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EXPLANATION OF SPURT FOR A NON-NEWTONIAN FLUID BY A DIFFUSION TERM

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1. Introduction. A surprising feature of the flow of polymers is associated with a sudden increase in the volumetric flow rate when the pressure gradient is gradually increased beyond a critical value. This striking phenomenon, called "spurt", was apparently first observed by Vinogradov et al. [15] in rheological experiments involving the flow through thin capillaries of highly elastic and very viscous non-Newtonian fluids like some synthesized polybutadienes and polyisoprenes. The interested reader is referred to [15, Table 1] for more detailed information about microstructure characteristics of samples. The spurt phenomenon is a kind of a flow instability in pressure-driven shear flows of viscoelastic fluids.

Much effort is being spent to explain spurt and related phenomena mathematically. Several authors have considered mathematical models based on differential constitutive equations due to Johnson, Sagelman, and Oldroyd exhibiting local extrema of the steady shear stress as a function of steady strain rate (see [6-8, 10-13]). These papers show that the spurt phenomenon is dynamic and, hence, cannot be explained in a satisfactory manner by only studying the steady-state equations. Dynamical theory can explain phenomena observed in experiments and in numerical simulations, and it can also predict phenomena like latency, shape memory, and hysteresis which should be observable in future experiments.

In this paper we modify the models of [6] and [13] by adding a diffusion term to the constitutive equation. The resulting system of equations (in dimensionless units) governing planar shear flow has the form

$$\begin{aligned} \alpha v_t &= v_{xx} + \sigma_x + f, \\ \sigma_t &= -\sigma + g(v_x) + \nu^2 \sigma_{xx} \end{aligned} \tag{1.1}$$

where $v(t, x)$ is the velocity of the planar flow, $\sigma(t, x)$ is the polymer contribution to the shear stress, $g : \mathfrak{R} \to \mathfrak{R}$ is a given smooth function, and $f > 0$ is the pressure gradient driving the flow.

Unlike the models investigated in [13] and [6] and the other models in [10-12], system (1.1) contains the spatial diffusion term $\nu^2 \sigma_{xx}$. Spatial diffusion is usually

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neglected in non-Newtonian models because of the spatial homogeneity of the structure. In the model of [4] (also see [3]), Brownian motion prevents polymer molecules (treated as dumb-bells) from being completely independent of each other, giving rise to a diffusion term in constitutive equations. Typical values of ν^2 will be described in Sec. 6. The structure of steady states of system (1.1) is determined by treating $\nu^2 > 0$ as a small parameter and by applying the singular perturbation theory of [9]. This theory enables us to select steady states that appear to be appropriate for capturing the spurt phenomenon.

System (1.1) with $\nu^2 = 0$ exhibits the same behavior in steady shear as the more realistic models studied in [10–12], where the differential constitutive equations also involve normal stresses (in particular, the first normal stress difference), giving rise to a governing system of three quasi-linear parabolic-hyperbolic PDEs in place of the two in system (1.1). The dimensionless parameter α representing the ratio of Reynolds number to Deborah number is very small. The analytical study in [11–13] is based on treating the respective governing equations as singular perturbation problems with α as a singular parameter. Their approach is to determine the complete dynamics when $\alpha = 0$ and then to show that the dynamics of the full system is similar for $\alpha > 0$ sufficiently small. By contrast, our quasi-linear system (1.1) with $\nu^2 > 0$ is parabolic, and the theory of parabolic systems can be exploited to determine the global dynamics for $\alpha > 0$ sufficiently small. In particular, the existence of a global compact attractor and an inertial manifold can be established. It should be noted that the feature of mathematical models studied in [11–13] that makes their qualitative analysis (asymptotic behavior as $t \to \infty$, stability properties, etc.) particularly difficult is that the governing equations possess uncountably many isolated steady states. From this fact one can deduce that these governing systems can admit neither a compact global attractor nor a finite-dimensional inertial manifold.

The paper is organized as follows. In Sec. 2, we use general ideas from [6] to derive a non-Newtonian model of shearing motions incorporating spatial diffusion. Basic properties of the model (existence and long-time behavior of solutions, qualitative properties of steady states) are established in Sec. 3. It is shown that in the case of a generic g, the asymptotic behavior of solutions is very simple—each solution tends to some steady state and the number of steady states is finite. We also prove exponential stability of two particular steady states playing a crucial role in the explanation of spurt. In Secs. 4 and 5, spurt and hysteresis phenomena in our mathematical model are established. The phenomenon of spurt is associated with extinction of a stable steady state when the pressure gradient increases beyond a critical (bifurcation) value. The results of numerical simulations for small values of $\alpha, \nu > 0$ are presented in Sec. 6. We have performed numerical simulations of spurt and hysteresis phenomena for sample PI-3 (see [15]). Numerical results match the data observed experimentally by Vinogradov et al.

2. Non-Newtonian model of shearing motions including diffusion. In this section, we derive a mathematical model for shearing motion of a fluid leading to a system of governing equations including a diffusion term in the constitutive equation.

We consider the planar shear flow of a viscoelastic fluid in an infinite narrow strip: $x \in [-h, h]$ and $y \in (-\infty, \infty)$, with the flow directed along the y-axis. We suppose the fluid to be non-Newtonian, incompressible, and the motion to take place under isothermal conditions. We restrict ourselves to motions that are symmetric with respect to the centerline. Under our assumptions the flow variables will depend only on the transversal variable x. Hence, the velocity vector $\vec{v}$ has the form $\vec{v} = (0, v(t, x))$ with $v(t, x) = v(t, -x)$. It is easy to verify that the mass balance is then automatically satisfied. The equation governing the motion of the fluid is the balance of linear momentum

$$\varrho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} \right) = \nabla \vec{S} \tag{2.1}$$

where ϱ is the constant fluid density and $\vec{S}$ is the total stress which can be decomposed as

$$\vec{S} = p \cdot \vec{\mathrm{Id}} + \varepsilon \cdot \vec{D} + \vec{\Sigma}. \tag{2.2}$$

Here p is the isotropic pressure of the form $p = p_0(t, x) + f \cdot y$ where f is the pressure gradient driving the flow, ε is the Newtonian viscosity, and $\vec{D}$ is the rate of deformation tensor, i.e., $\vec{D} = (\nabla\vec{v} + (\nabla\vec{v})^{\mathrm{T}})/2$. According to [6, Sec. 2] the extra stress

$$\vec{\Sigma} = \begin{pmatrix} \sigma^{xx}, & \sigma^{xy} \\ \sigma^{yx}, & \sigma^{yy} \end{pmatrix}$$

satisfies

$$\begin{aligned} \sigma^{xy} &= \sigma^{yx} = \mathscr{S}_{0\,s=0}^{\;\infty}[\Lambda_t(s)], \\ \sigma^{xx} - \sigma^{yy} &= \mathscr{S}_{1\,s=0}^{\;\infty}[\Lambda_t(s)], \\ \sigma^{xx} + \sigma^{yy} &= 0 \end{aligned} \tag{2.3}$$

where $\mathscr{S}_0, \mathscr{S}_1$ are generally nonlinear operators acting on the relative shearing history

$$\Lambda_t(s) = -\int_{t-s}^{t} v_x(\tau, x)\, d\tau. \tag{2.4}$$

Since we assume the flow to be planar, Eq. (2.1) reduces to

$$\varrho v_t = \varepsilon v_{xx} + \sigma_x + f \tag{2.5}$$

where $\sigma := \sigma^{xy}$.

We specify the operator $\mathscr{S}_0$ in such a way that it takes into account long-range molecular forces. According to [4], the latter provide the constitutive equations by a diffusion term $\nu^2\sigma_{xx}$. The first normal stress difference determined by the operator $\mathscr{S}_1$ plays no role in our model.

Let A denote the selfadjoint closure in $L_2(0, h)$ of the operator defined on $C_B^2(0, h)$ by $Au = -u_{xx}$ for any $u \in C_B^2(0, h) := \{u \in C^2(0, h);\ u(0) = u_x(h) = 0\}$; its domain $D(A)$ is the Sobolev space $W_B^{2,2}(0, h) = \{u \in W^{2,2}(0, h);\ u(0) = u_x(h) = 0\}$. Let $\lambda, \nu > 0$ be fixed. Then the operator $-(\lambda + \nu^2 A)$ generates an analytic semigroup $\exp(-(\lambda + \nu^2 A)t)$, $t \geq 0$; (see [5, Chapter 1]).

Assume that $g : \mathfrak{R} \to \mathfrak{R}$ is an odd Lipschitz continuous function. As usual, we identify g with the Nemitsky operator $g : W^{1,2}(0, h) \to L_2(0, h)$ defined by $g(u)(x) = g(u(x))$ for a.e. $x \in [0, h]$. Due to the assumptions on g the nonlinear operator g is well defined and Lipschitz continuous.

Let $\tilde{f} \in L_2(0, h)$ be defined as

$$\tilde{f} : x \mapsto f \cdot x \quad \text{for any } x \in [0, h]. \tag{2.6}$$

We define

$$\begin{gathered}\mathscr{S}_0(\Lambda_t) = \int_0^\infty \exp(-(\lambda + \nu^2 A)s) \cdot \left[g\left(-\frac{d}{ds}\Lambda_t(s) \right) + \lambda \cdot \tilde{f} \right] ds - \tilde{f} \\ \text{for any } v \in C(\mathfrak{R} : W^{1,2}(0, h)),\ \sup_{t\in\mathfrak{R}} \|v(t)\|_{W^{1,2}} < \infty,\ \text{ and } t \geq 0\end{gathered} \tag{2.7}$$

where $\Lambda_t(s)$ is defined by Eq. (2.4), i.e., $\Lambda_t(s) = -\int_{t-s}^t v_x(\tau, x)\, d\tau$.

Clearly,

$$\mathscr{S}_0(\Lambda_t) = \int_0^\infty \exp(-(\lambda + \nu^2 A)s)[g(v_x(t-s, \cdot)) + \lambda\tilde{f}\,]\, ds - \tilde{f}. \tag{2.8}$$

In case $\nu = 0$, the definition of the functional $\mathscr{S}_0$ coincides with that of [6, formula (5)]. However, since the operator $\lambda + \nu^2 A$, $\nu > 0$, is a diffusion operator generating an analytic semigroup, the operator $\exp(-(\lambda + \nu^2 A)s)$, $s > 0$, smooths out solutions, i.e., $\exp(-(\lambda + \nu^2 A)s)w \in D(A)$ for any $w \in L_2(0, h)$ and $s > 0$ (see [5, Chapter 1]).

Differentiating Eq. (2.8) with respect to t and substituting $u := \sigma + \tilde{f} = \mathscr{S}_0(\Lambda_t) + \tilde{f}$, we obtain the following constitutive equation of rate type:

$$u_t + (\lambda + \nu^2 A)u = g(v_x) + \lambda\tilde{f} \tag{2.9a}$$

with boundary conditions

$$u(t, 0) = u_x(t, h) = 0 \tag{2.9b}$$

or, equivalently,

$$\sigma_t + \lambda\sigma - \nu^2\sigma_{xx} = g(v_x) \tag{2.10a}$$

with boundary conditions

$$\sigma(t, 0) = 0, \qquad \sigma_x(t, h) = -f, \tag{2.10b}$$

respectively.

We note that $\sigma_x(t, h) = -f$ implies $v_{xx}(t, h) = 0$ which is the boundary condition appearing in the theory of multipolar fluids (see, [2, Sec. 3]). The boundary condition $u(t, 0) = 0$ $(\sigma(t, 0) = 0)$ implies that the function $u(t, \cdot)$ $(\sigma(t, \cdot))$ can be extended as an odd function to the interval $[-h, h]$ for all t. It ensures the symmetry of the flow about the centerline.

Summarizing, our model leads to the initial-boundary value problem

$$\begin{gathered}\varrho v_t = \varepsilon v_{xx} + \sigma_x + f; \\ \sigma_t = \nu^2\sigma_{xx} + g(v_x) - \lambda\sigma; \\ v(0, x) = v_0(x) \text{ and } \sigma(0, x) = \sigma_0(x) \quad \text{for a.e. } x \in [0, h]; \\ v_x(t, 0) = v(t, h) = 0,\ \sigma(t, 0) = 0,\ \text{ and } \sigma_x(t, h) = -f \quad \text{for } t \geq 0.\end{gathered} \tag{2.11}$$

To facilitate the discussion, we scale the space variable x by h, times t by λ^{-1}, v by $h\lambda$, σ by $\varepsilon\lambda$, f by $\varepsilon\lambda/h$, and ν^2 by $h^2\lambda$, and replace $g(\xi)$ by $g(\lambda\xi)/\varepsilon\lambda^2$. The resulting system is

$$\begin{gathered}\alpha v_t = v_{xx} + \sigma_x + f, \\ \sigma_t = \nu^2\sigma_{xx} + g(v_x) - \sigma \\ \text{for } (t, x) \in [0, \infty] \times [0, 1]\end{gathered} \tag{2.12}$$

with boundary conditions

$$\begin{gathered}v_x(t, 0) = v(t, 1) = 0, \\ \sigma(t, 0) = 0, \qquad \sigma_x(t, 1) = -f\end{gathered} \tag{2.13}$$

and initial data

$$v(0, x) = v_0(x) \text{ and } \sigma(0, x) = \sigma_0(x) \quad \text{for a.e. } x \in [0, 1]. \tag{2.14}$$

There are two dimensionless parameters:

$$\alpha = \frac{\varrho h^2\lambda}{\varepsilon} \quad \text{and} \quad \nu > 0.$$

According to [15] and [4], the typical values of α and ν are

$$\alpha = O(10^{-9}) \quad \text{and} \quad \nu^2 = O(10^{-4}).$$

Hence, we may treat α and ν as small parameters.

3. Existence of solutions, asymptotic behavior, steady-state solutions and their stability. In this section, we study the problem of existence of solutions, their long-time behavior, and some qualitative properties of steady states of the system (2.12). Using the abstract theory developed in [5] we establish local and global solvability. For g real analytic we furthermore prove that the asymptotic behavior of the solutions is simple—each trajectory approaches some steady state and the number of steady state solutions is finite. To single out the appropriate stationary solutions, we apply the results of the theory of singularly perturbed boundary value problems of [9].

3.1. *Existence of solutions.* In terms of the variables v and u the initial boundary value problem (2.12) takes the form

$$\begin{gathered}\alpha v_t = v_{xx} + u_x, \\ u_t = \nu^2 u_{xx} - u + g(v_x) + fx,\end{gathered} \tag{3.1}$$

$$\begin{gathered}v_x(t, 0) = v(t, 1) = 0 \text{ and } u(t, 0) = u_x(t, 1) = 0 \quad \text{for } t \geq 0, \\ v(0, x) = v_0(x) \text{ and } u(0, x) = u_0(x) \quad \text{for } x \in [0, 1].\end{gathered}$$

To facilitate the discussion, let

$$S = v_x + u = v_x + \sigma + \tilde{f}. \tag{3.2}$$

Obviously,

$$\alpha S_t = S_{xx} + \alpha u_t. \tag{3.3}$$

In terms of S and u, the system (3.1) takes the form

$$\begin{aligned} \alpha S_t &= S_{xx} + \alpha\nu^2 u_{xx} + \alpha(g(S-u) + fx - u), \\ u_t &= \nu^2 u_{xx} - u + g(S-u) + fx \end{aligned} \tag{3.4}$$

with boundary conditions

$$u(t, 0) = u_x(t, 1) = 0, \qquad S(t, 0) = S_x(t, 1) = 0$$

and initial data

$$S(0, x) = S_0(x) = v_{0x}(x) + u_0(x), \text{ and } u(0, x) = u_0(x) \quad \text{for } x \in [0, 1]. \tag{3.5}$$

Throughout this paper we will assume that α and ν are small parameters. The pressure gradient f is assumed to be positive. The function $h(u) := u + g(u)$ is assumed to be C^2 with a single loop as shown in Fig. 1.

More precisely, we make the following hypotheses:

(i) $g: \mathfrak{R} \to \mathfrak{R}$ is an odd C^2 function with bounded first and second derivatives satisfying $g(u)u \geq 0$ for any $u \in \mathfrak{R}$;

(ii) there exist constants $0 < c_1 < c_2$ such that

$$\begin{aligned} h'(u) &= 1 + g'(u) > 0, \; h'' < 0 \quad && \text{on } [0, c_1), \\ h'(u) &= 1 + g'(u) < 0 && \text{on } (c_1, c_2), \\ h'(u) &= 1 + g'(u) > 0, \; h'' > 0 && \text{on } (c_2, \infty). \end{aligned} \tag{W}$$

Under assumptions (W), there exists a $\gamma_0 > 0$ such that

$$\int_{\min h^{-1}(\gamma_0)}^{\max h^{-1}(\gamma_0)} (h(u) - \gamma_0)\, du = 0.$$

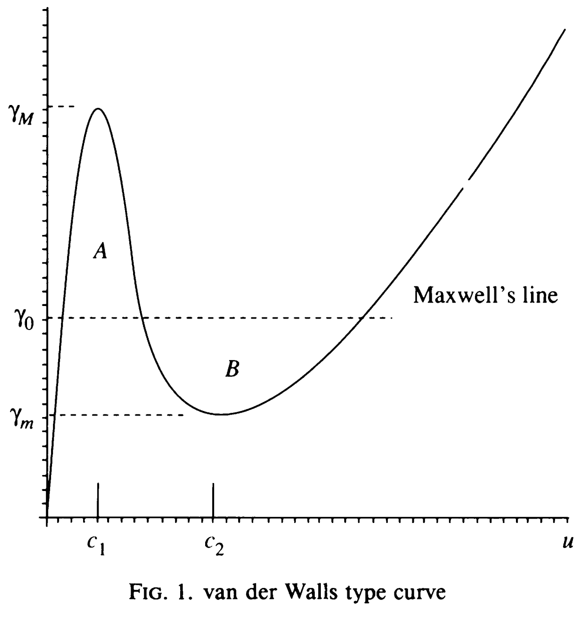

FIG. 1. van der Walls type curve

The last integral condition is commonly known as *Maxwell's equal area rule* (the area A equals B). In Fig. 1 the line $u = \gamma_0$ is called *Maxwell's line*. We also note that the function $h(u) = u + g(u)$ satisfying (W) is sometimes called *van der Walls type curve*.

In what follows, we let X denote the real Hilbert space $L_2(0, 1)$ with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. Recall that the operator A defined in the previous section is sectorial and positive in X with domain $D(A) = \{w \in W^{2,2}(0, 1);\ w(0) = w_x(1) = 0\}$. Hence, fractional powers of A can be defined. Let X^γ, $\gamma \geq 0$, be the Hilbert space consisting of the domain $D(A^\gamma)$ endowed with the graph norm

$$\|w\|_\gamma = \|A^\gamma w\| \text{ for any } w \in X^\gamma = D(A^\gamma). \tag{3.6}$$

The operator A has a compact resolvent $A^{-1}: X \to X$.

Now one can treat the governing equations (3.4), (3.5) as abstract differential equations in the Hilbert space

$$\mathscr{X} = X \times X. \tag{3.7}$$

To do so, we let $\Phi = [\begin{smallmatrix} S \\ u \end{smallmatrix}]$. The system (3.4) then becomes

$$\frac{d}{dt}\Phi + L\Phi = F(\Phi), \qquad \Phi(0) = \Phi_0 = \begin{bmatrix} S_0 \\ u_0 \end{bmatrix} \tag{3.8}$$

where the linear operator L is defined by

$$L\begin{bmatrix} S \\ u \end{bmatrix} := \begin{bmatrix} A(\frac{1}{\alpha}S + \nu^2 u) \\ \nu^2 A u \end{bmatrix} = \begin{pmatrix} \frac{1}{\alpha}A & \nu^2 A \\ 0 & \nu^2 A \end{pmatrix}\begin{bmatrix} S \\ u \end{bmatrix} \tag{3.9}$$

on its domain $D(L) = D(A) \times D(A)$. The nonlinearity F is given by

$$F\left(\begin{bmatrix} S \\ u \end{bmatrix}\right) = \begin{bmatrix} g(S-u) - u + fx \\ g(S-u) - u + fx \end{bmatrix}. \tag{3.10}$$

It is routine to verify that $L: D(L) \subset \mathscr{X} \to \mathscr{X}$ is a sectorial operator generating an analytic semigroup $\exp(-Lt)$, $t \geq 0$. Since A^{-1} is compact, it is easy to show that L has a compact resolvent $L^{-1}: \mathscr{X} \to \mathscr{X}$. The fractional power $L^{1/2}$ is then easily computed as

$$\begin{pmatrix} \frac{1}{\sqrt{\alpha}}A^{1/2} & \frac{\sqrt{\alpha}\nu^2}{1+\nu\sqrt{\alpha}}A^{1/2} \\ 0 & \nu A^{1/2} \end{pmatrix}$$

and $D(L^{1/2}) = D(A^{1/2}) \times D(A^{1/2})$. Hence there is an equivalent norm in $\mathscr{X}^{1/2}$ such that

$$\mathscr{X}^{1/2} \cong X^{1/2} \times X^{1/2}, \tag{3.11}$$

and it can easily be verified that

$$X^{1/2} = \{w \in W^{1,2}(0, 1);\ w(0) = 0\}. \tag{3.12}$$

Since we have assumed that the first and second derivative of g are bounded, the nonlinearity F is a C^1 mapping from $\mathscr{X}^{1/2}$ into $\mathscr{X}$.

Now we can apply the general theory of abstract parabolic equations [5]. According to [5, Theorems 3.3.3, 3.3.4, 3.4.1, and 3.5.2], for any initial condition $\Phi_0 \in \mathscr{X}^{1/2}$ the abstract equation (3.8) has a unique solution $\Phi(t)$ defined on $[0, \infty)$ by the property

$$\Phi \in C_{\mathrm{loc}}([0, \infty), \mathscr{X}^{1/2}) \cap C^1_{\mathrm{loc}}((0, \infty), \mathscr{X}^{1/2}),$$
$$\Phi(t) \in D(L) \text{ for } t > 0 \quad \text{and } \Phi(0) = \Phi_0.$$

Hence, Eq. (3.8) defines a C^1-semidynamical system $(T(t),\ t \geq 0)$ in $\mathscr{X}^{1/2}$ defined by

$$T(t)\Phi_0 = \Phi(t, \Phi_0) \quad \text{for any } t \geq 0$$

where $\Phi(t, \Phi_0)$ is the solution of Eq. (3.8) with $\Phi(0) = \Phi_0 \in \mathscr{X}^{1/2}$.

3.2. *Asymptotic behavior of solutions.* We now turn our attention to the asymptotic behavior of solutions of Eq. (3.8). First, we will study the set of steady states, i.e., stationary solutions of Eq. (3.8) which we denote by $\mathscr{E}$. Clearly,

$$\mathscr{E} = \left\{ \begin{bmatrix} 0 \\ \bar{u} \end{bmatrix} ;\ \bar{u} \in D(A) \text{ is a solution of } \nu^2 A\bar{u} = -\bar{u} + g(-\bar{u}) + fx \right\}. \tag{3.13}$$

In fact, $[\begin{smallmatrix} 0 \\ \bar{u} \end{smallmatrix}] \in \mathscr{E}$ iff

$$\bar{u} \in C^4(0, 1), \qquad \nu^2 \bar{u}_{xx} + \bar{u} + g(\bar{u}) - fx, \qquad \bar{u}(0) = \bar{u}_x(1) = 0. \tag{3.14}$$

Here we have used the assumption that g is an odd C^2 function.

The system (3.8) admits a global Lyapunov function $V : \mathscr{X}^{1/2} \to \mathfrak{R}$ defined by

$$V\left(\begin{bmatrix} S \\ u \end{bmatrix}\right) = \frac{1}{2}\left\{ \frac{1}{\alpha}\|S\|^2_{1/2} + \nu^2\|S - u\|^2_{1/2} + \|S - u\|^2 + J(S - u) \right\}$$

where

$$J(w) = 2\int_0^1 \int_0^{w(x)} (g(s) + fx)\, ds\, dx. \tag{3.15}$$

Indeed, a simple calculation shows that for any solution $[\begin{smallmatrix} S(t) \\ u(t) \end{smallmatrix}]$ the following formula holds:

$$\frac{d}{dt}V\left(\begin{bmatrix} S(t) \\ u(t) \end{bmatrix}\right) + \frac{1}{\alpha}\|S(t)\|^2_{1/2} + \frac{1 + \alpha\nu^2}{\alpha^2}\|S(t)\|^2_1 = 0 \quad \text{for any } t > 0. \tag{3.16}$$

Due to the assumption $g(u)u \geq 0$ for any $u \in \mathfrak{R}$ it follows that the functional V is bounded from below. From Eqs. (3.14), (3.16) it follows that the real-valued function $t \mapsto V([\begin{smallmatrix} S(t) \\ u(t) \end{smallmatrix}])$, $t \geq 0$, is strictly decreasing unless $[\begin{smallmatrix} S(t) \\ u(t) \end{smallmatrix}] = [\begin{smallmatrix} 0 \\ \bar{u} \end{smallmatrix}] \in \mathscr{E}$ is a steady-state solution of Eq. (3.8). Then a standard argument (see, e.g., [16, Theorem 4.1]) enables us to conclude that the omega-limit set

$$\Omega(\Phi_0) := \{\Phi \in \mathscr{X}^{1/2},\ \text{there exists } t_n \to \infty \text{ such that } T(t_n)\Phi_0 \to \Phi\}$$

satisfies

$$\Omega(\Phi_0) \subseteq \mathscr{E}, \tag{3.17}$$

for any $\Phi_0 \in \mathscr{X}^{1/2}$. Since the operator L has a compact resolvent L^{-1}, it follows from [5, Theorems 3.3.6 and 4.3.3] and Eq. (3.17) that

$$\lim_{t\to\infty} \operatorname{dist}(T(t)\Phi_0, \mathscr{E}) = 0, \tag{3.18}$$

where $\operatorname{dist}(\Phi, \mathscr{E}) = \inf(\|\Phi - \Psi\|_{\mathscr{X}^{1/2}}, \ \Phi \in \mathscr{E})$. In the following simple proposition, we obtain bounds on steady states, and we show for g real analytic that the number of possible steady states is finite.

Proposition 3.1. Let $u_0 \geq c_2$ be such that $h(u_0) \geq f$. Then $0 \leq u(x) \leq u_0$ for any solution $u(x)$ of Eq. (3.14). Moreover, there exists a constant $M = M(g, f) > 0$ such that

$$\nu \sup_{x\in[0,1]} |u_x(x)| + \sup_{x\in[0,1]} |u(x)| \leq M.$$

If g is real analytic, then the number of solutions of Eq. (3.14) is finite.

Proof. Let u be an arbitrary solution of Eq. (3.14). Since $h(u) := u + g(u)$ is nondecreasing on $[u_0, \infty)$ and $h(u_0) \geq f$, it follows that $u(x) \geq u_0$ implies $\nu^2 u_{xx}(x) = h(u(x)) - fx \geq h(u_0) - fx \geq f(1-x)$. Thus the function $u(x)$ is strictly convex whenever $u(x) \geq u_0$. Since $u(0) = 0$, if $u(x_0) > u_0$ for some $x_0 \in (0, 1]$, then there exists $x_1 \in (0, 1)$ such that $u(x_1) = u_0$, $u(x) > u_0$, and $u_x(x) > 0$ on $(x_1, 1)$. This means that u cannot satisfy $u_x(1) = 0$. Hence, $u(x) \leq u_0$ for every $x \in [0, 1]$ and $\nu > 0$. The inequality $0 \leq u(x)$ can be obtained in a similar way. The estimates for $u(x)$ and $\nu u_x(x)$ follow from the well-known interpolation inequality

$$\nu \sup_{x\in[0,1]} |u_x(x)| \leq 2\left(\sup_{x\in[0,1]} |u(x)| + \nu^2 \sup_{x\in[0,1]} |u_{xx}(x)|\right)$$

$$\text{for any } u \in C^2([0, 1]) \text{ and } \nu > 0.$$

Now we assume that g is real analytic. We fix a $\nu > 0$ and define the map $\mu \mapsto \phi(\mu)$ as $\phi(\mu) = u_x^\mu(1)$ where $u^\mu(x)$ is the solution of the initial-value problem $\nu^2 u_{xx} = u + g(u) - fx$, $u^\mu(0) = 0$, $u_x^\mu(0) = \mu$. Since g is Lipschitz continuous and analytic, the function $\phi(\mu)$ is well defined and analytic on $\mathfrak{R}$. Furthermore, $\phi(\mu) = 0$ if and only if $u^\mu(x)$ is a solution of the BVP (3.14). Suppose to the contrary, the existence of infinitely many solutions of the BVP (3.14). Then the set $\{\mu \in [-M/\nu, M/\nu];\ \phi(\mu) = 0\}$ must have an accumulation point. Because of analyticity of ϕ, we have $\phi \equiv 0$ on $\mathfrak{R}$. Hence, there is a solution $u^\mu(x)$ of the BVP (3.14) for $\mu > M/\nu$ which is inconsistent with $u_x^\mu(0) = \mu$. $\square$

The omega-limit set $\Omega(\Phi_0)$ is connected [5, Theorem 4.3.3]. Thus, by Eq. (3.17), $\Omega(\Phi_0)$ is a singleton whenever $\mathscr{E}$ is finite. We have thus established the following.

Theorem 3.2. Assume the hypotheses (W). Then, for any initial condition $\Phi_0 \in \mathscr{X}^{1/2}$, the evolution problem (3.8) has the unique solution $\Phi = \Phi(t, \Phi_0)$, $t \geq 0$, its omega-limit set $\Omega(\Phi_0)$ being contained in the set of steady-state solutions $\mathscr{E}$. If, in addition, g is real analytic, then each trajectory tends to a single steady state.

3.3. *Steady-state solutions.* We now examine steady-state solutions of Eq. (3.8). Recall that $[\frac{\overline{S}}{\overline{u}}]$ is a steady state if and only if $\overline{S} \equiv 0$ and $\overline{u} \in C^4(0, 1)$ is a solution of the BVP

$$\begin{aligned} \nu^2 u_{xx} &= u + g(u) - fx, \\ u(0) &= u_x(1) = 0. \end{aligned} \tag{3.19}$$

The steady-state velocity profile $\overline{v}$ is then calculated as $\overline{v}(x) = \int_x^1 \overline{u}(\xi)\, d\xi$. Since ν is assumed to be small, the problem (3.19) can be viewed as a singular perturbation of the reduced problem

$$0 = u + g(u) - fx. \tag{3.20}$$

From now on, we assume

$$f \in [f_{\min}, f_{\max}],$$

where $0 < f_{\min} < \gamma_m$ and $\gamma_M < f_{\max} < \infty$. From Fig. 1 it is clear that the problem (3.20) has a unique C^1 solution $u = \phi_1(x)$, $x \in [0, 1]$, whenever $f \in [f_{\min}, \gamma_m)$. When $f \in [\gamma_m, f_{\max}]$ there exist C^1 functions $\phi_i(x)$ defined on two overlapping intervals I_i contained in $[0, 1]$, where $0 \in I_1$, $1 \in I_2$, $i = 1, 2$, and such that $h(\phi_i(x)) - fx = 0$, $x \in I_i$, and $\phi_2(x) > \phi_1(x)$ on $I_1 \cap I_2$. Hence, there also exist discontinuous solutions of (3.20). Indeed, any function $u = u(x)$ where $u = \phi_1(x)$ on $[0, 1]\backslash I_2$, $u(x) \in \{\phi_1(x), \phi_2(x)\}$ on $I_1 \cap I_2$ and $u = \phi_2(x)$ on $[0, 1]\backslash I_1$ is the solution of (3.20); the number of discontinuities of u is unlimited. Inevitably, each solution of (3.20) is discontinuous whenever $f \in (\gamma_M, f_{\max}]$. In the case $f \in (\gamma_0, f_{\max}]$ and ν small we expect the existence of a solution of (3.19) having an abrupt transition at some interior point $x_0 \in (0, 1)$. When ϕ_1 is defined on the whole interval $[0, 1]$ we also expect that (3.19) has a solution that is close to ϕ_1 on $[0, 1]$ for ν small.

To make the above discussion precise, we employ general results of singularly perturbed equations due to Lin [9]. To this end, let us consider (3.19) as the equivalent 2×2 system

$$\begin{aligned} \nu u_x &= w, \\ \nu w_x &= u + g(u) - fx, \\ u(0) &= w(1) = 0. \end{aligned} \tag{3.21}$$

In case $f \in [f_{\min}, \gamma_M)$ the piecewise continuous function

$$\overline{U}_\nu^1 = \begin{cases} (0, 0), & x \in [0, \nu^{1/2}), \\ (\phi_1(x), 0), & x \in [\nu^{1/2}, 1 - \nu^{1/2}), \\ (\phi_1(1), 0), & x \in [1 - \nu^{1/2}, 1] \end{cases} \tag{3.22}$$

is a formal approximation of the system (3.21) in the sense of [9, Theorem 2.1]. When $f \in (\gamma_0, f_{\max}]$ (γ_0 is determined by Maxwell's equal area rule), there is another formal approximation of system (3.21) given by

$$\overline{U}_\nu^2 = \begin{cases} (0, 0), & x \in [0, \nu^{1/2}); \\ (\phi_1(x), 0), & x \in [\nu^{1/2}, x_0 - \nu^{1/2}]; \\ (z(\frac{x - x_0}{\nu}),\ z'(\frac{x - x_0}{\nu})), & x \in (x_0 - \nu^{1/2}, x_0 + \nu^{1/2}); \\ (\phi_2(x), 0), & x \in [x_0 + \nu^{1/2}, 1 - \nu^{1/2}); \\ (\phi_2(1), 0), & x \in [1 - \nu^{1/2}, 1]. \end{cases} \tag{3.23}$$

Here $x_0 \in (0, 1)$ is determined by $fx_0 = \gamma_0$ and $z = z(\tau)$ is the heteroclinic solution of the second-order autonomous ODE

$$z'' = z + g(z) - \gamma_0 \tag{3.24}$$

such that $\lim_{\tau\to-\infty} z(\tau) = \phi_1(x_0)$, $\lim_{\tau\to\infty} z(\tau) = \phi_2(x_0)$, $z > 0$, and $z' > 0$. The existence of such a solution follows (by phase-plane analysis) from the fact that (due to the hypothesis (W)) $\phi_1(x_0)$ and $\phi_2(x_0)$ lie on the same level curve of an integral for the system (3.21). We note that $\phi_1(x_0) = \min h^{-1}(\gamma_0)$, $\phi_2(x_0) = \max h^{-1}(\gamma_0)$ for any $f \in [\gamma_0, f_{\max}]$, and hence the solution z does not depend on f.

It is now easy to verify that the formal approximations $\overline{U}_\nu^{(1)}$ and $\overline{U}_\nu^{(2)}$ satisfy the hypotheses (H1)–(H3) of [9]. We omit this detail. Then the main result of [9] adapted to the BVP (3.19) reads

THEOREM 3.3 [9, Theorem 2.2]. *Let $\overline{U}_\nu$ be a formal approximation of (3.19) given by (3.22) or (3.23). Then there exists $\nu_0 > 0$ and $\delta_0 > 0$ such that for $0 < \nu \le \nu_0$ there exists a unique true solution $u = u_\nu(x)$ of system (3.19) with $r := \sup_{x\in[0,1]} |U_\nu(x) - \overline{U}(x)| \le \delta_0$, where $U_\nu(x) = (u(x), \nu u_x(x))$. The remainder r is of order $O(\nu^{1/2})$ when $\nu \to 0^+$.*

REMARK 3.4. Theorem 2.2 of [9], however, does not specify the explicit dependence of the remainder r on the coefficients of Eq. (3.19). The decay of the remainder r may depend on the parameter f. Nevertheless, for any fixed $\eta > 0$ small enough, using the implicit function theorem and following the lines of the proof of [9, Theorems 2.2, 4.3, and 4.4], one can show that the remainder $r = r(\nu, f)$ for the formal approximation $\overline{U}_\nu^{(1)}$ ($\overline{U}_\nu^{(2)}$) is $O(\nu^{1/2})$ uniformly with respect to $f \in [f_{\min}, \gamma_M - \eta]$ and $f \in [\gamma_0 + \eta, f_{\max}]$, respectively, when $\nu \to 0^+$.

For $f \in [f_{\min}, \gamma_M)$, Theorem 3.3 asserts the existence of a true solution $u_\nu^{(1)}$ of Eq. (3.19) approximating the given formal approximation $\overline{U}_\nu^1$. We have

$$u_\nu^{(1)}(x) \xrightarrow{\text{unif}} \phi_1(x) \text{ and } v_\nu^{(1)}(x) \xrightarrow{\text{unif}} \int_x^1 \phi_1(\xi)\, d\xi \quad \text{for any } x \in [0, 1] \text{ as } \nu \to 0^+. \tag{3.25}$$

Again, by Theorem 3.3, for any $f \in (\gamma_0, f_{\max}]$, there exists a solution $u_\nu^{(2)}$ of Eq. (3.19) such that

$$\lim_{\nu\to 0^+} u_\nu^{(2)}(x) = \phi_1(x) \quad \text{for any } x \in [0, x_0),$$
$$\lim_{\nu\to 0^+} u_\nu^{(2)}(x) = \phi_2(x) \quad \text{for any } x \in (x_0, 1]. \tag{3.26}$$

Hence, for small $\nu > 0$ the solution $u_\nu^{(2)}$ has a graph as in Fig. 2 (see p. 412).

By the Lebesgue dominated convergence theorem we have the uniform convergence

$$v_\nu^{(2)} \xrightarrow{\text{unif}} v_0^{(2)} \equiv \begin{cases} \int_x^1 \phi_2(\xi)\, d\xi, & x \in [x_0, 1]; \\ \int_x^{x_0} \phi_1(\xi)\, d\xi + \int_{x_0}^1 \phi_2(\xi)\, d\xi, & x \in [0, x_0] \end{cases}$$

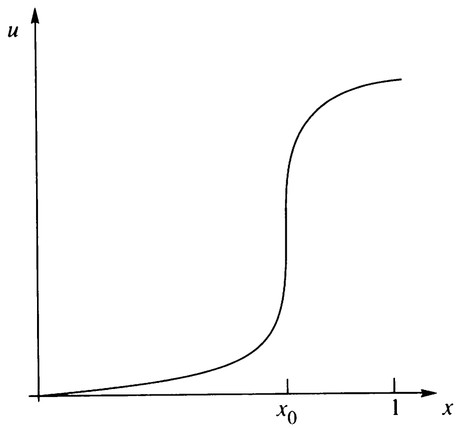

FIG. 2

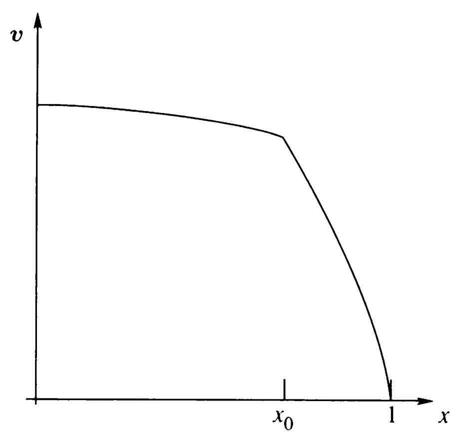

FIG. 3

when $\nu \to 0^+$. Hence, the family $(v_\nu^{(2)})_{\nu>0}$ converges uniformly to the velocity profile $v_0^{(2)}$ with a kink located at x_0 as shown in Fig. 3.

It is now clear that given a pressure gradient $f \in (\gamma_0, \gamma_M)$, for any ν sufficiently small there exist at least two solutions $u_\nu^{(1)}$, $u_\nu^{(2)}$ of Eq. (3.19) satisfying Eqs. (3.25) and (3.26), respectively.

Integrating the velocity $\overline{v}$ with respect to x yields the steady-state flow rate per cross section

$$Q = 2\int_0^1 \overline{v}(x)\, dx. \tag{3.27}$$

Denote by Q_ν^i the volumetric flow rate corresponding to the velocity $v_\nu^{(i)}$ given by Eqs. (3.25) and (3.26), respectively. Clearly, for any $\eta > 0$ there is $d = d(g, \eta) > 0$ such that

$$Q_\nu^{(2)} - Q_\nu^{(1)} \geq d \quad \text{for any } f \in [\gamma_0 + \eta, \gamma_M) \text{ and } \nu > 0 \text{ sufficiently small.} \tag{3.28}$$

We conclude this section by discussing the stability of steady states. We first show that linearized stability of a solution $\overline{u}$ of system (3.19) extends to that of the steady-state solution $[\begin{smallmatrix}0\\ \overline{u}\end{smallmatrix}]$ of Eq. (3.8).

LEMMA 3.5. Let $0 < \alpha < 1/\sup_{u\in\mathfrak{R}} |g'(u)|$. A steady-state solution $[\begin{smallmatrix}0\\ \overline{u}\end{smallmatrix}]$ of Eq. (3.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space $\mathscr{X}^{1/2} = X^{1/2} \times X^{1/2}$, provided the principal eigenvalue μ_0 of the linearized Sturm-Liouville problem $B_1[u] = \nu^2 u_{xx} - u - g'(-\overline{u}(x))u = \mu u$, $u(0) = u_x(1) = 0$ is negative.

Using Lemma 3.5 we are able to prove the theorem below establishing stability of the solutions $[\begin{smallmatrix}0\\ u_\nu^i\end{smallmatrix}]$, $i = 1, 2$, as well as their uniqueness for certain parameter values. The details of the proofs of Lemma 3.5 and Theorem 3.6 are given in the appendix.

THEOREM 3.6. Assume that $0 < \alpha < 1/\sup_{u\in\mathfrak{R}} |g'(u)|$ and g satisfies the hypotheses (W).

(a) If $f \in [f_{\min}, \gamma_M)$ and $\nu > 0$ is sufficiently small, then the principal eigenvalue μ_0 of the linearized Sturm-Liouville problem $B_1[u] = \mu u$ at $u_\nu^{(2)}$ is negative. Consequently, the steady-state solution $[\begin{smallmatrix}0\\ u_\nu^{(1)}\end{smallmatrix}]$ of Eq. (3.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space $\mathscr{X}^{1/2} = X^{1/2} \times X^{1/2}$.

(b) If $f \in (\gamma_0, f_{\max}]$ and $\nu > 0$ is sufficiently small, then the principal eigenvalue μ_0 of the linearized Sturm-Liouville problem $B_1[u] = \mu u$ at $u_\nu^{(1)}$ is negative. Consequently, the steady-state solution $[\begin{smallmatrix}0\\ u_\nu^{(2)}\end{smallmatrix}]$ of Eq. (3.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space $\mathscr{X}^{1/2} = X^{1/2} \times X^{1/2}$.

(c) There exists a unique steady-state solution of Eq. (3.8) whenever $f \in [f_{\min}, \gamma_m)$ or $f \in (\gamma_M, f_{\max}]$ and $\nu > 0$ is sufficiently small.

4. Spurt. Having developed the mathematical background we are in position to explain the occurrence of spurt for a fluid governed by the system of equations (3.8).

Suppose that we are loading the pressure gradient quasi-statically from $f_{\min}$ to $f_{\max}$ allowing the system to settle down to its equilibrium state at each step.

Since $v_\nu^{(1)} = v_\nu^{(1)}(f)$ depends continuously on f, the volumetric flow rate $Q_\nu^{(1)} = Q_\nu^{(1)}(f)$ of the steady-state velocity $v_\nu^{(1)} = v_\nu^{(1)}(f)$ for $f < \gamma_M$ forms a continuous curve. At each step of the "loading-stabilization" procedure, the volumetric flow rate corresponding to the velocity $v(T)$ is close to $Q_\nu^{(1)} = Q_\nu^{(1)}(f)$ when T is large enough.

The situation changes dramatically when the pressure gradient f passes γ_M. For $f > \gamma_M$ the solution has no other possibility than to settle down to the unique steady-state solution

$$\begin{bmatrix} 0 \\ u_\nu^{(2)}(\cdot, f) \end{bmatrix}$$

of system (3.8) which is globally asymptotically stable by Theorem 3.6. Hence, by Eq. (3.28), this small change of the pressure gradient causes a jump of size $d > 0$ in the volumetric flow rate as shown in Fig. 4. This jump is equal to the area between the two equilibrium solutions $v_\nu^{(1)}$ and $v_\nu^{(2)}$ (see Fig. 4).

For f varying in the interval $(\gamma_M, f_{\max}]$, the "loading-stabilization" can be repeated. The corresponding volumetric flow rates are close to the continuous curve $f \mapsto Q_\nu^{(2)}(f)$ of the steady-state volumetric flow rates in Fig. 5.

Let us note that earlier models that did not include the diffusion terms in their constitutive relations also captured the spurt phenomenon [10–12]. For $f > \gamma_M$ the principal difference between our explanation of spurt and that of papers mentioned is: the change in volumetric flow rate as f passes through the critical value γ_M on loading is much more drastic in our model than the earlier ones; here the "kink" develops at the point $0 < \gamma_0/\gamma_M < 1$ very suddenly and then moves slowly with a definite speed toward the centerline. In [10, 11], the kink develops at the wall; for $f > \gamma_M$, the layer position is $x^* = \gamma_M/f$. The phenomenon of latency that occurs on loading described in [10, 11] is not discussed here.

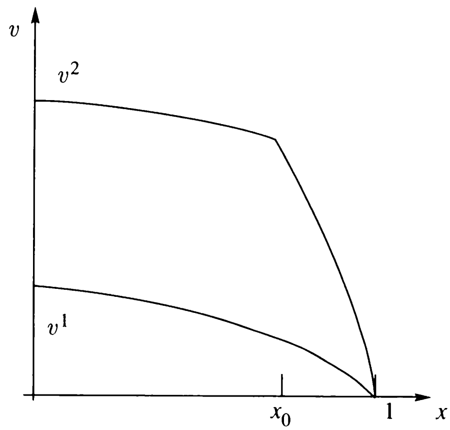

FIG. 4

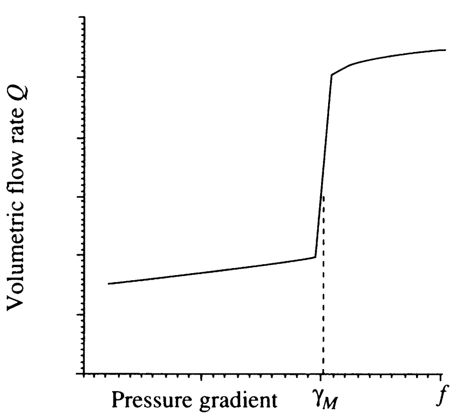

FIG. 5. Spurt

5. Hysteresis. We now consider the loading-unloading cyclic process. The behavior of the volumetric flow rate during the loading period has been described in the previous section. Recall that the volumetric flow rate increased rapidly when the pressure gradient passed the value γ_M. Now let us unload the pressure gradient starting from $f = f_{\max}$. By convention, as long as f stays larger than γ_0, the solution still settles down on

$$\begin{bmatrix} v_\nu^{(2)}(\cdot, f) \\ u_\nu^{(2)}(\cdot, f) \end{bmatrix}.$$

On the other hand, for any $f < \gamma_m$ there exists the unique solution

$$\begin{bmatrix} v_\nu^{(1)}(\cdot, f) \\ u_\nu^{(1)}(\cdot, f) \end{bmatrix}.$$

Therefore, the solution

$$\begin{bmatrix} v_\nu^{(2)}(\cdot, f) \\ u_\nu^{(2)}(\cdot, f) \end{bmatrix}$$

ceases to exist at some critical value near γ_0. Figure 6 shows two branches of the bifurcation diagram corresponding to the stable steady states

$$\begin{bmatrix} v_\nu^{(i)}(\cdot, f) \\ u_\nu^{(i)}(\cdot, f) \end{bmatrix}, \qquad i = 1, 2.$$

By Eq. (3.28), $Q_\nu^{(2)}(f) - Q_\nu^{(1)}(f) \geq d(\eta) > 0$ for any $f \subset [\gamma_0 + \eta, \gamma_M)$ where $\eta > 0$ is fixed. Hence, there is a hysteresis loop as shown in Fig. 7 (see p. 416).

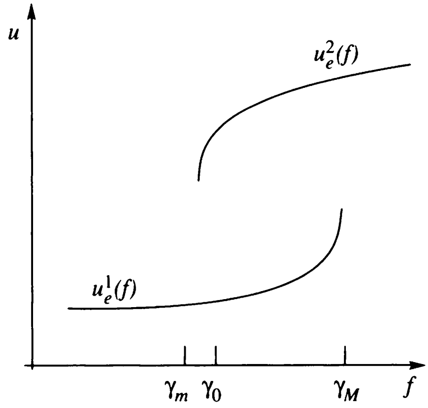

FIG. 6

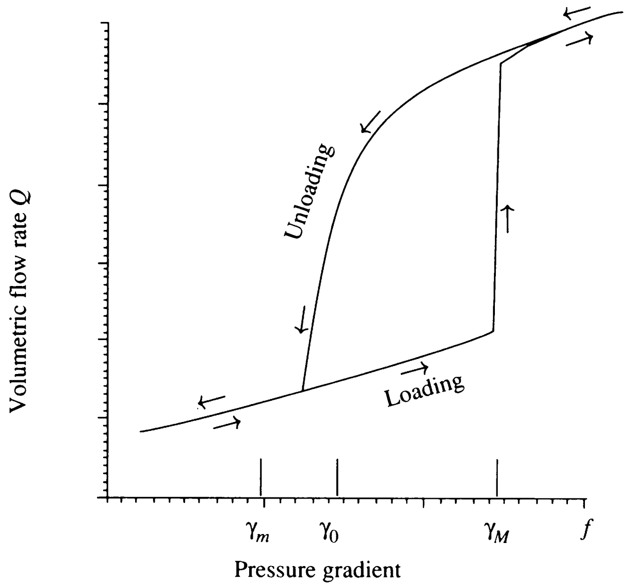

FIG. 7. Hysteresis

6. Numerical simulations. In this section we present some numerical results exhibiting spurt and hysteresis. Recall that our model leads to the system of governing equations

$$\begin{aligned} \varrho v_t &= \varepsilon v_{xx} + \sigma_x + f; \\ \sigma_t &= \nu^2 \sigma_{xx} + g(v_x) - \lambda\sigma \\ &\qquad \text{for } (t, x) \in [0, \infty] \times [0, r_{\text{cap}}] \end{aligned} \tag{6.1}$$

with boundary conditions

$$v_x(t, 0) = v(t, r_{\text{cap}}) = 0, \quad \sigma(t, 0) = 0, \qquad \sigma_x(t, r_{\text{cap}}) = -f$$

and initial data

$$v(0, x) = v_0(x) \text{ and } \sigma(0, x) = \sigma_0(x) \quad \text{for a.e. } x \in [0, r_{\text{cap}}]. \tag{6.2}$$

We will consider an analytic function g of a particular form

$$g(u) = \mu \frac{u}{1 + (1 - a^2)u^2/\lambda^2} \tag{6.3}$$

where $\mu > 0$ is the elastic modulus, a is the dimensionless slip parameter, and λ is the relaxation time of the polymer. The particular choice of the function g is taken from [11, Sec. 3].

First, we determine the magnitude of the coefficient $\nu > 0$ in Eqs. (6.1). Following [4]

$$\nu^2 \approx \frac{k \cdot \theta}{2\xi} \tag{6.4}$$

where θ is the absolute temperature, k is the Boltzmann constant, and ξ is the hydrodynamic resistance of one dumb-bell bead (assumed to be constant). If we take typical values of $\theta \approx 10^2$K, $\xi \approx 10^{-9}$kg s^{-1} and recall that $k \approx 10^{-23}$J K^{-1}, we obtain $\nu^2 \approx 10^{-12}\mathrm{m}^2\ \mathrm{s}^{-1}$. In our numerical simulations we have chosen the fixed value

$$\nu^2 = 4 \times 10^{-12}\mathrm{m}^2\ \mathrm{s}^{-1}. \tag{6.5}$$

We next turn to the Vinogradov et al. rheological data. In all experiments, the radius of the capillary was

$$r_{\mathrm{cap}} = 0.48 \times 10^{-3}\mathrm{m}.$$

The elastic modulus μ and the density ϱ have been taken constant for all samples and equal to

$$\mu = 6 \times 10^4\mathrm{Pa}, \qquad \varrho = 10^3\mathrm{kg\ m}^{-3}, \tag{6.6}$$

respectively.

Numerical experiments were performed for the polyisoprene PI-3 which was the first sample for which spurt was observed [15, Fig. 3b]. According to [15] and [8, p. 323] we have

$$\lambda = 0.1\ \mathrm{s}^{-1}, \qquad \varepsilon = 0.01484\frac{\mu}{\lambda} = 8.9 \times 10^3\mathrm{Pa\ s}^{-1} \qquad a = 0.98. \tag{6.7}$$

We see that the constants $\alpha = \varrho r_{\mathrm{cap}}^2\lambda/\varepsilon = 2.58 \times 10^{-9}$ and $\nu^2/r_{\mathrm{cap}}^2\lambda = 10^{-4}$ introduced in Sec. 2 can be treated as small parameters. It is easy to verify that the real analytic function

$$h(u) = \lambda u + \frac{\mu}{\varepsilon}\cdot\frac{u}{1 + (1 - a^2)u^2/(\varepsilon^2\lambda^2)}$$

is of van der Walls type (see the hypothesis (W)).

As our first numerical experiment, we simulated spurt. In S. I. units, we choose

$$f_{\mathrm{min}} = 9.3 \times 10^7\mathrm{kg\ m}^{-2}\ \mathrm{s}^{-2}, \qquad f_{\mathrm{max}} = 51.2 \times 10^7\mathrm{kg\ m}^{-2}\ \mathrm{s}^{-2},$$
$$\Delta f = 1.8 \times 10^7\mathrm{kg\ m}^{-2}\ \mathrm{s}^{-2}.$$

The startup initial condition (for $f = f_{\mathrm{min}}$) was chosen to be $(v_0, u_0) = (0, 0)$. At each loading step, the solutions were followed for a sufficiently long time $T_{\mathrm{max}} = 150$ sec to allow them to settle down. Since $\alpha > 0$ was very small, we could use the Crank-Nicholson implicit time-space discretization scheme. The spatial mesh contained a total of 40 nodes. The time step was chosen as $\Delta t = 0.005$ sec.

Figure 8 (see p. 418) shows the results obtained (Fig. 8(a)) and compares them with Vinogradov et al.'s experimental data (Fig. 8(b), the flow curve for PI-3 is labeled by 3). Following [15] c-g-s units are employed and axes are in the logarithmic scale. The nominal shear stress τ is defined by $\tau = r_{\mathrm{cap}}f$ (see [8, Eq. (48)]). Since we have considered a planar flow instead of a capillary flow the corresponding definition of a volumetric flow rate is

$$Q = \frac{3}{r_{\mathrm{cap}}^2}\int_0^{r_{\mathrm{cap}}} v(x)\,dx$$

(see [8, Eq. (47)]).

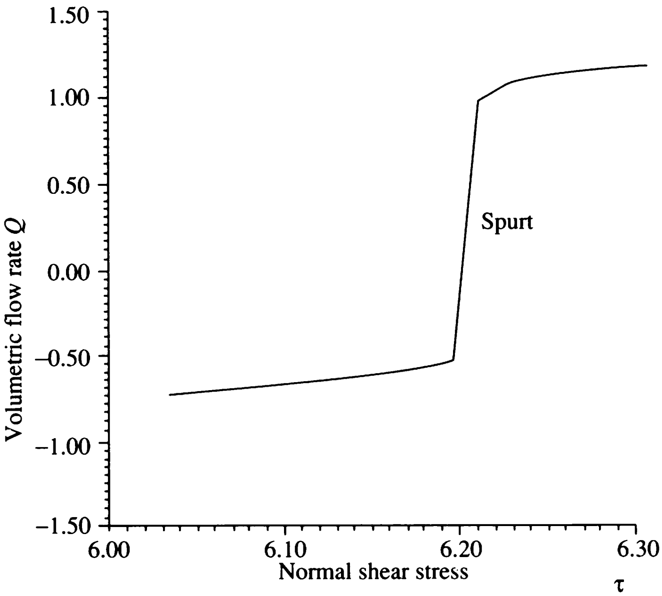

FIG. 8(a). The spurt phenomenon for the sample PI-3.

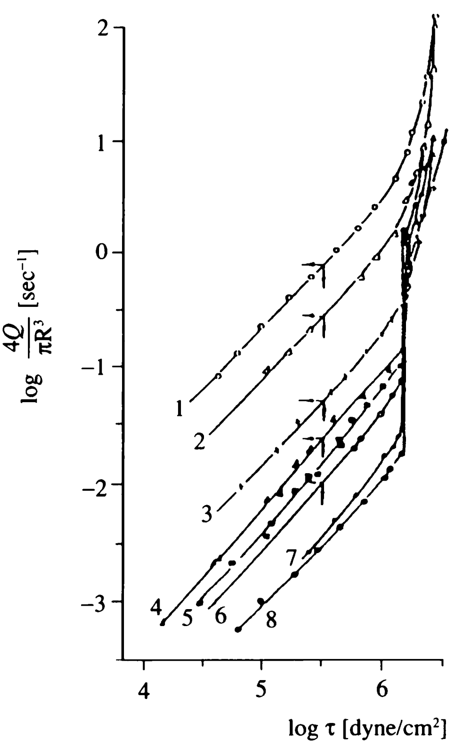

FIG. 8(b).

Finally, we have performed numerical simulations of a loading-unloading cycle. The hysteresis loop under the cyclic load is displayed in Fig. 9.

Figure 10 shows the steady, kinked velocity profile for the spurt value of the nominal shear stress $\tau = 1.61 \times 10^6 \text{dyne cm}^{-2}$ ($\log \tau = 6.21$).

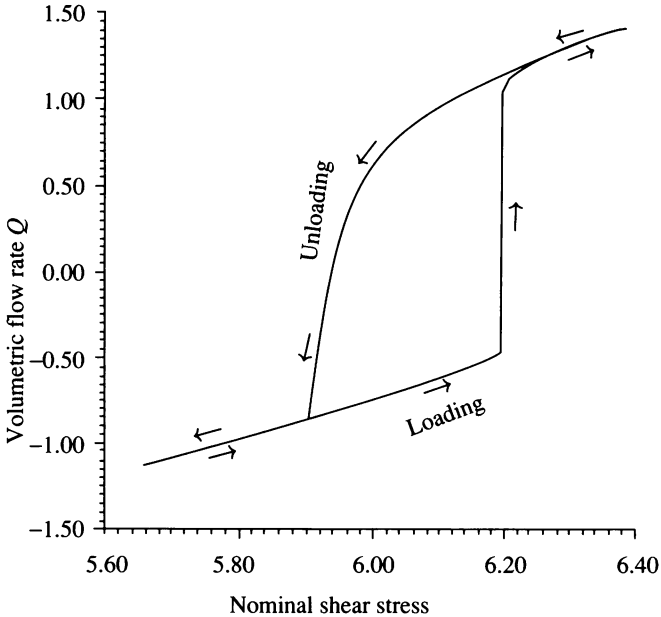

FIG. 9. The hysteresis loop under cyclic load

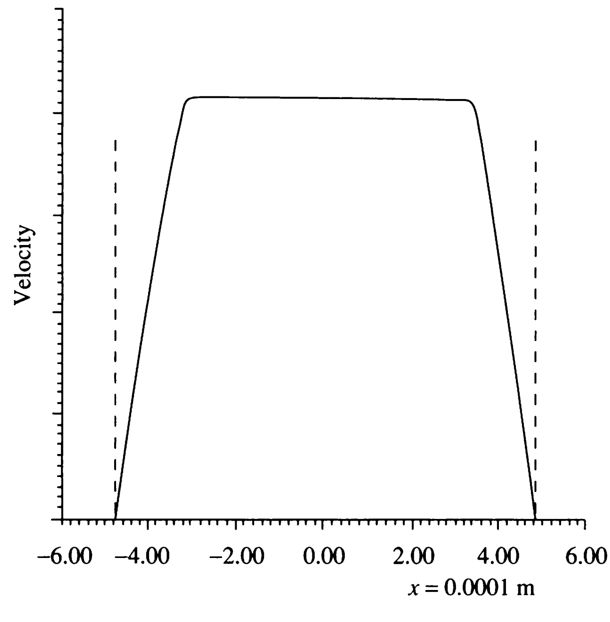

FIG. 10. The velocity profile at the critical value of pressure

7. Discussion. We have proposed a modification of the mathematical model of shearing motions leading to a system of governing equations including a diffusion term $\nu^2\sigma_{xx}$ in the constitutive equation. In addition, we have described the asymptotic behavior of solutions which is simple in typical situations—each solution tends to some steady state and the number of steady states is finite.

The diffusion term makes the system of governing equations parabolic. As a consequence of the resulting parabolic smoothing effect the system will admit a finite-dimensional inertial manifold as well as a compact global attractor. In a subsequent paper we will study singular limits when $\alpha = \varrho r_{\text{cap}}^2 \lambda/\varepsilon$ tends to zero.

Acknowledgments. The authors are thankful to J. A. Nohel and A. Tzavaras for introducing them to the subject and helpful discussions.

Appendix.

Proof of Lemma 3.5. Let $[{0 \atop \bar{u}}]$ be an arbitrary steady state solution of Eq. (3.8). The linearization of Eq. (3.8) at $[{0 \atop \bar{u}}]$ has the form

$$\frac{d}{dt}\begin{bmatrix} S \\ u \end{bmatrix} = B\begin{bmatrix} S \\ u \end{bmatrix}$$

where the linear operator B is given by

$$B\begin{bmatrix} S \\ u \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha}S_{xx} + \nu^2 u_{xx} - u + g'(-\bar{u}(x))(S-u) \\ \nu^2 u_{xx} - u + g'(-\bar{u}(x))(S-u) \end{bmatrix}, \tag{A.1}$$

its domain being $D(B) = \{[{S \atop u}],\ S, u \in W^{2,2}(0,1);\ S(0) = S_x(1) = u(0) = u_x(1) = 0\} \subset L_2(0,1) \times L_2(0,1)$. Denote by B_1 the Sturm-Liouville operator

$$B_1[u] = \nu^2 u_{xx} - u - g'(-\bar{u}(x))u \tag{A.2}$$

on its domain $D(B_1) = \{w \in W^{2,2}(0,1);\ w(0) = w_x(1) = 0\} \subset L_2(0,1)$.

Assume that the principal eigenvalue μ_0 of the linear problem $B_1[u] = \mu u$, $u \in D(B_1)$ is negative. Since B_1 is a selfadjoint Sturm-Liouville operator, we have

$$\frac{(B_1[u], u)}{\|u\|^2} \le \mu_0 < 0 \tag{A.3}$$

for any $u \in D(B_1)$, $u \neq 0$. Moreover, B_1 is invertible and $B_1^{-1}: L_2 \to L_2$ is compact. Hence, the operator B is also invertible and

$$B^{-1}\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \alpha A^{-1}(\psi - \phi) \\ B_1^{-1}(\psi - \alpha g'(-\bar{u}(\cdot))A^{-1}(\psi - \phi)) \end{bmatrix}$$

where the linear operator A was defined in Sec. 2. Since, by Eq. (3.6), $A^{-1}: L_2 \to L_2$ is compact, $B^{-1}: \mathscr{X} \to \mathscr{X}$ is compact as well. Therefore, the spectrum $\sigma(B)$ consists of eigenvalues.

We will show that $\operatorname{Re}\lambda < 0$ for any $\lambda \in \sigma(B) = \sigma_p(B)$. Suppose to the contrary that there exists an eigenvalue $\lambda \in \sigma(B)$ such that $\operatorname{Re}\lambda \geq 0$. Let $[{S \atop u}]$ denote the eigenvector of the linear problem

$$B\begin{bmatrix} S \\ u \end{bmatrix} = \lambda \begin{bmatrix} S \\ u \end{bmatrix}. \tag{A.4}$$

Subtracting the equations for S and u we obtain $\frac{1}{\alpha}S_{xx} = \lambda(S-u)$. Thus,

$$S_x(x) = -\alpha\lambda \int_x^1 (S-u)(\xi)\,d\xi. \tag{A.5}$$

Taking the inner product of (A.5) with $-\int_x^1 (S-u)(\xi)\,d\xi$ we obtain

$$-\|S-u\|^2 - (u\,,\,S-u) = \alpha\lambda \left\| \int_x^1 (S-u)(\xi)\,d\xi \right\|^2 .$$

Since $\operatorname{Re}\lambda \geq 0$, we have $\|S-u\|^2 \leq -\operatorname{Re}(u\,,\,S-u) \leq \|u\|\,\|S-u\|$ and hence,

$$\|S-u\| \leq \|u\|. \tag{A.6}$$

From (A.5) we have $S(x) = -\alpha\lambda \int_0^x \int_r^1 (S-u)(\xi)\,d\xi\,dr$. Thus $S = \alpha\lambda\, J(S-u)$ where $J\colon L_2 \to L_2$ is a linear bounded operator with $\|J\| \leq 1$ Therefore, u satisfies the equation

$$B_1[u] + \alpha\lambda g'(-\overline{u}(\cdot))J(S-u) = \lambda u. \tag{A.7}$$

Take the inner product of (A.7) with u to obtain

$$(B_1[u]\,,\,u) = \lambda(\|u\|^2 - \alpha(g'(-\overline{u}(\cdot))J(S-u)\,,\,u)).$$

Since B_1 is selfadjoint, we have $\operatorname{Im}(\lambda - \alpha\lambda(g'(-\overline{u}(\cdot))J(S-u)\,,\,u)/\|u\|^2) = 0$ and

$$\mu_0 \geq \frac{(B_1[u]\,,\,u)}{\|u\|^2} = \lambda\left(1 - \alpha\frac{(g'(-\overline{u}(\cdot))J(S-u)\,,\,u)}{\|u\|^2}\right).$$

According to (A.6) we have

$$\alpha\left|\frac{(g'(-\overline{u}(\cdot))J(S-u)\,,\,u)}{\|u\|^2}\right| \leq \alpha \sup_{s\in\Re} |g'(s)| \frac{\|J(S-u)\|\,\|u\|}{\|u\|^2} \leq \alpha \sup_{s\in\Re} |g'(s)| < 1$$

because $\|J\| \leq 1$. Therefore,

$$\mu_0 \geq \lambda\left(1 - \alpha\frac{(g'(-\overline{u}(\cdot))J(S-u)\,,\,u)}{\|u\|^2}\right) \geq 0,$$

a contradiction. Hence, $\operatorname{Re}\lambda < 0$ for any $\lambda \in \sigma(B)$. By [5, Theorem 5.1.1], the steady-state solution $[{0 \atop \overline{u}}]$ of Eq. (3.8) is exponentially asymptotically stable with respect to small perturbations of initial data in the phase space $\mathscr{X}^{1/2} = X^{1/2} \times X^{1/2}$. □

Proof of Theorem 3.6. (a) For any $u \in D(B_1)$, $u \neq 0$, we have

$$\begin{aligned}\frac{(B_1[u]\,,\,u)}{\|u\|^2} &= \frac{1}{\|u\|^2}\left(-\nu^2\int_0^1 u_x^2(x)\,dx - \int_0^1 h'(u_\nu^{(1)}(x))u^2(x)\,dx\right)\\ &\leq -\frac{1}{\|u\|^2}\int_0^1 h'(u_\nu^{(1)}(x))u^2(x)\,dx.\end{aligned} \tag{A.8}$$

We have $h'(\phi_1(x)) > 0$ for $x \in [0, 1]$. Therefore, $h'(u_\nu^{(1)}(x)) > 0$ for any $x \in [0, 1]$ and ν small. Hence, the principal eigenvalue μ_0 of B_1 satisfies

$$\mu_0 = \sup_{u \in D(B_1),\ u \neq 0} \frac{(B_1[u], u)}{\|u\|^2} < 0. \tag{A.9}$$

(b) Let us now consider the solution $u_\nu^{(2)}$ of Eq. (3.19) having an abrupt transition at the point $x_0 = \gamma_0/f \in (0, 1)$.

First we prove that $u_\nu^{(2)}$ is increasing on $[0, 1)$. The curve $h(u) - fx = 0$ splits the first quadrant into two parts (Fig. A.1).

The function $u_\nu^{(2)}$ is convex or concave at x depending on whether the point $(x, u_\nu^{(2)}(x))$ belongs to the left-hand or to the right-hand component labeled by $+, -$, respectively. According to Theorem 3.3 we have

$$\sup\{|u_\nu^{(2)}(x) - \phi_1(x)|,\ x \in [0, x_0 - \nu^{1/2}]\} = O(\nu^{1/2}),$$

$$\sup\{|u_\nu^{(2)}(x) - \phi_2(x)|,\ x \in [x_0 + \nu^{1/2}, 1]\} = O(\nu^{1/2})$$

as $\nu \to 0^+$. Since $u_\nu^{(2)}$ is a solution of Eq. (3.19) and $0 \le u_\nu^{(2)}$ (by Proposition 3.1), we have $\frac{d}{dx} u_\nu^{(2)}(0) > 0$. Indeed, $\frac{d}{dx} u_\nu^{(2)}(0) \le 0$ would imply

$$\frac{d^3}{dx^3} u_\nu^{(2)}(0) = \frac{1}{\nu^2}\left(h'(u_\nu^{(2)}(0)) \frac{d}{dx} u_\nu^{(2)}(0) - f\right) < 0.$$

Since $u_\nu^{(2)}(0) = \frac{d^2}{dx^2} u_\nu^{(2)}(0) = 0$, we have $u_\nu^{(2)}(x) < 0$ for some $x > 0$, a contradiction. By an obvious indirect argument, one can show that $\frac{d}{dx} u_\nu^{(2)}(x)$ cannot

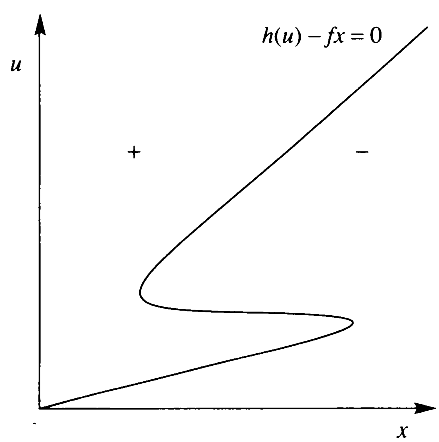

FIG. A.1.

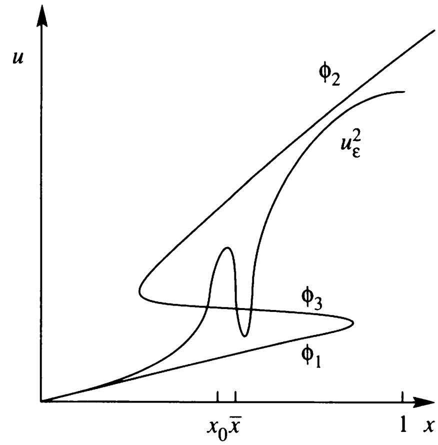

FIG. A.2.

become negative in $[0, x_0 - \nu^{1/2}] \cup [x_0 + \nu^{1/2}, 1]$. To prove that $\frac{d}{dx} u_\nu^{(2)}$ is positive in $(x_0 - \nu^{1/2}, x_0 + \nu^{1/2})$ suppose the contrary. Since $u_\nu^{(2)}$ is convex in $+$ and concave in $-$, this is possible only if there exists an $\overline{x} \in (x_0 - \nu^{1/2}, x_0 + \nu^{1/2})$ such that $\frac{d}{dx} u_\nu^{(2)}(\overline{x}) < 0$ and $u_\nu^{(2)}(\overline{x}) = \phi_3(\overline{x})$, ϕ_3 being the middle branch solution of $h(u) - fx = 0$ is shown in Fig. A.2.

Let us introduce the "fast-time" variable $\tau = (x - x_0)/\nu$ for $x \in (x_0 - \nu^{1/2}, x_0 + \nu^{1/2})$ and put $u(\tau) = u_\nu^{(2)}(x_0 + \nu\tau)$. Then $\frac{d}{d\tau} u(\tau) = \nu \frac{d}{dx} u_\nu^{(2)}(x_0 + \nu\tau)$. According to Theorem 3.3 we have

$$\sup_{\tau \in (-\nu^{-1/2}, \nu^{-1/2})} \left| \frac{d}{d\tau}(u(\tau) - z(\tau)) \right| = O(\nu^{1/2}) \quad \text{as } \nu \to 0^+,$$

z being the heteroclinic solution of the problem (3.24). Since $\overline{x} - x_0 = O(\nu^{1/2})$, we have $|\phi_3(\overline{x}) - \phi_3(x_0)| = O(\nu^{1/2})$ as $\nu \to 0^+$. Therefore, $\frac{d}{dx} u_\nu^{(2)}(\overline{x}) = \nu \frac{d}{d\tau} u((\overline{x} - x_0)/\nu)$ must have the same sign as $\frac{d}{d\tau} z((\overline{x} - x_0)/\nu)$ for any ν small. Hence $\frac{d}{dx} u_\nu^{(2)}(\overline{x}) > 0$, a contradiction.

Knowing that for any $f \in (\gamma_0, f_{\max}]$, $u_\nu^{(2)}$ is increasing in $[0, 1)$ for ν small we return to the linearized eigenvalue problem $B_1[u] = \mu u$ where $B_1[u] = \nu^2 u_{xx} - h'(u_\nu^{(2)}(x))u$, $u(0) = u_x(1) = 0$. First we prove the following useful lemma.

LEMMA A. Assume $f \in [f_{\min}, f_{\max}]$. Let $\overline{u}$ be any nondecreasing solution of (3.19) such that $|h(\overline{u}(1)) - f| < (1 - a)f$ and $h'(\overline{u}(x)) \geq 0$ on $[a, 1]$ for some $a \in (0, 1)$. Then the principal eigenvalue μ_0 of the linear operator $B_1[w] = \nu^2 w_{xx} - h'(\overline{u}(x))w$, $w \in D(B_1)$, is negative.

Proof. Denote $\phi(x) = \frac{d}{dx} \overline{u}(x)$. Then ϕ satisfies

$$\nu^2 \phi_{xx} - h'(\overline{u}(x))\phi = -f, \qquad \phi_x(0) = \phi(1) = 0, \tag{A.10}$$

and $\phi > 0$ on $[0, 1)$. Let w be a solution of

$$B_1[w] = \nu^2 w_{xx} - h'(\overline{u}(x))w = \mu_0 w, \qquad w(0) = w_x(1) = 0 \tag{A.11}$$

corresponding to the principal eigenvalue μ_0 of B_1. Since (A.11) is a Sturm-Liouville problem, there exists w satisfying (A.11) such that $w > 0$ on $(0, 1)$ and $\int_0^1 w(x)\,dx = 1$. If we multiply (A.11) by ϕ and integrate over $[0, 1]$, we obtain

$$\begin{aligned} \mu_0 \int_0^1 w(x)\phi(x)\,dx &= \nu^2(w_x\phi - w\phi_x)|_0^1 - f\int_0^1 w(x)\,dx \quad [\text{because } w_x(0)\phi(0) \geq 0] \\ &\leq -w(1)(h(\overline{u}(1) - f) - f \leq w(1)|h(\overline{u}(1)) - f| - f. \end{aligned} \tag{A.12}$$

Now suppose to the contrary that $\mu_0 \geq 0$. Since $w > 0$ on $(0, 1)$, $w_x(1) = 0$, we have $\nu^2 w_{xx} = h'(\overline{u}(x))w + \mu_0 w \geq 0$ on $[a, 1]$. Hence, $w(x) \geq w(1)$ on $[a, 1]$ and, consequently,

$$1 = \int_0^1 w(x)\,dx \geq \int_a^1 w(x)\,dx \geq (1-a)w(1).$$

From (A.12) we obtain

$$\mu_0 \int_0^1 w(x)\phi(x)\,dx < 0.$$

Since $w \geq 0$, $\phi \geq 0$, we have $\mu_0 < 0$, a contradiction. □

Now it is easy to complete the proof of part (b). We fix an $a > x_0$. Then, by Theorem 3.3, $\sup\{|u_\nu^{(2)}(x) - \phi_2(x)|,\ x \in [a, 1]\} = O(\nu^{1/2})$ as $\nu \to 0^+$. Therefore, $|h(u_\nu^{(2)}(1)) - f| < (1-a)f$ and $h'(u_\nu^{(2)}(x)) > 0$ on $[a, 1]$ for any $\nu > 0$ sufficiently small. Lemma A completes the proof.

Note that, for certain singularly perturbed problems, an asymptotic estimate of the form $\mu_0(\nu) = O(\nu)$ as $\nu \to 0^+$ is proved in [1].

(c) Our next goal is to prove uniqueness of solutions of (3.19) for $f \in [f_{\min}, \gamma_m) \cup (\gamma_M, f_{\max}]$ and ν small. Let us consider the case $f \in (\gamma_M, f_{\max}]$. First, we show linearized stability of an arbitrary nondecreasing solution $\overline{u}$ of (3.19). By Lemma A it is sufficient to prove that $|h(\overline{u}(1)) - f| < (1-a)f$ and $h'(\overline{u}(x)) \geq 0$ on $[a, 1]$ for some $a \in (0, 1)$. To this end, we recall first that according to Proposition 3.1 there exists an $M > 0$ such that

$$\nu \sup_{x\in[0,1]} |\overline{u}_x(x)| + \sup_{x\in[0,1]} |\overline{u}(x)| \leq M \tag{A.13}$$

for any solution $\overline{u}$ of (3.19) and $\nu > 0$.

Let $\overline{u}$ be a nondecreasing solution for (3.19). Let $1 > \tilde{a} > \gamma_M/f$. Then for any $x \in [\tilde{a}, 1]$ we have $fx > \gamma_M$; so $\overline{u}$ is concave on $[\tilde{a}, 1]$. Thus, by (A.13)

$$0 \leq \overline{u}_x(x) \leq \int_{\tilde{a}}^x \overline{u}_x(\xi)\,d\xi \cdot \frac{1}{x-\tilde{a}} \leq \frac{4M}{1-\tilde{a}} \tag{A.14}$$

for any $x \in [a, 1]$ where $a = (\tilde{a}+1)/2$. Therefore, there exists a constant $M_1 > 0$ such that

$$0 \leq fx - h(\overline{u}(x)) \leq f\xi - h(\overline{u}(\xi)) + M_1(\xi - x) \tag{A.15}$$

for any $\xi, x \in [a, 1]$, $x \leq \xi$. Thus, by (A.14) and (A.15)

$$\begin{aligned} 0 &\leq \nu^{1/2}(fx - h(\overline{u}(x))) \\ &\leq \int_x^{x+\nu^{1/2}} (f\xi - h(\overline{u}(\xi)) + M_1(\xi - x))\, d\xi \\ &= -\nu^2 \int_x^{x+\nu^{1/2}} \overline{u}_{xx}(\xi)\, d\xi + \frac{M_1\nu}{2} \leq \left(2M + \frac{M_1}{2}\right)\nu =: M_2\nu. \end{aligned}$$

Hence $|fx - h(\overline{u}(x))| \leq M_2\nu^{1/2}$ for any $x \in [a, 1]$, $\nu > 0$, and any nondecreasing solution $\overline{u}$ of (3.19).

For $\nu \leq ((fa - \gamma_M)/M_2)^2$ we have

$$h(\overline{u}(x)) \geq fx - |fx - h(\overline{u}(x))| \geq fa - |fa - \gamma_M| = \gamma_M \quad \text{for any } x \in [a, 1].$$

Since $h(u) \leq \gamma_M$ for $u \leq c_2$ (see Fig. 1), we have $\overline{u}(x) \geq c_2$ on $[a, 1]$, hence $h'(\overline{u}(x)) \geq 0$ for $x \in [a, 1]$. By Lemma A, the principal eigenvalue μ_0 of the problem $B_1[w] = \nu^2 w_{xx} - h'(\overline{u}(x))w = \mu w$, $w \in D(B_1)$, is negative.

Now, consider the parabolic equation

$$u_\tau = \nu^2 u_{xx} - h(u) + fx,$$
$$u(\tau, 0) = u_x(\tau, 1) = 0, \qquad \tau \geq 0, \; u(0, x) = u_0(x), \; x \in [0, 1].$$

This equation generates a gradient-like semidynamical system $\mathscr{S}(\tau)$, $\tau \geq 0$, in the Hilbert space $X^{1/2} = \{u \in W^{1,2}(0, 1), \; u(0) = 0\}$ defined by $\mathscr{S}(\tau)u_0 = u(\tau, \cdot)$, where $u(0, \cdot) = u_0(\cdot)$ (see [5, Chapter 4]). The set $\mathscr{K} = \{u \in X^{1/2}, \; u_x(x) \geq 0$, a.e. on $[0, 1]\}$ is a closed convex cone in $X^{1/2}$. Moreover, $\mathscr{K}$ is invariant under $\mathscr{S}$, i.e.,

$$u(\tau, \cdot) \in \mathscr{K} \quad \text{whenever } u(0, \cdot) \in \mathscr{K} \text{ for any } \tau \geq 0.$$

Indeed, the function

$$w(\tau, x) = \begin{cases} -u_x(\tau, x), & x \in [0, 1], \; \tau \geq 0; \\ -u_x(\tau, -x), & x \in [-1, 0], \; \tau \geq 0, \end{cases}$$

is the solution of the scalar parabolic equation

$$w_\tau = \nu^2 w_{xx} - h'(u(x))w - f,$$
$$w(\tau, -1) = w(\tau, 1) = 0.$$

Therefore, $w(\tau, x) \leq 0$ whenever $w(0, x) \leq 0$ by the Maximum Principle (see [14]). Hence, $\mathscr{S}$ is a semidynamical system on the complete metric space $\mathscr{K}$ with the topology induced by $X^{1/2}$.

To complete the proof we argue similarly as in [1, Theorem 4]. Since $\mathscr{K}$ is invariant, it is the union of (disjoint) attraction domains of the nondecreasing stationary solutions of (A.14). Because those solutions are asymptotically stable, these attraction domains are open in $\mathscr{K}$. Since the set $\mathscr{K}$ is connected, it cannot be a union of two nonempty disjoint open sets; hence, $u_\nu^{(2)}$ is the unique stationary solution in $\mathscr{K}$.

Now, let $\overline{u}$ be an arbitrary solution of (3.19) (not necessarily nondecreasing). By Proposition 3.1, $\overline{u}$ is bounded and $\overline{u} \geq 0$. Then there exist $\overline{u}^-$, $\overline{u}^+ \in \mathscr{K} \cap D(A)$ such that $\overline{u}^-(x) \leq \overline{u}(x) \leq \overline{u}^+(x)$, $x \in [0, 1]$. With regard to the Maximum Principle [14, Chapter 3, Theorem 3] we obtain $\mathscr{S}(\tau)\overline{u}^-(x) \leq \mathscr{S}(\tau)\overline{u}(x) \leq \mathscr{S}(\tau)\overline{u}^+(x)$ for any $\tau \geq 0$ and $x \in [0, 1]$. Since $\mathscr{S}(\tau)\overline{u}^{\pm} \in \mathscr{K}$, for any $\tau \geq 0$, we have $\mathscr{S}(\tau)\overline{u}^{\pm} \to u_\nu^{(2)}$ as $\tau \to \infty$. Thus, $\overline{u} = u_\nu^{(2)}$.

Hence, the solution $u_\nu^{(2)}$ is unique, provided ν is small and $f \in (\gamma_M, f_{\max}]$. The proof of uniqueness of solutions of (3.19) for $f \in [f_{\min}, \gamma_m)$ is similar. □

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