QUARTERLY OF APPLIED MATHEMATICS
VOLUME LI, NUMBER 4
DECEMBER 1993, PAGES 751-763

GLOBAL SOLVABILITY AND EXPONENTIAL STABILITY IN ONE-DIMENSIONAL NONLINEAR THERMOELASTICITY

By

REINHARD RACKE (*Institute of Applied Mathematics, University of Bonn, Bonn, Germany*)

YOSHIHIRO SHIBATA (*Institute of Mathematics, University of Tsukuba, Ibaraki, Japan*)

AND

SONGMU ZHENG (*Institute of Mathematics, Fudan University, Shanghai, People's Republic of China*)

Abstract. We are mainly concerned with the Dirichlet initial boundary value problem in one-dimensional nonlinear thermoelasticity. It is proved that if the initial data are close to the equilibrium then the problem admits a unique, global, smooth solution. Moreover, as time tends to infinity, the solution is exponentially stable. As a corollary we also obtain the existence of periodic solutions for small, periodic right-hand sides.

1. Introduction. This paper is concerned with global existence, uniqueness, and asymptotic behavior of solutions to the equations of one-dimensional nonlinear thermoelasticity subject to Dirichlet boundary conditions for both temperature difference and displacement. Moreover, the question of existence of periodic solutions is addressed.

The reference configuration is represented by the interval $\Omega := (0, l)$, l being a fixed positive number.

The equations for the displacement $u = u(t, x)$ from the reference configuration and the temperature difference $\theta = \theta(t, x) = T_a(t, x) - \tau_0$, where T_a is the absolute temperature and τ_0 is the constant reference temperature, read as follows:

$$u_{tt} - S(u_x, \theta)_x = f_1 \quad \text{in } [0, \infty) \times \Omega, \tag{1.1}$$

$$(\theta + \tau_0)N(u_x, \theta)_t - Q(u_x, \theta_x, \theta)_x = f_2 \quad \text{in } [0, \infty) \times \Omega. \tag{1.2}$$

Here S is the Piola-Kirchhoff stress tensor, N is the specific entropy, and Q is the heat flux.

Received September 17, 1991.
1991 *Mathematics Subject Classification.* Primary 35Q99, 73C25; Secondary 35B40, 73C50.
Key words and phrases. Nonlinear thermoelasticity, global existence and uniqueness, asymptotic behavior, periodic solution.
Research supported by the Sonderforschungsbereich 256 of the Deutsche Forschungsgemeinschaft at the University of Bonn.

Introducing

$$a := \frac{\partial S}{\partial u_x}, \qquad b := -\frac{\partial S}{\partial \theta}, \qquad c := \frac{\partial N}{\partial \theta}, \qquad f := f_1,$$
$$d := \frac{1}{(\theta + \tau_0)} \frac{\partial Q}{\partial \theta_x}, \qquad g := \frac{f_2}{\theta + \tau_0} \qquad (\text{for } |\theta| < \tau_0),$$

observing $-\partial S/\partial\theta = \partial N/\partial u_x$ (cf. [12]), and assuming $Q = Q(\theta_x)$ for simplicity, we may rewrite the equations as

$$u_{tt} - a(u_x, \theta)u_{xx} + b(u_x, \theta)\theta_x = f, \tag{1.3}$$
$$c(u_x, \theta)\theta_t + b(u_x, \theta)u_{tx} - d(\theta, \theta_x)\theta_{xx} = g. \tag{1.4}$$

Considering first the Dirichlet initial boundary value problem, u and θ are subject to the following boundary and initial conditions respectively:

$$u(t, 0) = u(t, l) = \theta(t, 0) = \theta(t, l) = 0 \quad \text{in } [0, \infty], \tag{1.5}$$
$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(t = 0) = \theta_0(x) \quad \text{in } \Omega, \tag{1.6}$$

with prescribed data u_0, u_1, and θ_0.

Slemrod proved in [15] the global existence and asymptotic stability of solutions if u and θ satisfy Neumann-Dirichlet ($u_x = \theta = 0$) or Dirichlet-Neumann ($u = \theta_x = 0$) boundary conditions (see also Zheng [16]). The Dirichlet-Dirichlet case (1.5) remained open until Racke and Shibata recently proved in [12] the global existence of small smooth solutions using spectral analysis methods. Polynomial decay rates as $t \to \infty$ depending on the smoothness of the initial data were also obtained.

On the other hand, Muñoz Rivera recently proved in [1] the exponential decay of solutions to the Dirichlet-Dirichlet initial boundary value problem in *linear* thermoelasticity, using the energy method and a tricky treatment of some boundary terms.

The main aim of this paper is to use Muñoz Rivera's idea to improve the results of Racke and Shibata [12]. The improvement is threefold: we obtain an exponential decay result, less regularity of the data is required, and the proof is much simpler. Moreover, the existence of periodic solutions is obtained as a corollary.

We wish to mention that Shibata [13] and Jiang [6] recently discussed the corresponding Neumann problem, the latter also for Ω being the half-line, where decay rates are only given in [13] proving polynomial decay. Previously, the Cauchy problem $\Omega = \mathbb{R}$ was investigated by Kawashima and Okada [8], Kawashima [7], Zheng and Shen [17], and Hrusa and Tarabek [3], proving the global existence of small solutions. The blow-up for large data was shown for the Cauchy problem by Dafermos and Hsiao [1] and Hrusa and Messaoudi [2]. The case of a half-line assuming the same boundary conditions as Slemrod was discussed by Jiang [4] as well as recently the Dirichlet-Dirichlet case in [5], without giving any decay rates.

The following assumption is made throughout the paper.

ASSUMPTION 1.1. a, b, c, d are C^2-functions of their arguments. There exist positive constants a_0, c_0, d_0, K, with $K < \tau_0$, such that if $|u_x| \le K$, $|\theta| \le K$, $|\theta_x| \le K$ we have

$$a(u_x, \theta) \ge a_0, \qquad c(u_x, \theta) \ge c_0, \qquad d(\theta, \theta_x) \ge d_0, \tag{1.7}$$

$$b(u_x, \theta) \neq 0. \tag{1.8}$$

f_1 and f_2 satisfy:

$$f_1, f_2 \in C^2([0, \infty), L^2) \cap C^1([0, \infty), H^1). \tag{1.9}$$

REMARK. The conclusion of the Main Theorem still remains valid if f_1, f_2 also depend on higher-order (quadratic , ...) terms in u and θ up to their second derivatives.

Since we are looking for the solution in a K-neighborhood of the origin, we can assume without loss of generality that the functions a, b, c, d and their derivatives are bounded.

Let

$$u_2 := u_{tt}(t = 0), \qquad \theta_1 := \theta_t(t = 0)$$

be given formally through the differential equation, explicitly in terms of the initial data u_0, u_1, θ_0 :

$$u_2 = a(u_{0,x}, \theta_0)u_{0,xx} + b(u_{0,x}, \theta_0)\theta_{0,xx} + f(t = 0), \tag{1.10}$$

$$\theta_1 = \frac{1}{c(u_{0,x}, \theta_0)}\{d(\theta_0, \theta_{0,x})\theta_{0,xx} - b(u_{0,x}, \theta_0)u_{1,x} + g(t = 0)\}. \tag{1.11}$$

ASSUMPTION 1.2. Suppose $u_0 \in H^3$, $u_1 \in H^2$, $u_2 \in H^1$, $\theta_0 \in H^3$, $\theta_1 \in H^2$, and $|\theta_0(x)| < \tau_0$ in $\overline{\Omega}$,

$$u_0 = u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \tag{1.12}$$

$$\theta_0 = \theta_1 = 0 \quad \text{on } \partial\Omega. \tag{1.13}$$

Then we have the

MAIN THEOREM. Let $\lambda(t) := \sum_{j=0}^{1} \|D^j(f_1, f_2)(t, \cdot)\|^2 + \|\partial_t^2(f_1, f_2)(t, \cdot)\|^2 + \|\partial_t\partial_x(f_1, f_2)(t, \cdot)\|^2$, and suppose that Assumptions 1.1 and 1.2 are satisfied. Then there exists a small constant $\varepsilon_0 > 0$ such that if

$$\|u_0\|_2^2 + \|u_1\|_2^2 + \|u_2\|_1^2 + \|\theta_0\|_2^2 + \|\theta_1\|_2^2 + \sup_{t\geq 0} \lambda(t) \leq \varepsilon_0$$

then the initial boundary value problem (1.1), (1.2), (1.5), (1.6) admits a unique global solution

$$u \in \bigcap_{j=0}^{3} C^j([0, \infty), H^{3-j}), \qquad \theta \in \bigcap_{j=0}^{1} C^j([0, \infty), H^{3-j}), \qquad \theta \in C^2([0, \infty), L^2).$$

Moreover, there are constants c_1, $c_2 > 0$ such that for $t \geq 0$

$$\begin{aligned} &\sum_{j=0}^{3} \|D^j u(t)\|^2 + \sum_{j=0}^{2} \|D^j \theta(t)\|^2 + \|\theta_{txx}(t)\|^2 + \|\theta_{xxx}(t)\|^2 \\ &\leq c_1 e^{-c_2 t}\left(\|u_0\|_2^2 + \|u_1\|_2^2 + \|u_2\|_1^2 + \|\theta_0\|_2^2 + \|\theta_1\|_2^2 + \int_0^t e^{c_2 r}\lambda(r)\, dr\right). \end{aligned} \tag{1.14}$$

REMARK. It should be noticed that

$$\sup_{t\geq 0} e^{-c_2 t}\int_0^t e^{c_2 r}\lambda(r)\,dr \leq \varepsilon_0/c_2 \tag{1.15}$$

holds. This is needed for the corollary on periodic solutions below.

By using the technique of Matsumura from [10], exploiting the exponential stability, we obtain the existence of global, small periodic solutions provided f_1 and f_2 satisfy the following additional assumption.

ASSUMPTION 1.3. f_1 and f_2 are periodic with respect to t with period $\omega > 0$ arbitrary:

$$f_k(t+\omega\,,\,\cdot) = f_k(t\,,\,\cdot)\,, \qquad k = 1\,,\,2\,,\ \text{for all } t \geq 0. \tag{1.16}$$

Then we have the following

COROLLARY. Suppose Assumptions 1.1 and 1.3 are satisfied. Then there exists a small constant $\varepsilon_1 > 0$ such that if

$$\sup_{t\geq 0}\sum_{j=0}^{1}\|D^j(f_1\,,\,f_2)(t\,,\,\cdot)\| + \|\partial_t^2(f_1\,,\,f_2)(t\,,\,\cdot)\| + \|\partial_t\partial_x(f_1\,,\,f_2)(t\,,\,\cdot)\|^2 < \varepsilon_1$$

then the problem (1.1), (1.2), (1.5) admits a unique solution

$$u \in \bigcap_{j=0}^{3} C^j([0\,,\,\infty)\,,\,H^{3-j})\,, \qquad \theta \in \bigcap_{j=0}^{1} C^j([0\,,\,\infty)\,,\,H^{3-j})\,,$$

$$\theta \in C^2([0\,,\,\infty)\,,\,L^2)\,,$$

which is periodic with period ω.

The notation in this paper is as follows: The usual L^2-space on Ω and its norm are denoted by L^2 and $\|\cdot\|$ respectively. H^m, $m \in \mathbb{N}$, denotes the usual Sobolev space

$$H^m := \{v \in L^2 \mid \|v\|_m \equiv \|v\| + \|v^{(1)}\| + \cdots + \|v^{(m)}\| < \infty\}\,,$$

where $v^{(j)}(x) := (d/(dx))^j v(x)$, $1 \leq j \leq m$. $D^L v(t\,,\,x) = \partial_t^j\partial_x^k v(t\,,\,x)$, $j+k = L \in \mathbb{N}_0$, where $\partial_t^j = (\partial/(\partial t))^j$, $\partial_x^k = (\partial/(\partial x))^k$. Differentiation is mainly indicated by indices: $u_t = \partial_t u$, $u_x = \partial_x u$, and so on. The L^∞-norm is denoted by $|\cdot|_\infty$. $C^L(I\,,\,B)$ (resp. $L^2(I\,,\,B)$) denotes the space of B-valued functions that are L-times continuously differentiable (resp. square integrable) in I, $I \subset \mathbb{R}$ an interval, B a Banach space, $L \in \mathbb{N}_0$.

2. Proof of the Main Theorem. The proof of the Main Theorem consists in combining the following local existence and uniqueness theorem with uniform a priori estimates.

THEOREM 2.1 (Local existence and uniqueness). Suppose Assumptions 1.1 and 1.2 are satisfied. Then there exists $T > 0$ depending only on a bound for $\sum_{j=0}^{2}\|u_i\|_{3-j} + \sum_{j=0}^{1}\|\theta_j\|_{3-j}$, $|\theta_0|_\infty$ and on

$$\sup_{t\geq 0}\left\{\sum_{j=0}^{1}\|D^j(f_1\,,\,f_2)(t\,,\,\cdot)\| + \|\partial_t^2(f_1\,,\,f_2)(t\,,\,\cdot)\| + \|\partial_t\partial_x(f_1\,,\,f_2)(t\,,\,\cdot)\|^2\right\}$$

such that the initial boundary value problem (1.3)–(1.6) admits a unique solution (u, θ) in $[0, T] \times \Omega$.

u and θ satisfy

$$u \in \bigcap_{j=0}^{3} C^j([0, T], H^{3-j}), \qquad \theta \in \bigcap_{j=0}^{1} C^j([0, T], H^{3-j}), \qquad \theta \in C^2([0, T], L^2),$$

$$|\theta(t, x)| < \tau_0 \quad \text{for } (t, x) \in [0, T] \times \overline{\Omega}.$$

If

$$|u_{0,x}| < K/2, \quad |\theta_0| < K/2, \quad |\theta_{0,x}| < K/2 \qquad \text{in } \overline{\Omega},$$

then

$$|u_x| < K, \quad |\theta| < K, \quad |\theta_x| < K \qquad \text{in } [0, T] \times \overline{\Omega}. \tag{2.1}$$

Since Theorem 2.1 can be proved by standard methods—energy method and contraction mapping principle—we omit its proof (cf. [12] and Slemrod [15] for the case of the Dirichlet-Neumann boundary condition).

We now proceed to get uniform a priori estimates.

Multiplying (1.3) by u_t and (1.4) by θ, then adding together and integrating with respect to x, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2 + \int_0^l a u_x^2\,dx + \int_0^l c\theta^2\,dx\right) + \int_0^l d\theta_x^2\,dx \\
= \int_0^l [\tfrac{1}{2}a_t u_x^2 - a_x u_x u_t + b_x u_t \theta - d_x \theta\theta_x + \tfrac{1}{2}c_t\theta^2]\,dx + \int_0^l (f u_t + g\theta)\,dx \\
\equiv R_1 + \int_0^l (f u_t + g\theta)\,dx.
\end{aligned} \tag{2.2}$$

(The parameters t, x are mostly dropped here and in the sequel.)

Differentiating (1.3), (1.4) with respect to t once and twice, respectively, we obtain with the same (energy) method as before:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|u_{tt}\|^2 + \int_0^l a u_{tx}^2\,dx + \int_0^l c\theta_t^2\,dx\right) + \int_0^l d\theta_{tx}^2\,dx \\
= \int_0^l [\tfrac{1}{2}a_t u_{tx}^2 - a_x u_{tt}u_{tx} + a_t u_{tt}u_{xx} - b_t\theta_x u_{tt} + b_x u_{tt}\theta_t - b_t\theta_x u_{tt} \\
- b_t\theta_t u_{tx} - d_x\theta_t\theta_{tx} + d_t\theta_t\theta_{xx} - \tfrac{1}{2}c_t\theta_t^2]\,dx \\
+ \int_0^l (f_t u_{tt} + g_t\theta_t)\,dx \\
\equiv R_2 + \int_0^l (f_t u_{tt} + g_t\theta_t)\,dx,
\end{aligned} \tag{2.3}$$

respectively

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|u_{ttt}\|^2+\int_0^l au_{ttx}^2\,dx+\int_0^l c\theta_{tt}^2\,dx\right)+\int_0^l d\theta_{ttx}^2\,dx\\
&=\frac{1}{2}\int_0^l [a_tu_{ttx}^2-a_xu_{ttt}u_{ttx}+2a_tu_{ttt}u_{txx}+a_{tt}u_{ttt}u_{xx}-2b_t\theta_{tx}u_{ttt}-b_{tt}\theta_xu_{ttt}\\
&\qquad -\tfrac{3}{2}c_t\theta_{tt}^2-c_{tt}\theta_t\theta_{tt}+b_x\theta_{tt}u_{ttt}-b_{tt}u_{tx}\theta_{tt}-2b_tu_{ttx}\theta_{tt}\\
&\qquad\qquad -d_x\theta_{tt}\theta_{ttx}+d_{tt}\theta_{xx}\theta_{tt}+2d_t\theta_{txx}\theta_{tt}]\,dx\\
&\quad+\int_0^l (f_{tt}u_{ttt}+g_{tt}\theta_{tt})\,dx\\
&\equiv R_3+\int_0^l (f_{tt}u_{ttt}+g_{tt}\theta_{tt})\,dx.
\end{aligned}\tag{2.4}$$

REMARK. The differentiation with respect to t twice is formally not allowed, for example, u_{tttt} is not defined. But smoothing the initial data here and going to the limit in the final energy estimates justifies (2.4) and similar calculations below. Observe, for example, that only $\|u_{ttt}\|$ will appear in the energy estimate.

Differentiating (1.3), (1.4) with respect to x, multiplying it by u_{tx} and θ_x, respectively, then adding together, and integrating with respect to x, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|u_{tx}\|^2+\int_0^l au_{xx}^2\,dx+\int_0^l c\theta_x^2\,dx\right)+\int_0^l d\theta_{xx}^2\,dx\\
&=\int_0^l [\tfrac{1}{2}a_tu_{xx}^2+\tfrac{1}{2}c_t\theta_x^2-c_x\theta_t\theta_x-b_x\theta_xu_{tx}]\,dx\\
&\quad+d\theta_x\theta_{xx}|_0^l+fu_{tx}|_0^l+\int_0^l (f_xu_{tx}+g_x\theta_x)\,dx\\
&=\int_0^l [\tfrac{1}{2}a_tu_{xx}^2+\tfrac{1}{2}c_t\theta_x^2-c_x\theta_t\theta_x-b_x\theta_xu_{tx}]\,dx\\
&\quad+bu_{tx}\theta_x|_0^l+fu_{tx}|_0^l-g\theta_x|_0^l+\int_0^l (f_xu_{tx}+g_x\theta_x)\,dx\\
&\equiv R_4+bu_{tx}\theta_x|_0^l+fu_{tx}|_0^l-g\theta_x|_0^l+\int_0^l (f_xu_{tx}+g_x\theta_x)\,dx.
\end{aligned}\tag{2.5}$$

Here we used

$$bu_{tx}-g-d\theta_{xx}|_0^l=-c\theta_t|_0^l=0.$$

In order to deal with the boundary terms we use a technique due to Muñoz Rivera [11] (for the linear case). We remark that an important aspect in [11] is to consider the once-differentiated equation. This leads to second-order energy terms, which are needed to get decay information for the first-order energy and then for the second-order energy too. Here we shall also consider the third-order energy terms because of the regularity assumptions in the local existence theorem.

We differentiate (1.3) with respect to t, multiply it by $(x-l/2)u_{tx}$, and integrate with respect to x to obtain

$$\int_0^l \left(x-\frac{l}{2}\right) u_{tx}(u_{ttt}-au_{txx}-a_tu_{xx}+b_t\theta_x+b\theta_{tx}-f_t)\,dx=0, \tag{2.6}$$

$$\begin{aligned}\frac{d}{dt}\int_0^l \left(x-\frac{l}{2}\right) u_{tx}u_{tt}\,dx-\int_0^l \left(x-\frac{l}{2}\right) u_{tt}u_{ttx}\,dx-\frac{1}{2}\int_0^l \left(x-\frac{l}{2}\right) a(u_{tx}^2)_x\,dx\\ =\int_0^l \left(x-\frac{l}{2}\right)[a_tu_{tx}u_{xx}-b_t\theta_xu_{tx}-u_{tx}b\theta_{tx}]\,dx+\int_0^l \left(x-\frac{l}{2}\right) f_tu_{tx}\,dx.\end{aligned} \tag{2.7}$$

It turns out that we have

$$\begin{aligned}&\frac{l}{4}(au_{tx}^2|_{x=l}+au_{tx}^2|_{x=0})\\ &=\frac{d}{dt}\int_0^l \left(x-\frac{l}{2}\right) u_{tx}u_{tt}\,dx+\frac{1}{2}\|u_{tt}\|^2+\frac{1}{2}\int_0^l au_{tx}^2\,dx\\ &\quad+\int_0^l \left(x-\frac{l}{2}\right)[\tfrac{1}{2}a_xu_{tx}^2-a_tu_{tx}u_{xx}+b_t\theta_xu_{tx}+bu_{tx}\theta_{tx}]\,dx\\ &\quad-\int_0^l \left(x-\frac{l}{2}\right) f_tu_{tx}\,dx\\ &=\frac{d}{dt}\int_0^l \left(x-\frac{l}{2}\right) u_{tx}u_{tt}dx+\frac{1}{2}\|u_{tt}\|^2+\frac{1}{2}\int_0^l au_{tx}^2\,dx\\ &\quad+\int_0^l \left(x-\frac{l}{2}\right) bu_{tx}\theta_{tx}-\int_0^l \left(x-\frac{l}{2}\right) f_tu_{tx}\,dx+R_5\end{aligned} \tag{2.8}$$

with

$$R_5:=\int_0^l \left(x-\tfrac{l}{2}\right)[\tfrac{1}{2}a_xu_{tx}^2-a_tu_{tx}u_{xx}+b_x\theta_xu_{tx}]\,dx. \tag{2.9}$$

Similarly, differentiating (1.3), (1.4) with respect to t and x, then multiplying it by u_{ttx} and θ_{tx}, respectively, adding together, and integrating with respect to x, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u_{ttx}\|^2+\int_0^l au_{txx}^2\,dx+\int_0^l c\theta_{tx}^2\,dx\right)+\int_0^l d\theta_{txx}^2\,dx\\ &=\frac{1}{2}\int_0^l a_tu_{txx}^2\,dx+au_{txx}u_{ttx}|_0^l\\ &\quad+\int_0^l [a_{tx}u_{xx}u_{ttx}-b_x\theta_{tx}u_{ttx}+a_tu_{xxx}u_{ttx}-b_{tx}\theta_xu_{ttx}\\ &\qquad-b_t\theta_{xx}u_{ttx}-\tfrac{1}{2}c_t\theta_{tx}^2-c_{tx}\theta_{tx}\theta_t-c_x\theta_{tt}\theta_{tx}\\ &\qquad\qquad+b_tu_{tx}\theta_{txx}-d_t\theta_{xx}\theta_{txx}]\,dx\\ &\quad+\int_0^l (f_{tx}u_{ttx}+g_{tx}\theta_{tx})\,dx+g_t\theta_{tx}|_0^l.\end{aligned} \tag{2.10}$$

In deriving (2.10) we used the relation

$$d\theta_{txx}-bu_{ttx}+d_t\theta_{xx}-b_tu_{tx}|_0^l=\frac{d}{dt}(d\theta_{xx}-bu_{xx})|_0^l=-g_t|_0^l.$$

Differentiating (1.3) with respect to t, we obtain

$$u_{ttt} - au_{txx} - a_t u_{xx} + b_t\theta_x + b\theta_{tx} = f_t. \tag{2.11}$$

Thus it turns out with the help of the boundary condition that

$$u_{txx}|_0^l = \left(-\frac{a_t}{a}u_{xx} + \frac{b_t}{a}\theta_x + \frac{b}{a}\theta_{tx} - \frac{f_t}{a}\right)\Big|_0^l. \tag{2.12}$$

Similarly to (2.8) we have

$$\begin{aligned}\frac{l}{4}&(au_{ttx}^2|_{x=l} + au_{ttx}^2|_{x=0})\\ &= \frac{d}{dt}\int_0^l\left(x-\frac{l}{2}\right)u_{ttx}u_{ttt}\,dx + \tfrac{1}{2}\|u_{ttt}\|^2 + \frac{1}{2}\int_0^l au_{ttx}^2\,dx\\ &\quad+\int_0^l\left(x-\frac{l}{2}\right)bu_{ttx}\theta_{ttx}\,dx + \int_0^l\left(x-\frac{l}{2}\right)f_{tt}u_{ttx}\,dx + R_6\end{aligned} \tag{2.13}$$

with

$$R_6 := \int_0^l\left(x-\frac{l}{2}\right)[\tfrac{1}{2}a_x u_{ttx}^2 - a_{tt}u_{xx}u_{ttx} - 2a_t u_{ttx}u_{txx} + 2b_t\theta_{tx}u_{ttx} + b_{tt}\theta_x u_{ttx}]\,dx. \tag{2.14}$$

The strategy in the sequel consists in estimating u in terms of θ and its derivatives.

Using the second differential equation (1.4) and Poincaré's inequality we have

$$\|u_{tx}\|^2 \le K_1(\|\theta_{tx}\|^2 + \|\theta_{xx}\|^2 + \|g\|^2). \tag{2.15}$$

Here K_1 and in the sequel $K_2, K_3, \ldots$ denote positive constants.

$$\|u_t\|^2 \le K_2(\|\theta_{tx}\|^2 + \|\theta_{xx}\|^2 + \|g\|^2). \tag{2.16}$$

In order to produce a $\int_0^l au_x^2\,dx$-term, we use the differential equation (1.3):

$$\begin{aligned}\frac{d}{dt}&\int_0^l uu_t\,dx + \int_0^l au_x^2\,dx\\ &= \|u_t\|^2 - \int_0^l a_x uu_x\,dx - \int_0^l bu\theta_x\,dx + \int_0^l uf\,dx\\ &\le K_3(\|u_t\|^2 + \|\theta_x\|^2) + \frac{1}{2}\int_0^l au_x^2\,dx - \int_0^l a_x uu_x\,dx + \int_0^l uf\,dx.\end{aligned} \tag{2.17}$$

This implies

$$\begin{aligned}\frac{d}{dt}&\int_0^l uu_t\,dx + \frac{1}{2}\int_0^l au_x^2\,dx\\ &\le K_3(\|u_t\|^2 + \|\theta_x\|^2) - \int_0^l a_x uu_x\,dx + \int_0^l uf\,dx\\ &\le K_4(\|\theta_{tx}\|^2 + \|\theta_x\|^2 + \|\theta_{xx}\|^2 + \|f\|^2) + R_7\end{aligned} \tag{2.18}$$

with

$$R_7 := -\int_0^l a_x uu_x\,dx. \tag{2.19}$$

Similarly we have

$$\frac{d}{dt}\int_0^l u_t u_{tt}\,dx - \|u_{tt}\|^2 + \int_0^l a u_{tx}^2\,dx + \int_0^l b\theta_{tx}u_t\,dx + \int_0^l u_t f_t\,dx + R_8 = 0 \quad (2.20)$$

with

$$R_8 := \int_0^l [a_x u_t u_{tx} - a_t u_{xx} u_t + b_t \theta_x u_t]\,dx. \quad (2.21)$$

Thus it turns out that

$$\begin{aligned}\|u_{tt}\|^2 &\le K_4\left(\int_0^l a u_{tx}^2 dx + \|\theta_{tx}\|^2\right) + \frac{d}{dt}\int_0^l u_t u_{tt}\,dx + \int_0^l u_t f_t\,dx + R_8 \\ &\le K_5(\|\theta_{tx}\|^2 + \|\theta_{xx}\|^2) + \frac{d}{dt}\int_0^l u_t u_{tt}\,dx + \int_0^l u_t f_t\,dx + R_8,\end{aligned} \quad (2.22)$$

where we have used (2.15) and Poincaré's inequality. Using the differential equation (1.3) once more, we get

$$\|u_{xx}\|^2 \le K_6(\|\theta_x\|^2 + \|\theta_{tx}\|^2 + \|\theta_{xx}\|^2 + \|f\|^2) + \frac{d}{dt}\int_0^l u_t u_{tt} dx + R_8. \quad (2.23)$$

With the help of the differential equation (1.4) we conclude

$$\|u_{ttx}\|^2 \le K_7(\|\theta_{tt}\|^2 + \|\theta_{txx}\|^2 + \|g_t\|^2 + R_9) \quad (2.24)$$

with

$$R_9 := \| - c_t\theta_t + b_t u_{tx} + d_t \theta_{xx}\|^2. \quad (2.25)$$

Similarly we have

$$\|u_{ttt}\|^2 = \frac{d}{dt}\int_0^l u_{tt}u_{ttt}\,dx + \int_0^l a u_{ttx}^2\,dx + \int_0^l b u_{tt}\theta_{ttx}\,dx + \int_0^l u_{tt} f_{tt}\,dx + R_{10} \quad (2.26)$$

with

$$R_{10} := \int_0^l [a_x u_{ttx} - 2a_t u_{txx} - a_{tt}u_{xx} + b_{tt}\theta_x + 2b_t\theta_{tx}]u_{tt}\,dx. \quad (2.27)$$

This implies

$$\begin{aligned}\|u_{ttt}\|^2 \le{}& \frac{d}{dt}\int_0^l u_{tt}u_{ttt}\,dx \\ &+ K_8\left(\|\theta_{tt}\|^2 + \|\theta_{txx}\|^2 + \|\theta_{tx}\|^2 + \|\theta_{xx}\|^2 + \frac{d}{dt}\int_0^l u_t u_{tt}\,dx + R_8\right) \\ &+ \int_0^l u_{tt} f_{tt}\,dx + R_{10}.\end{aligned} \quad (2.28)$$

The differential equation (1.4) differentiated with respect to t and to x, respectively, yields

$$\|u_{txx}\|^2 + \|u_{xxx}\|^2 \le K_9(\|u_{ttt}\|^2 + \|u_{ttx}\|^2 + \|\theta_{tx}\|^2 + \|\theta_{xx}\|^2 + \|f_t\|^2 + \|f_x\|^2 + R_{11}) \quad (2.29)$$

with

$$R_{11} := \|a_t u_{xx} - b_t\theta_x + a_x u_{xx} - b_x\theta_x\|. \quad (2.30)$$

Differentiating the differential equation (1.4) with respect to x leads to

$$\|\theta_{xxx}\|^2 \le K_{10}(\|\theta_{xx}\|^2 + \|u_{txx}\|^2 + \|g_x\|^2 + R_{12}) \tag{2.31}$$

with

$$R_{12} := \|c_x\theta_t + b_x u_{tx} - d_x\theta_{xx}\|^2. \tag{2.32}$$

The boundary term $bu_{tx}\theta_x|_0^l$ in (2.5) can be estimated as

$$\begin{aligned} bu_{tx}\theta_x|_0^l &\le \varepsilon^{1/2}\frac{l}{4}(au_{tx}^2|_{x=l} + au_{tx}^2|_{x=0}) + K_{11}\varepsilon^{-1/2}(\theta_x^2|_{x=0} + \theta_x^2|_{x=l}) \\ &\le \varepsilon^{1/2}\frac{l}{4}(au_{tx}^2|_{x=l} + au_{tx}^2|_{x=0}) + K_{12}(\varepsilon^{1/2}\|\theta_{xx}\|^2 + \varepsilon^{-3/2}\|\theta_x\|^2). \end{aligned} \tag{2.33}$$

Using (2.12) we can estimate the boundary term $au_{txx}u_{ttx}|_0^l$ in (2.10) as

$$au_{txx}u_{ttx}|_0^l \le \varepsilon^{1/2}\frac{l}{4}(au_{ttx}^2|_{x=l}+au_{ttx}^2|_{x=0})+K_{13}(\varepsilon^{1/2}\|\theta_{txx}\|^2+\varepsilon^{-3/2}\|\theta_{tx}\|^2+R_{13}) \tag{2.34}$$

with

$$R_{13} := \|b_t\theta_x - a_t u_{xx}\|_1^2. \tag{2.35}$$

We now define

$$\begin{aligned} X_1(t) := {}& \varepsilon^{-2}[\|u_t\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \int_0^l a(u_x^2 + u_{tx}^2 + u_{ttx}^2)\,dx + \int_0^l c(\theta^2+\theta_t^2+\theta_{tt}^2)] \\ &+ \|u_{tx}\|^2 + \|u_{ttx}\|^2 + \int_0^l a(u_{xx}^2 + u_{txx}^2)\,dx + \int_0^l c(\theta_x^2 + \theta_{tx}^2)\,dx \\ &- \varepsilon^{1/2}\int_0^l \left(x - \frac{l}{2}\right) u_{tx}u_{tt}\,dx - \varepsilon^{1/2}\int_0^l \left(x - \frac{l}{2}\right) u_{ttx}u_{ttt}\,dx \\ &+ \varepsilon^{1/4}\int_0^l uu_t\,dx - \varepsilon^{1/4}\int_0^l u_t u_{tt}\,dx - \varepsilon^{1/4}\int_0^l u_{tt}u_{ttt}\,dx \end{aligned} \tag{2.36}$$

with a small positive constant ε.

It is easy to see that there exists $\varepsilon^* > 0$ such that for $0 < \varepsilon \le \varepsilon^*$ there are constants $\widetilde{K}_1$, $\widetilde{K}_2$ such that

$$\widetilde{K}_1 X_2(t) \le X_1(t) \le \widetilde{K}_2 X_2(t) \tag{2.37}$$

where

$$X_2(t) := \sum_{j=0}^{2}\|D^j u\|^2 + \sum_{j=0}^{1}\|D^j\theta\|^2 + \|u_{ttt}\|^2 + \|u_{ttx}\|^2 + \|u_{txx}\|^2 + \|\theta_{tx}\|^2 + \|\theta_{tt}\|^2. \tag{2.38}$$

Multiplying (2.2), (2.3), (2.4) by ε^{-2}, adding together with (2.5), (2.10), multiplying (2.8), (2.13) with $\varepsilon^{1/2}$ and (2.18), (2.22), (2.26) with $\varepsilon^{1/4}$, and taking ε small enough, we obtain, using the estimates on the boundary terms in (2.33), (2.34) (resp. (2.8), (2.13)) as well as the estimates (2.15), (2.16), (2.22), (2.23), (2.24), (2.28), (2.29), (2.31):

$$\frac{dX_1(t)}{dt} + K_{14}X_2(t) + K_{15}(\|u_{xxx}\|^2 + \|\theta_{xx}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{txx}\|^2 + \|\theta_{xxx}\|^2)$$
$$\leq K_{16}\left(\sum_{j=1}^{13}|R_j| + \sum_{j=0}^{1}\|D^j(f_1, f_2)(t, \cdot)\|^2 + \|\partial_t^2(f_1, f_2)(t, \cdot)\|^2 \right. \tag{2.39}$$
$$\left. + \|\partial_t\partial_x(f_1, f_2)(t, \cdot)\|^2\right),$$

or, using (2.37),

$$\frac{dX_1(t)}{dt} + K_{17}X_1(t) + K_{18}(\|u_{xxx}\|^2 + \|\theta_{xx}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{txx}\|^2 + \|\theta_{xxx}\|^2)$$
$$\leq K_{19}\left(\sum_{j=1}^{13}|R_j| + \sum_{j=0}^{1}\|D^j(f_1, f_2)(t, \cdot)\|^2 \right. \tag{2.40}$$
$$\left. \|\partial_t^2(f_1, f_2)(t, \cdot)\|^2 + \|\partial_t\partial_x(f_1, f_2)(t, \cdot)\|^2\right).$$

By Sobolev's embedding theorem $(H^1 \hookrightarrow L^\infty)$ we can easily get

$$\sum_{j=1}^{13}|R_j| \leq K_{20}(X_1^{3/2} + X_1^2 + X_1^{1/2}X_3) \tag{2.41}$$

where

$$X_3(t) := \|u_{xxx}\|^2 + \|\theta_{xxx}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{txx}\|^2 + \|\theta_{xxx}\|^2. \tag{2.42}$$

Observe that all terms in R_j, $j = 1, \ldots, 13$, contain derivatives of the coefficients a, b, c, or d; hence, they are at least cubic terms. As examples for the proof of (2.41) we consider a few typical terms in R_1, R_6, and R_{10}:

In R_1: $\int_0^l a_t u_x^2\,dx$ (cf. (2.2)). $a_t = a_{u_x}u_{tx} + a_\theta\theta_t$.

This implies

$$\left|\int_0^l a_t u_x^2\,dx\right| \leq K_{21}|u_x|_\infty H(\|u_{tx}\| + \|u_x\| + \|\theta_t\|) \leq K_{22}X_1^{3/2}. \tag{2.43}$$

In R_6: $\int_0^l b_{tt}\theta_x u_{ttx}\,dx$ (cf. (2.14)). $b_{tt} = b_{u_xu_x}u_{tx}^2 + b_{u_x}u_{ttx} + b_{\theta\theta}\theta_t^2 + b_\theta\theta_{tt}$. This implies

$$\left|\int_0^l b_{tt}\theta_x u_{ttx}\,dx\right| \leq K_{23}(|u_{tx}|_\infty^2\|\theta_x\|\,\|u_{ttx}\|$$
$$+ |\theta_x|_\infty\|u_{ttx}\|^2 + |\theta_t|_\infty^2\|\theta_x\|\,\|u_{ttx}\| + |\theta_x|_\infty\|u_{ttx}\|\,\|\theta_{tt}\|)$$
$$\leq K_{24}(X_1^2 + X_1^{3/2} + X_1X_3^{1/2})$$
$$\leq K_{25}(X_1^2 + X_1^{3/2} + X_1^{1/2}X_3). \tag{2.44}$$

In R_{10}: $\int_0^l a_x u_{ttx} u_{tt}\,dx$ (cf. (2.27)). $a_x = a_{u_x} u_{xx} + a_\theta \theta_x$. This implies

$$\begin{aligned}\left|\int_0^l a_x u_{ttx} u_{tt}\,dx\right| &\le K_{26}(|u_{tt}|_\infty \|u_{ttx}\|\,\|u_{xx}\| + |\theta_x|_\infty |u_{tt}|_\infty \sqrt{l}\|u_{ttx}\|)\\ &\le K_{27}(X_1^{3/2} + X_1 X_3^{1/2} + X_1^{1/2} X_3)\\ &\le K_{28}(X_1^{3/2} + X_1^{1/2} X_3).\end{aligned} \tag{2.45}$$

If we now assume *a priori*

$$X_1(s) \le 1 \quad \text{for } 0 \le s \le t, \tag{2.46}$$

we conclude from (2.40)

$$\frac{dX_1(s)}{ds} + K_{17} X_1(s) \le K_{29}(X_1^{3/2}(s) + \lambda(s)), \tag{2.47}$$

where

$$\lambda(s) = \sum_{j=0}^{1} \|D^j(f_1, f_2)(s, \cdot)\|^2 + \|\partial_t^2(f_1, f_2)(s, \cdot)\|^2 + \|\partial_t \partial_x (f_1, f_2)(s, \cdot)\|^2.$$

If additionally *a priori*

$$X_1(s) \le \left(\frac{K_{17}}{2K_{29}}\right)^2 \tag{2.48}$$

is assumed, then we obtain

$$\frac{dX_1(s)}{ds} + \frac{K_{17}}{2} X_1(s) \le K_{29} \lambda(s) \tag{2.49}$$

and, hence,

$$X_1(s) \le e^{-c_2 s} X_1(0) + K_{29} e^{-c_2 s} \int_0^s e^{c_2 r} \lambda(r)\,dr, \tag{2.50}$$

where

$$c_2 := K_{17}/2. \tag{2.51}$$

Now let

$$\lambda_1 := \sup_{s \ge 0} e^{-c_2 s} \int_0^s e^{c_2 r} \lambda(r)\,dr.$$

If

$$X_1(0) + K_{29} \lambda_1 < \frac{1}{2}\left(\frac{K_{17}}{2K_{29}}\right)^2, \tag{2.52}$$

where the right-hand side is assumed to be less than 1 without loss of generality, then (2.46) and (2.48) hold for $0 \le s \le t_1$ for some $t_1 > 0$; hence, (2.50) holds in $s = t_1$, which is the desired a priori estimate.

Now the usual combination of the local existence and uniqueness theorem with the above uniform a priori estimate yields the Main Theorem (cf., e.g., [9, 12, 14] for this standard argument); the exponential decay claimed there follows from (2.50) observing that $\|u_{xxx}\|^2 + \|\theta_{xx}\|^2 + \|\theta_{txx}\|^2 + \|\theta_{xxx}\|^2$ can be bounded by X_1 and that

$$X_1(0) \le K_{30}(\|u_0\|_2^2 + \|u_1\|_2^2 + \|u_2\|_1^2 + \|\theta_0\|_2^2 + \|\theta_1\|_2^2 + \|f_t(t=0)\|^2 + \|g_t(t=0)\|^2)$$

holds. □

REFERENCES

[1] C. M. Dafermos and L. Hsiao, *Development of singularities in solutions of the equations of nonlinear thermoelasticity*, Quart. Appl. Math. **44**, 463–474 (1986)

[2] W. H. Hrusa and S. A. Messaoudi, *On formation of singularities in one-dimensional nonlinear thermoelasticity*, Arch. Rational Mech. Anal. **111**, 135–151 (1990)

[3] W. J. Hrusa and M. A. Tarabek, *On smooth solutions of the Cauchy problem in one-dimensional nonlinear thermoelasticity*, Quart. Appl. Math. **47**, 631–644 (1989)

[4] S. Jiang, *Global existence of smooth solutions in one-dimensional nonlinear thermoelasticity*, Proc. Roy. Soc. Edinburgh Sect. A **115**, 257–274 (1990)

[5] S. Jiang, *Global solutions of the Dirichlet problem in one-dimensional nonlinear thermoelasticity*, SFB 256 preprint 138, Universität Bonn, 1990

[6] S. Jiang, *Global solutions of the Neumann problem in one-dimensional nonlinear thermoelasticity*, Nonlinear Anal. **19**, 107–121 (1992)

[7] S. Kawashima, *Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics*, Thesis, Kyoto University, 1983

[8] S. Kawashima and M. Okada, *Smooth global solutions for the one-dimensional equations in magnetohydrodynamics*, Proc. Japan Acad. Ser. A Math. Sci. **53**, 384–387 (1982)

[9] S. Klainerman and G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math. **36**, 133–141 (1983)

[10] A. Matsumura, *Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with the first-order dissipation*, Publ. Res. Inst. Math. Sci. **13**, 349–379 (1977)

[11] J. E. Muñoz Rivera, *Energy decay rates in linear thermoelasticity*, Funkcial. Ekvac **35**, 19–30 (1992)

[12] R. Racke and Y. Shibata, *Global smooth solutions and asymptotic stability in one-dimensional nonlinear thermoelasticity*, Arch. Rational Mech. Anal. **116**, 1–34 (1991)

[13] Y. Shibata, *Neumann problem for one-dimensional nonlinear thermoelasticity*, SFB 256 preprint **145**, Universität Bonn, 1990

[14] Y. Shibata and Y. Tsutsumi, *On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain*, Math. Z. **191**, 165–199 (1986)

[15] M. Slemrod, *Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional non-linear thermoelasticity*, Arch. Rational Mech. Anal. **76**, 97–133 (1981)

[16] S. Zheng, *Global solutions and applications to a class of quasilinear hyperbolic-parabolic coupled systems*, Sci. Sinica Ser. A **127**, 1274–1286 (1984)

[17] S. Zheng and W. Shen, *Global solutions to the Cauchy problem of quasilinear hyperbolic parabolic coupled systems*, Sci. Sinica Ser. A **30**, 1133–1149 (1987)