Vol.* 1, Fluid Mechanics, 2nd ed., Wiley, New York, 1987

[10] C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monoatomic Gas*, Academic Press, New York, 1980

[11] R. Zwanzig, *Nonlinear shear viscosity of a gas*, J. Chem. Phys. **71**, 4416–4420 (1979)

[12] A. Santos, J. J. Brey, and J. M. Dufty, *Divergence of the Chapman-Enskog expansion*, Phys. Rev. Lett. **56**, 1571–1574 (1986); J. Gomez Ordonez, J. J. Brey, and A. Santos, *Velocity distribution function of a dilute gas under uniform shear flow. A comparison between a Monte Carlo simulation method and the BGK equation*, Phys. Rev. A (3) **41**, 810–815 (1990)

[13] J. Gomez Ordonez, J. J. Brey, and A. Santos, *Shear-rate dependence of the viscosity for dilute gases*, Phys. Rev. A (3) **39**, 3038–3040 (1989)