QUARTERLY OF APPLIED MATHEMATICS
VOLUME LII, NUMBER 1
MARCH 1994, PAGES 103-112

ROOTS OF $J_\gamma(z) \pm iJ_{\gamma+1}(z) = 0$ AND THE EVALUATION OF INTEGRALS WITH CYLINDRICAL FUNCTION KERNELS

By

SRINIVAS TADEPALLI AND COSTAS EMMANUEL SYNOLAKIS

University of Southern California, Los Angeles, California

Abstract. An elementary proof is presented showing that the function $f(z) = J_\gamma(z) \pm iJ_{\gamma+1}(z)$, where γ is a natural number, has no zeroes in the lower and upper half-planes respectively. The roots of $f(z)$ are given for certain values of γ and their locations are plotted. Cartesian maps (mappings of constant coordinate lines) of $f(z)$ are obtained, and special features of these maps are discussed. Some integrals with cylindrical kernels involving $f(z)$ are obtained in terms of the zeroes of $J_\gamma(z)$.

I. Introduction. The function

$$J_0(z) - iJ_1(z) = 0$$

frequently arises in certain solutions of problems in coastal hydrodynamics [1]. Its behaviour in the upper half-plane has been well established by Synolakis [1] and Rawlins [2], while its zeroes in the lower half-plane were first calculated by Macdonald [3]. In this paper we will present a theorem discussing the behaviour of the function

$$f(z) = J_\gamma(z) \pm iJ_{\gamma+1}(z) = 0 \tag{1}$$

and certain applications. This function arises in problems of wave reflection off composite beaches, i.e., beaches with multiple slopes.

II. THEOREM. The function $f(z) = J_\gamma(z) \pm iJ_{\gamma+1}(z)\ \forall \gamma \varepsilon N$, where N is a natural number has no zeroes in the lower and upper half-planes respectively.

Proof. Let the sequence $\zeta_{\gamma,n}$, where γ is a natural number, denote the real zeroes of $J_\gamma(z)$ which lie on the positive real axis arranged in order of nondecreasing magnitude other than the origin. Using the result of Bateman [4]

$$\frac{J_{\gamma+1}(z)}{J_\gamma(z)} = -2z \sum_{n=1}^{\infty} \frac{1}{(z^2 - \zeta_{\gamma,n}^2)}, \tag{2}$$

and using the fact that $J_\gamma(z)$ and $J_{\gamma+1}(z)$ have no common zeroes other than the origin [5], and generalizing the method of Rawlins [2], one can derive that the zeroes

Received January 13, 1992.
1991 *Mathematics Subject Classification*. Primary 33C10.

of the function $f(z)$ are the same as the zeroes of the function $h(z)$ which is defined by

$$h(z) = 1 \mp i \sum_{n=1}^{\infty} \left[\bar{z} \left[\frac{1}{|z-\zeta_{\gamma,n}|^2} + \frac{1}{|z+\zeta_{\gamma,n}|^2} \right] - \zeta_{\gamma,n} \left[\frac{1}{|z-\zeta_{\gamma,n}|^2} - \frac{1}{|z+\zeta_{\gamma,n}|^2} \right] \right]. \tag{3}$$

The real part of $h(z)$ is given by

$$\Re[h(z)] = 1 \mp \Im(z) \sum_{n=1}^{\infty} \left[\frac{1}{|z-\zeta_{\gamma,n}|^2} + \frac{1}{|z+\zeta_{\gamma,n}|^2} \right], \tag{4}$$

where $\Im(z)$ is the imaginary part of z. This function in Eq. (4) is nonzero in the lower and in the upper half-planes respectively, and therefore, $J_\gamma(z) \pm iJ_{\gamma+1}(z)$ has no zeroes in the lower and upper half-planes respectively.

COROLLARY. Using the theorem, it is easy to show that for any polynomial function $S(z)$,

$$\oint_{C_\pm} \frac{S(z)}{(J_\gamma(z) \mp iJ_{\gamma+1}(z))} \, dz = 0, \tag{5}$$

where $C_\pm$ is any contour in the upper and lower half-planes respectively.

III. Evaluation of roots. Using Eq. (2) in Eq. (1) and letting z_j be the zeroes of $f(z)$, one then obtains

$$\sum_{n=1}^{\infty} \frac{z_j}{z_j^2 - \zeta_{\gamma,n}^2} = \mp \frac{i}{2}. \tag{6}$$

For $|z_j| < \zeta_{\gamma,n}$ this equation can be written in the form of a power series which is an alternate representation for the zeroes of the function $f(z)$,

$$\sum_{n=1}^{\infty} S_{2n,\gamma} z_j^{2n-1} = \pm \frac{i}{2}, \tag{7}$$

where

$$S_{2n,\gamma} = \sum_{m=1}^{\infty} \zeta_{\nu,m}^{-2n} \tag{8}$$

and

$$S_{2,\gamma} = \frac{1}{4(\gamma+1)}, \qquad S_{4,\gamma} = \frac{1}{16(\gamma+1)^2(\gamma+2)},$$

$$S_{6,\gamma} = \frac{1}{32(\gamma+1)^3(\gamma+2)(\gamma+3)}, \qquad S_{8,\gamma} = \frac{(5\gamma+11)}{256(\gamma+1)^4(\gamma+2)^2(\gamma+3)(\gamma+4)}.$$

These coefficients were first given by Rayleigh [5]. Although elegant, unfortunately, this method does not provide a more direct way of obtaining z_j. On the other hand, using the power series expansion [6], one obtains the following series for the zeroes of $f(z)$:

$$\sum_{k=1}^{\infty} \frac{(-z_j^2)^{k-1}}{4^{k-1}(k-1)!(\gamma+k-1)!} \left[1 \pm \frac{iz_j}{2(\gamma+k)} \right] = 0. \tag{9}$$

Numerical evaluation of the roots of $J_\gamma(z) \mp iJ_{\gamma+1}(z)$. We performed a standard numerical computation using Newton-Raphson iterations directly on Eq. (9). The real part of the initial estimates for the Newton-Raphson iterations were chosen as multiples of π. We used 200 terms for computing the zeroes of $f(z)$. Examples of results for $\gamma = 1$ and for $\gamma = 2$ are shown in Tables 1 and 2. To check our numerical accuracy we also used Mathematica [7], which uses the computational method in Abramowitz and Stegun [6] to calculate the same zeroes; z_p refers to the zeroes using the power series directly, and z_a denotes the zeroes obtained from Mathematica. Since the zeroes of $J_\gamma(z) \pm iJ_{\gamma+1}(z) = 0$ are symmetric about the real axis, the zeroes of $J_\gamma(z) + iJ_{\gamma+1}(z) = 0$ are the reflection of zeroes of $J_\gamma(z) - iJ_{\gamma+1}(z) = 0$ about the real axis.

TABLE 1. Roots of $J_1(z) \mp iJ_2(z) = 0$.

$\mathrm{Re}(z_a)$	$\mathrm{Im}(z_a)$	$\mathrm{Re}(z_p)$	$\mathrm{Im}(z_p)$
4.41994	$\mp$0.93294	4.41994	$\mp$0.93294
7.66601	$\mp$1.18374	7.66602	$\mp$1.18374
10.85440	$\mp$1.34902	10.85440	$\mp$1.34903
14.02312	$\mp$1.47294	14.02311	$\mp$1.47294
17.18260	$\mp$1.57218	17.18260	$\mp$1.57218
20.33771	$\mp$1.65497	20.33770	$\mp$1.65496
23.48821	$\mp$1.72599	23.48822	$\mp$1.72599
26.63732	$\mp$1.78818	26.63732	$\mp$1.78815
29.78493	$\mp$1.84351	29.78492	$\mp$1.84353
32.93150	$\mp$1.89323	32.93150	$\mp$1.89323
36.07732	$\mp$1.93862	36.07731	$\mp$1.93865

TABLE 2. Roots of $J_2(z) \mp iJ_3(z) = 0$.

$\mathrm{Re}(z_a)$	$\mathrm{Im}(z_a)$	$\mathrm{Re}(z_p)$	$\mathrm{Im}(z_p)$
5.71609	$\mp$0.81917	5.71610	$\mp$0.81916
9.05387	$\mp$1.02167	9.05392	$\mp$1.02170
12.28672	$\mp$1.16306	12.28671	$\mp$1.16306
15.48223	$\mp$1.27272	15.48223	$\mp$1.27272
18.65960	$\mp$1.36245	18.65961	$\mp$1.36243
21.82732	$\mp$1.43851	21.82731	$\mp$1.43831
24.98791	$\mp$1.50449	24.98786	$\mp$1.50351
28.14471	$\mp$1.56279	28.14473	$\mp$1.56277
31.29850	$\mp$1.61501	31.29850	$\mp$1.61501
34.45010	$\mp$1.66229	34.45020	$\mp$1.66229
37.60002	$\mp$1.70539	37.59999	$\mp$1.70543

We attempted to generalize the method of Macdonald [3] to compute the zeroes. Using the standard asymptotic expansions of the Bessel functions and retaining only the first four terms, we obtain that $J_\gamma(z) - iJ_{\gamma+1}(z) = 0$ has approximately the same zeroes as:

$$2\zeta e^{2\zeta}\left[1 - \frac{(a_{1_{(\gamma)}} + a_{1_{(\gamma+1)}})}{2\zeta} + \frac{(a_{2_{(\gamma)}} + a_{2_{(\gamma+1)}})}{2\zeta^2} + \frac{(a_{3_{(\gamma)}} + a_{2_{(\gamma+1)}})}{2\zeta^3}\right]$$
$$= (-1)^{(\gamma)} i\left[(a_{1_{(\gamma+1)}} - a_{1_{(\gamma)}}) + \frac{(a_{2_{(\gamma+1)}} - a_{2_{(\gamma)}})}{\zeta} + \frac{(a_{3_{(\gamma+1)}} - a_{3_{(\gamma)}})}{\zeta^2} + \frac{(a_{4_{(\gamma+1)}} - a_{4_{(\gamma)}})}{\zeta^3}\right], \tag{10}$$

where

$$a_{n_{(\gamma)}} = \prod_{r=1}^{n} \frac{[4\gamma^2 - (2r-1)^2]}{r!2^{3r}}.$$

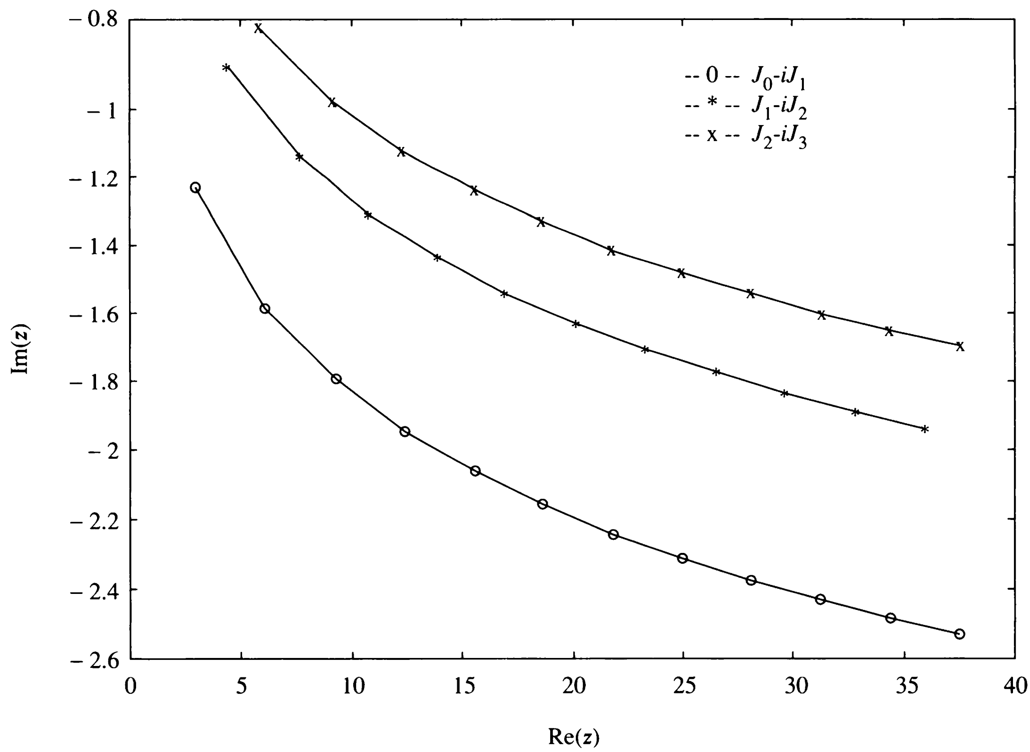

FIG. 1. Location of zeroes of $J_\gamma(z) - iJ_{\gamma+1}(z)$ in the fourth quadrant.

Taking the logarithm after rearranging and expanding for $\zeta \gg 1$, we obtain

$$\zeta = \ln(a_{1_{(\gamma+1)}} - a_{1_{(\gamma)}}) - \frac{1}{2}\ln(2\zeta) + \frac{(2\gamma+1)\pi i}{4} + \frac{A_\gamma}{\zeta} + \frac{B_\gamma}{\zeta^2} + \frac{C_\gamma}{\zeta^3}, \tag{11}$$

where A_γ, B_γ, C_γ are rational functions of $a_{1_{(\gamma)}}$, $a_{2_{(\gamma)}}$ and $a_{3_{(\gamma)}}$. In attempting to implement the Newton-Raphson iteration method on Eq. (10), we noted that it is very sensitive to the initial estimate and the iterations often do not converge. We conclude that this method, even though more direct, is not as computationally effective as direct iterations in Eq. (9).

Fig. 1 shows the location of zeroes of $J_\gamma(z) - iJ_{\gamma+1}(z) = 0$ using direct iterations which are plotted for $\gamma = 0, 1$, and 2 in the fourth quadrant.

IV. **Cartesian maps.** It is difficult to visualize complex functions with two-dimensional graphs. Cartesian maps show how a rectangular grid in the z-plane is mapped on the w-plane, where $w = f(z)$. The abscissa and the ordinate are the real and imaginary parts of $f(z)$ respectively. Cartesian maps for $J_0(z) + iJ_1(z)$ and for $J_1(z) - iJ_2(z)$ are plotted for domains of real and imaginary parts of z which are multiples of π. By visual inspection of the Cartesian maps it is possible to determine whether there are any zeroes in that domain, simply by checking whether the lines of constant coordinates pass through, enclose, or do not pass through the origin in the complex plane. Examples of Cartesian maps are given in Figs. 2 through 9 (see pp. 108–109). In Figs. 2 and 3, the Cartesian maps of $J_0(z) + iJ_1(z)$ in the upper half-plane enclose the origin indicating the presence of zeroes; while they do not in Figs. 4 and 5, in the lower half-plane, as expected from the theorem. Similarly, Figs. 6 through 9 show the Cartesian maps for the function $J_1(z) - iJ_2(z)$. It is clear that Cartesian maps can be a useful tool for a preliminary screening for the zeroes of functions in the complex plane especially when the behaviour of the function is not known a priori. We note that the Cartesian maps shown in Figs. 10 and 11 (see p. 110), obtained from $J_0(z) \pm iJ_1(z)$, suggest that it is possible to generate body-fitted grids over portions of bodies such as over cylinders and over the leading edge of an airfoil.

Cartesian maps of $J_0(z) + iJ_1(z)$

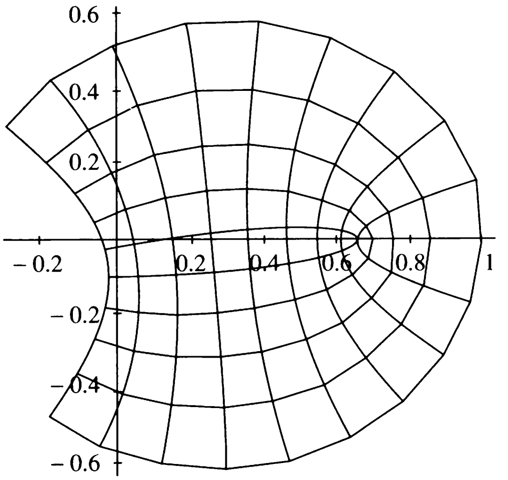

FIG. 2. $x \in (0, \pi)$, $y \in (0, \pi)$.

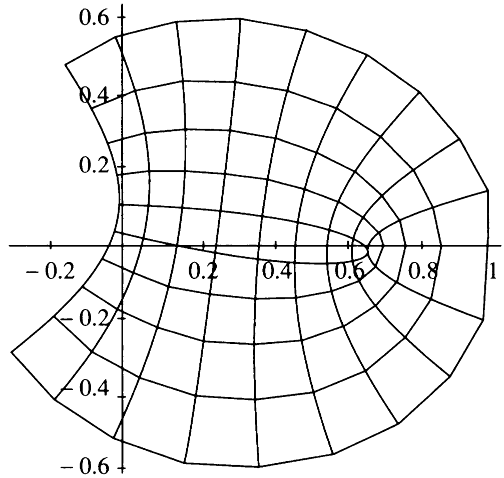

FIG. 3. $x \in (-\pi, 0)$, $y \in (0, \pi)$.

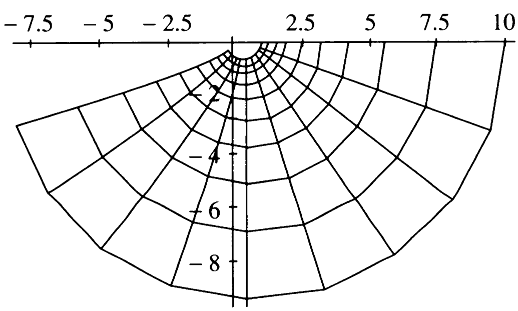

FIG. 4. $x \in (-\pi, 0)$, $y \in (-\pi, 0)$.

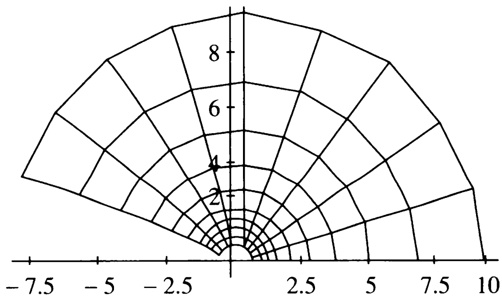

FIG. 5. $x \in (0, \pi)$, $y \in (-\pi, 0)$.

Cartesian maps of $J_1(z) - iJ_2(z)$

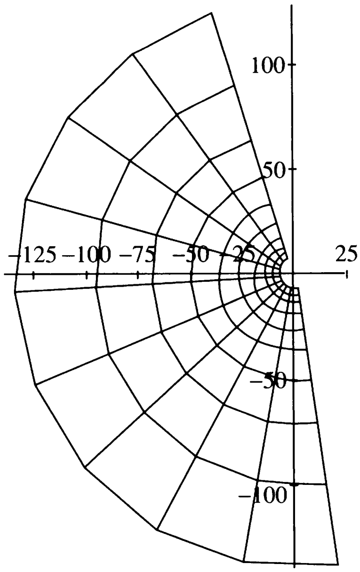

FIG. 6. $x \in (\pi, 2\pi)$, $y \in (\pi, 2\pi)$.

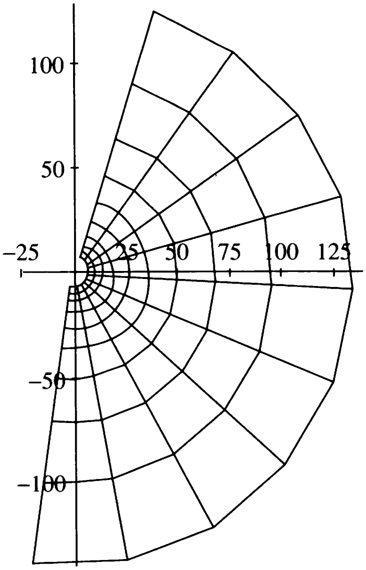

FIG. 7. $x \in (-2\pi, -\pi)$, $y \in (\pi, 2\pi)$.

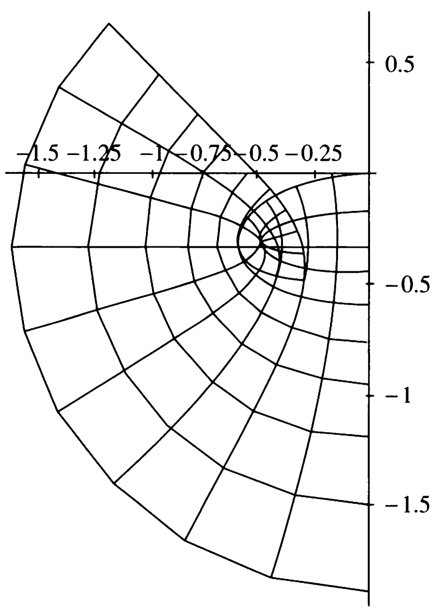

FIG. 8. $x \in (-\pi, 0)$, $y \in (-\pi, 0)$.

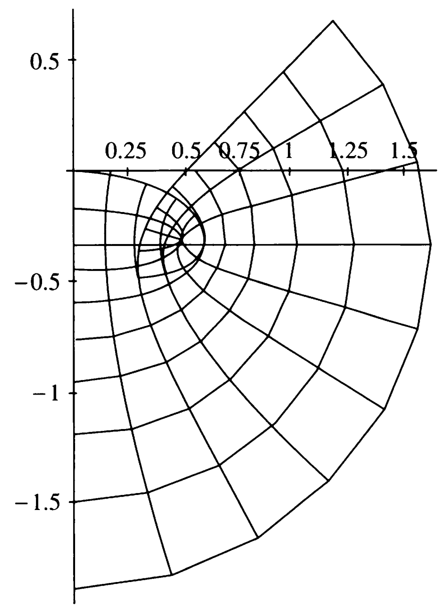

FIG. 9. $x \in (0, \pi)$, $y \in (\pi, 0)$.

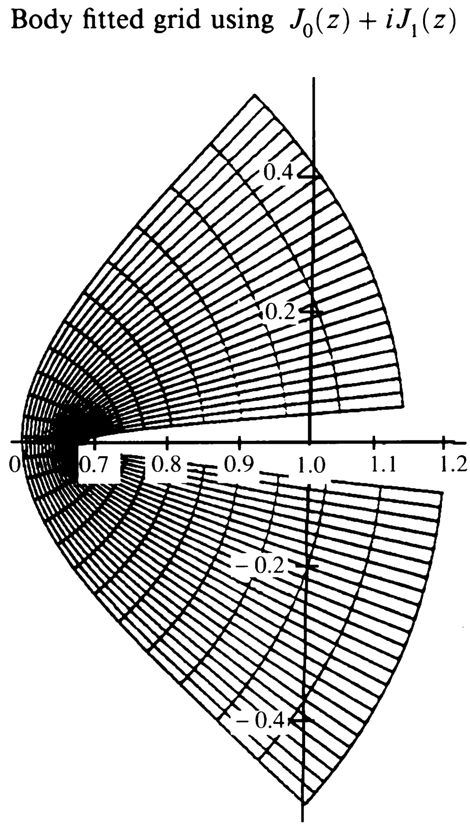

FIG. 10. $x \in (.1, .9)$, $y \in (-.25, 3.55)$

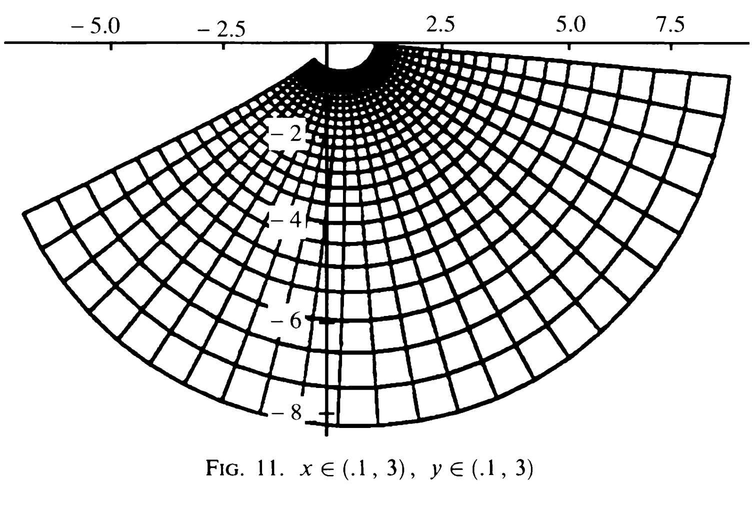

FIG. 11. $x \in (.1, 3)$, $y \in (.1, 3)$

V. **Conclusions.** The function $J_\gamma(z) \pm iJ_{\gamma+1}(z) = 0$ has no zeroes in the lower and upper half-planes respectively. Cartesian maps are helpful in determining the preliminary location of zeroes of $f(z)$.

VI. **Acknowledgment.** This work is supported by the National Science Foundation grant (no. 8818634) and a Presidential Young Investigator's Award (no. 8957853). We are indebted to Larry Glasser for his suggestions.

Appendix. Some integrals of the Bessel functions of the first kind, which are of interest in hydrodynamics are now solved analytically. Let

$$S_\gamma = \int \frac{J_\gamma^2(z) + J_{\gamma+1}^2(z)}{J_\gamma(z)J_{\gamma+1}(z)}\, dz. \tag{12}$$

Using

$$\frac{d}{dz}\left(\log \frac{J_{\gamma+1}(z)}{J_\gamma(z)}\right) = \frac{J_\gamma(z)J'_{\gamma+1}(z) - J_{\gamma+1}(z)J'_\gamma(z)}{J_\gamma(z)J_{\gamma+1}(z)} \tag{13}$$

and also using the recurrence relationships of the Bessel functions and Eq. (2) we obtain

$$J_\gamma(z)J'_{\gamma+1}(z) - J_{\gamma+1}(z)J'_\gamma(z) = J_\gamma^2(z) + J_{\gamma+1}^2(z) - (2\gamma+1)J_\gamma(z)J_{\gamma+1}(z)/z. \tag{14}$$

Hence,

$$S_\gamma = 2(\gamma+1)\log(z) + \log 2 + i\pi + \log\left(\sum_{n=1}^{\infty} \frac{1}{(z^2 - \zeta_{\gamma,n}^2)}\right). \tag{15}$$

Let

$$S_\gamma = \int \frac{J_\gamma(z)J_{\gamma+1}(z)}{z(J_\gamma^2(z) + J_{\gamma+1}^2(z))}\, dz. \tag{16}$$

It is easy to see that

$$\frac{J_\gamma(z)J_{\gamma+1}(z)}{z(J_\gamma^2(z) + J_{\gamma+1}^2(z))} = \frac{1}{(2\gamma+1)}\left[1 - \frac{d}{dz}\left[\tan^{-1}\left(\frac{J_{\gamma+1}(z)}{J_\gamma(z)}\right)\right]\right]. \tag{17}$$

Therefore,

$$S_\gamma = \frac{1}{(2\gamma+1)}\left[z - \tan^{-1}\left(\sum_{n=1}^{\infty} \frac{-2z}{(z^2 - \zeta_{\gamma,n}^2)}\right)\right]. \tag{18}$$

Similarly, it can be shown that

$$\int \frac{(J_\gamma(z)J_{\gamma+2}(z) + J_{\gamma+1}(z)J_{\gamma-1}(z))}{J_\gamma^2(z) + J_{\gamma+1}^2(z)}\, dz = z - 2\tan^{-1}\left(-2z\sum_{n=1}^{\infty} \frac{1}{(z^2 - \zeta_{\gamma,n}^2)}\right). \tag{19}$$

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