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CONSERVATION LAWS IN NONLINEAR ELASTICITY
I. ONE-DIMENSIONAL ELASTODYNAMICS

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Introduction. The conservation laws satisfied by the solutions of a previously given system of differential equations play a basic role in various problems of continuum physics: conservation laws such as the conservation laws of energy, momentum and others that one uses for the analysis of theoretical properties of differential equation solutions (a priori estimates, theorems for existence and uniqueness, stability, etc.), as well as the numerical discretizations—the method of integral relations, conservative difference schemes and finite elements.

In elasticity, in addition to the general applications mentioned above, the conservation laws (or, rather their path-independent integral forms) are of key importance in the study of cracks and dislocations [1, 2]. Instructive review of the conservation laws in elasticity and their numerous applications is presented by Olver in [3].

The results of Knowles, Sternberg [4], and Fletcher [5] obtained by variational principles being invariant at infinitesimal transformations are generalized for viscoelasticity in the paper [6].

The traditional method for obtaining conservation laws is based on Noether's theorem and its generalizations [7, 8]. For its application it is necessary to analyze in advance the group properties of the initial differential system of equations and to obtain the functional for which Euler's equations appear as equations to be analysed. Moreover, if the functional is degenerate (this is possible in elasticity [9]) one loses the one-to-one map between the groups which admit the initial equations and the conservation laws [7, 8]. Thus the investigation of conservation laws by previous determination of the groups of corresponding functionals sometimes leads to tedious computations.

In the present work we seek the conservation laws without using group analysis and a variational principle. In the case of an adiabatic, an elastic solid for the density, and the flux of an arbitrary conservation law, an overdetermined partial differential equations system of the first order is obtained, and the presence of the entropy permits one to perform a direct analysis for compatibility. Briefly, the main result that we obtain is the following. If the internal energy function does not satisfy a system of two partial differential equations of the second order with constant coefficients

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(the condition (iii) of Theorems 2.1, 3.1) then all linearly independent conservation laws in elasticity, which are explicitly independent of time and space are the classical conservation laws of momentum, energy, entropy, and mass. The two- and three-dimensional problems require a more delicate analysis.

Let us mention that with similar methods the conservation laws of other important differential equation systems of continuum physics are obtained [9, 10].

This paper is divided into three parts. In Sec. 1 are discussed the basic equations of one-dimensional elasticity in the case of compressible and incompressible solids. In Sec. 2 are obtained the conservation laws of compressible nonlinear elasticity and in Sec. 3 the conservation laws of incompressible elasticity. Also, two examples for applications of the basic theorems are considered, and the role of entropy for determining conservation laws is discussed for the equations of isothermic incompressible elasticity. The conclusion is that, like in the case of the shock waves theory [11, 12, 13], the presence of entropy does not allow conservation laws without physical meaning and vice versa.

1. Basic equations. Without loss of generality we can consider a motion of a solid in the form of waves parallel to the (x^2, x^3)-plane in the Lagrange coordinate system $X^1 = x, X^2, X^3$,

$$x_i = X^i + d_i(x, t), \qquad i = 1, 2, 3,$$

where the x_i are Euler's coordinates and the d_i are displacements. This motion generates a velocity field $v_i = \partial d_i/\partial t$, $i = 1, 2, 3$, and a deformation gradient field $p_i = \partial d_i/\partial x$, $i = 1, 2, 3$.

An adiabatic process in a compressible elastic solid without body forces is governed by the conservation laws of momentum and energy, [12, 15],

$$\rho_0 \frac{\partial v_i}{\partial t} - \frac{\partial T_i}{\partial x} = 0, \tag{1.1}$$

$$\frac{\partial p_i}{\partial t} - \frac{\partial v_i}{\partial x} = 0,$$

$$\frac{\partial E}{\partial t} - \frac{\partial}{\partial x} \sum_{i=1}^{3} v_i T_i = 0 \tag{1.2}$$

and the second law of thermodynamics

$$de = \Theta\, d\eta + \sum_{i=1}^{3} T_i\, dp_i. \tag{1.3}$$

Here, T_i, $i = 1, 2, 3$, are stresses, ρ_0 is constant density in the reference configuration, Θ is temperature, η is entropy,

$$E = e + \rho_0 \frac{v_1^2 + v_2^2 + v_3^2}{2}$$

is the total energy, and $e = \overline{e}(p_1, p_2, p_3, \eta)$ is the internal energy. The conservation law of the mass can be expressed by the continuity equation

$$\rho |1 + p_1| = \rho_0, \tag{1.4}$$

where ρ_0 is the density in the actual configuration.

Hence and henceforth we consider smooth solutions of the nonlinear elasticity equations and shall not specify explicitly the domain of the variables if it is not explicitly shown in adding.

Let us multiply the first equation of (1.1) by v_1, the second by v_2, the third by v_3 and subtract them from (1.2):

$$\frac{\partial e}{\partial t} - \sum_{i=1}^{3} T_i \frac{\partial v_i}{\partial x} = 0.$$

Using 4–6 of the equations of (1.1) we can write the last equation in the form

$$\frac{\partial e}{\partial t} - \sum_{i=1}^{3} T_i \frac{\partial p_i}{\partial t} = 0, \tag{1.5}$$

which by analogy with gas dynamics can be called the “entropy” equation of the energy. If we multiply the first equation of (1.1) by $q_1 = -v_1/\Theta$, the second by $q_2 = -v_2/\Theta$, the third by $q_3 = -v_3/\Theta$, the fourth by $q_4 = -T_1/\Theta$, the fifth by $q_5 = -T_2/\Theta$, the sixth by $q_6 = -T_3/\Theta$ and (1.2) by $q_0 = 1/\Theta$ and summarize the results, we get

$$\Theta\frac{\partial \eta}{\partial t} = \frac{\partial e}{\partial t} - \sum_{i=1}^{3} T_i \frac{\partial p_i}{\partial t} = 0.$$

From here and (1.5) the conservation law of entropy follows:

$$\frac{\partial \eta}{\partial t} = 0. \tag{1.6}$$

It is not too hard to prove that in the dependent variables q_i, $i = 0, \ldots, 6$, the elasticity equations can be written as a symmetric quasilinear system, [13].

If the solid is incompressible, then $\rho = \rho_0$ and from (1.4) it follows that $p_1 = 0$. With additional physical motivation, see [15], one can show that the equations of elasticity are reduced to

$$\begin{aligned} \rho_0\frac{\partial v_i}{\partial t} - \frac{\partial T_i}{\partial x} &= 0, \\ \frac{\partial p_i}{\partial t} - \frac{\partial v_i}{\partial x} &= 0, \qquad i = 2, 3, \end{aligned} \tag{1.7}$$

where

$$T_i = \frac{\partial e}{\partial p_i}, \qquad i = 2, 3, \qquad E = e + \rho_0\frac{v_2^2 + v_3^2}{2}.$$

Let us consider the equations of a compressible elastic solid. If we choose the velocities, deformations and entropy as basic unknown functions, the equations of one-dimensional nonlinear elasticity can be written by the quasilinear system

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0, \tag{1.8}$$

where

$$U = \begin{pmatrix} V \\ P \\ \eta \end{pmatrix}, \qquad A(U) = \operatorname{grad}\left(-\rho_0^{-1}\frac{\partial e}{\partial p}, -V, 0\right), \qquad V \equiv (v_1, v_2, v_3).$$

DEFINITION 1.1. The equation, [11],

$$\frac{\partial D(t, x, U)}{\partial t} + \frac{\partial F(t, x, U)}{\partial x} = 0 \tag{1.9}$$

in which the functions D, F are second-order smooth functions of their arguments is called a first-order conservation law of the system (1.8) if it is satisfied for every smooth solution of the system (1.8). The functions D, F are called density and flux of the conservation law (1.9). The conservation laws of the system (1.8) are linearly independent if and only if their corresponding functions $\mathbf{Z}(t, x, U) \equiv (D_j, F_j)$ are linearly independent.

In the same manner the conservation laws for incompressible elasticity can be defined.

2. Conservation laws in compressible nonlinear elasticity.

THEOREM 2.1. Let the function, $e = \bar{e}(P, \eta)$, $P \equiv (p_1, p_2, p_3)$, have continuous partial derivatives up to third order in a region $\Omega = Q \times (a, b)$. Let us denote

$$e_i = \frac{\partial \bar{e}}{\partial p_i} = +T_i, \quad T = (T_1, T_2, T_3), \quad e_{i\eta} = \frac{\partial^2 e}{\partial p_i \partial \eta}, \quad e_{ij} = \frac{\partial^2 e}{\partial p_i \partial p_j},$$
$$\Delta = \det \|e_{ij}\|, \qquad i, j = 1, 2, 3,$$

and by Δ_i $(i = 1, 2, 3)$ the determinant, which is obtained from Δ by substituting its ith column with $(e_1, e_2, e_3)^{\mathrm{T}}$. Let us suppose that the following conditions are fulfilled:

(i) $\Delta \neq 0$ if $(P, \eta) \in \Omega$;

(ii) $\exists \bar{\eta} \in (a, b)$, such that for almost every $P \in Q$

$$J(e) = \det \left\| \sum_{r=1}^{3} e_{ksr}\Delta_r - \Delta e_{ks\eta} \right\| \neq 0,$$

where

$$e_{ksr} = \frac{\partial e_{ks}}{\partial p_r}, \qquad e_{ks\eta} = \frac{\partial e_{ks}}{\partial \eta}, \qquad k, s, r = 1, 2, 3.$$

Then, all conservation laws of the system (1.9) corresponding to its solutions $U = (V, P, \eta)$, for which $(P, \eta) \in \Omega$ are the following:

$$\begin{aligned} D = &-c_{11}e - \int (c_{12}T_2 + c_{13}T_3)\, dp_1 - \sum_{i=1}^{3}(\rho_0 M_i x + L_i)P_i \\ &- \tfrac{1}{2}\rho_0 \sum_{i,j=1}^{3} c_{ij}v_i v_j - \sum_{i=1}^{3}(\rho_0 M_i t + N_i)v_i + \varphi(x, \eta) \\ &+ \int \psi(x, t)\, dt + \mu(x), \end{aligned}$$

$$F = \sum_{i,j=1}^{3} c_{ij} v_i T_j + \sum_{i=1}^{3} (\rho_0 M_i x + L_i) v_i + \sum_{i=1}^{3} (M_i t + N_i) e_i - \int \psi(x, t)\, dx + \nu(t),$$

where φ, ψ, μ, ν are arbitrary functions, M_i, N_i, L_i are arbitrary constants and the constants c_{ij} satisfy the conditions: $c_{ij} = c_{ji}$ and

$$\text{(iii)} \quad \begin{aligned} &(c_{11} - c_{22}) e_{12} + c_{12}(e_{22} - e_{11}) + c_{13} c_{32} - c_{23} e_{31} = 0, \\ &(c_{11} - c_{33}) e_{31} + c_{13}(e_{11} - e_{33}) + c_{12} e_{23} - c_{32} e_{21} = 0. \end{aligned}$$

Proof. In the equation (1.10) D and F are the required functions of the variables t, x, v_i, p_i, $i = 1, 2, 3$, η, and each of the variables v_i, p_i, $i = 1, 2, 3, \eta$ in their turn depend on t, x. We differentiate in (1.11) D, F as compound functions and using the system (1.1), (1.2) we eliminate the derivatives with respect to t. We get a linear relation with respect to space partial derivatives, which must be zero for arbitrary values of the solution derivatives. As a result the following system of eight equations for two functions arises:

$$\frac{\partial F}{\partial p_i} = -\rho_0^{-1} \sum_{j=1}^{3} e_{ij} \frac{\partial D}{\partial v_i}, \qquad i = 1, 2, 3, \tag{2.1$_i$}$$

$i = 1, 2, 3$

$$\frac{\partial F}{\partial v_k} = -\frac{\partial D}{\partial p_k}, \qquad k = 1, 2, 3, \tag{2.1$_{3+k}$}$$

$k = 1, 2, 3$

$$\frac{\partial F}{\partial \eta} = -\rho_0^{-1} \sum_{j=1}^{3} e_{j\eta} \frac{\partial D}{\partial v_j}, \tag{2.1$_7$}$$

$$\frac{\partial D}{\partial t} + \frac{\partial F}{\partial x} = 0. \tag{2.1$_8$}$$

At the first stage of the proof we will investigate for compatibility the equations (2.1_1)–(2.1_7). For this purpose we compute $\partial D/\partial v_j$, $j = 1, 2, 3$, from the equations (2.1_1)–(2.1_3):

$$-\frac{\partial D}{\partial v_j} = \frac{\rho_0}{\Delta} \sum_{i=1}^{3} (-1)^{i+j} \Delta_{ij} \frac{\partial F}{\partial p_i}, \qquad j = 1, 2, 3,$$

where Δ_{ij} is the subdeterminant of Δ corresponding to its (i, j)-th element. After substituting these expressions in the right side of (2.1_7) and suitable rearrangement we get the equation

$$\Delta \frac{\partial F}{\partial \eta} - \sum_{j=1}^{3} \Delta_j \frac{\partial F}{\partial p_j} = 0. \tag{2.2}$$

The general solution of (2.2) is

$$F = F(e_1, e_2, e_3, v_1, v_2, v_3, t, x). \tag{2.3}$$

For example, we shall check that e is a first integral of (2.2), i.e., $\partial e/\partial p = \text{const}$. Indeed, in view of the characteristic system of Eq. (2.2),

$$\frac{\partial \eta}{\Delta} = \frac{\partial p_1}{\Delta_1} = \frac{\partial p_2}{\Delta_2} = \frac{\partial p_3}{\Delta_3} = k,$$

we get

$$de_1 = k\left(e_{1\eta}\Delta + \sum_{j=1}^{3} e_{1j}\Delta_j\right).$$

But, using the notations for Δ, Δ_j, $j = 1, 2, 3$, it is easy to verify that the right side of this expression is zero.

Using the representation (2.3) for the function F we can rewrite (2.1_i) in the form

$$\left(\rho_0\frac{\partial F}{\partial e_1} + \frac{\partial D}{\partial v_1}\right)e_{i1} + \left(\rho_0\frac{\partial F}{\partial e_2} + \frac{\partial D}{\partial v_2}\right)e_{i2} + \left(\rho_0\frac{\partial F}{\partial e_3} + \frac{\partial D}{\partial v_3}\right)e_{i3} = 0, \qquad i = 1, 2, 3.$$

Taking into account that $\Delta \neq 0$, then from this homogeneous linear system of algebraic equations for the unknowns $\rho_0\frac{\partial F}{\partial e_i} + \frac{\partial D}{\partial v_i}$, $i = 1, 2, 3$, the equalities follow:

$$\rho_0\frac{\partial F}{\partial e_i} + \frac{\partial D}{\partial v_i} = 0, \qquad i = 1, 2, 3. \tag{2.4}$$

Let us differentiate each of the equalities (2.4) with respect to p_k, $k = 1, 2, 3$. Taking into account (2.1_{3+k}) we obtain

$$\rho_0\sum_{s=1}^{3} e_{ks}(P, \eta)\frac{\partial F}{\partial e_i\partial e_s} = -\frac{\partial D}{\partial v_i\partial p_k} = \frac{\partial F}{\partial v_i\partial v_k}, \qquad i, k = 1, 2, 3. \tag{2.5}$$

Since $\Delta \neq 0$ and

$$e_i = \frac{\partial \overline{e}}{\partial p_i} \equiv \tilde{e}_i(p_1, p_2, p_3, \eta), \qquad i = 1, 2, 3, \tag{2.6}$$

from the implicit functions theorem the (local) representation follows:

$$p_i = \tilde{p}_i(e_1, e_2, e_3, \eta), \qquad i = 1, 2, 3. \tag{2.7}$$

Now we can rewrite (2.5) in the form

$$\rho_0\sum_{s=1}^{3} \tilde{e}_{ks}(e_1, e_2, e_3, \eta)\frac{\partial^2 F(T, v)}{\partial e_i\partial e_s} = \frac{\partial^2 F(T, v)}{\partial v_i\partial v_k} = -\frac{\partial D}{\partial v_i\partial p_k} \tag{2.8}$$

where

$$\tilde{e}_{ks} = e_{ks}(\tilde{p}_1(e_1, e_2, e_3, \eta), \tilde{p}_2(e_1, e_2, e_3, \eta), \tilde{p}_3(e_1, e_2, e_3, \eta)). \tag{2.9}$$

Let us put e_i, $i = 1, 2, 3$, from (2.6) in (2.7) and differentiate the two sides of the equalities thus obtained with respect to η. As a result of this a system of linear algebraic equations arises for the unknowns $\partial p_i/\partial \eta$. Solving it we find

$$\frac{\partial p_i}{\partial \eta} = \frac{-\Delta_i}{\Delta}, \qquad i = 1, 2, 3.$$

Now, from (2.9) it follows

$$\frac{\partial \tilde{e}_{ks}}{\partial \eta} = \frac{1}{\Delta} \sum_{r=1}^{3} e_{ksr} \Delta_r - e_{ks\eta}. \tag{2.10}$$

Let us differentiate the two sides of (2.5) with respect to η. We get a homogeneous linear algebraic system of equations the determinant of which according to (2.10) and (ii) at $\eta = \overline{\eta}$ is nonzero. Hence

$$\frac{\partial^2 F}{\partial e_i \partial e_k} = 0, \qquad i, k = 1, 2, 3. \tag{2.11}$$

Now from (2.5), it follows that

$$\frac{\partial^2 F}{\partial v_j \partial v_k} = 0, \qquad i, k = 1, 2, 3, \tag{2.12}$$

and

$$\frac{\partial^2 D}{\partial v_i \partial p_k} = 0, \qquad i, k = 1, 2, 3. \tag{2.13}$$

Thus, the representation for F follows from (2.11), (2.12):

$$F = \sum_{i,j=1}^{3} c_{ij}(t, x) v_i e_j + \sum_{i=1}^{3} c_i(t, x) v_i + \sum_{i=1}^{3} K_i(t, x) e_i + l(t, x), \tag{2.14}$$

and from (2.13) for D:

$$D = D_1(p, \eta, t, x) + D_2(v, \eta, t, x). \tag{2.15}$$

In (2.14), (2.15) c_{ij}, c_i, K_i, D_1, D_2, l are arbitrary functions of their arguments.

In view of (2.14), from $(2, 1_{3+k})$ and (2.4) we get for D:

$$\frac{\partial D}{\partial p_i} = -\frac{\partial F}{\partial v_i} = -\sum_{j=1}^{3} c_{ij} e_j - c_i, \qquad i = 1, 2, 3, \tag{2.16}$$

$$\frac{\partial D}{\partial v_i} = -\rho_0 \left(\sum_{j=1}^{3} c_{ij} v_j + K_i \right), \qquad i = 1, 2, 3. \tag{2.17}$$

After a differentiation of the two sides of (2.16) with respect to p_i $(i = 1, 2, 3)$ and equalizing the mixed derivatives, we find that the functions c_{ij}, $i, j = 1, 2, 3$ must satisfy the conditions (iii).

From the first of the equations (2.16), after an integration with respect to p_i, we find that

$$D_1 = -c_{11} e_1 - \int (c_{12} e_2 + c_{13} e_3)\, dp_1 - c_1 p_1 + D_3(p_2, p_3, \eta, t, x) \tag{2.18}$$

where D_3 is an arbitrary function. Let us differentiate the two sides of (2.18) with respect to p_2 and replace the expression under the integral by its value from (iii). After some easy calculations we obtain

$$D_3 = -c_2 p_2 + D_4(p_3, \eta, t, x). \tag{2.19}$$

Taking D_3 from (2.19) we put D_1 from (2.18) in the third of the equations (2.16). After calculations similar to the above we get

$$D_4 = -c_3 p_3 + D_5(\eta, t, x), \tag{2.20}$$

where D_5 is an arbitrary function of its arguments. Finally, from (2.14)–(2.21) we get the formulas

$$\begin{aligned} D = &- c_{11}(t, x)e_1 - \int (c_{12}(t, x)e_2 + c_{13}(t, x)e_3)\, dp_1 \\ &- \sum_{i=1}^{3} c_i(t, x)p_i - \tfrac{1}{2}\rho_0 \sum_{i,j=1}^{3} c_{ij}(t, x)v_i v_j - \rho_0 \sum_{i=1}^{3} K_i(t, x) + D_5(\eta, t, x), \\ F = &\sum_{i,j=1}^{3} c_{ij}(t, x)v_i e_j + \sum_{i=1}^{3} c_i(t, x)v_i + \sum_{i=1}^{3} K_i(t, x)e_i + l(t, x). \end{aligned} \tag{2.21}$$

We put these expressions for D and F into Eq. (2.1_8) and equalize to zero the coefficients at the degrees of v_1, v_2, v_3:

$$\frac{\partial c_{ij}}{\partial t} = 0, \quad i, j = 1, 2, 3, \tag{2.22}$$

$$-\rho_0 \frac{\partial K_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial c_{ij}}{\partial x} e_i + \frac{\partial c_i}{\partial x} = 0, \quad i = 1, 2, 3, \tag{2.23}$$

$$-\sum_{i=1}^{3} \frac{\partial c_i}{\partial t} p_i + \frac{\partial D_5}{\partial t} + \sum_{i=1}^{3} \frac{\partial K_i}{\partial x} e_i + \frac{\partial l}{\partial x} = 0. \tag{2.24}$$

We differentiate (2.23) successively with respect to p_1, p_2, p_3 and get a linear homogeneous algebraic system of equations for the unknowns $\frac{\partial c_{i1}}{\partial x}, \frac{\partial c_{i2}}{\partial x}, \frac{\partial c_{i3}}{\partial x}$, with determinant $\Delta \neq 0$ (see (i)). Thus, (2.23) reduces to the equations

$$\frac{\partial c_{ij}}{\partial x} = 0, \qquad -\rho_0 \frac{\partial K_i}{\partial t} + \frac{\partial c_i}{\partial x} = 0. \tag{2.25}$$

In the same way, using the condition (ii) we can show that (2.24) reduces to the equations

$$\frac{\partial K_i}{\partial x} = 0, \qquad \frac{\partial c_i}{\partial t} = 0, \qquad \frac{\partial D_5}{\partial t} + \frac{\partial l}{\partial x} = 0. \tag{2.26}$$

An elementary analysis of (2.22), (2.25), (2.26) gives

$$\varphi_2 = -\int \psi(x, t)\, dt + \mu(x), \qquad l = -\int \psi(x, t)\, dx + \nu(x),$$

where φ, ψ, μ, ν are arbitrary functions. This completes the proof of the theorem.

COROLLARY 1. For an arbitrary function $e = \bar{e}(p_1, p_2, p_3, \eta)$ having continuous partial derivatives up to the third order there exist constants $c_{ij} = c_{ji}$ satisfying the conditions (iii), but if the function of the internal energy is such that these constants must satisfy the equalities

$$c_{ii} = c_{jj}, \quad c_{ij} = 0, \qquad i \neq j, \quad i, j = 1, 2, 3, \tag{2.27}$$

which do not depend explicitly on the time and the space, then we get the classical conservation laws: the conservation laws of mass (1.4), momentum (1.1), energy (1.2), and entropy (1.6).

Proof. Now, in view of (2.27) the formulas (2.21) take the form

$$D = ce + \sum_{i=1}^{3}(\rho_0 M_i x + L_i)p_i - \tfrac{1}{2}\rho_0 c\sum_{i=1}^{3} v_i^2 - \sum_{i=1}^{3}(\rho_0 M_i t + N_i)v_i + \varphi(x, \eta),$$

$$F = c\sum_{i=1}^{3} v_i T_i + \sum_{i=1}^{3}(\rho_0 M_i x + L_i)v_i + \sum_{i=1}^{3}(M_i t + N_i)e_i.$$

Setting successively in these formulas all constants except one to zero we get the classical conservation laws except the conservation law of mass.

The only trivial conservation laws (which give no information about the solutions) are the pairs

$$D = \int \psi(x, t)\,dt + \mu(x), \qquad F = -\int \psi(x, t)\,dx + \nu(t),$$

where ψ, μ, ν are arbitrary functions.

3. Conservation laws in incompressible nonlinear elastodynamics. An analogous theorem is valid for the equations (1.8).

THEOREM 3.1. Let the function $e = \overline{e}(P, \eta)$, $P = (p_2, p_3)$, have continuous partial derivatives up to third order in a region $\Omega = Q \times (a, b) \subset \mathbf{R}^3$. Let us denote

$$e_i = \frac{\partial \overline{e}}{\partial p_i} = T_i, \quad e_{i\eta} = \frac{\partial^2 \overline{e}}{\partial p_i \partial \eta}, \quad e_{ij} = \frac{\partial^2 \overline{e}}{\partial p_i \partial p_j}; \qquad i, j = 2, 3,$$

$\Delta = \det\|e_{ij}\|$, and let Δ_i $(i = 2, 3)$ be the determinant that is obtained from Δ by substituting its ith column with $(e_{2\eta}, e_{3\eta})$. Let us suppose that the following conditions are fulfilled:

(i) $\Delta \neq 0$ if $P \in Q$;
(ii) $\exists \overline{\eta} \in (a, b)$ such that for almost every $P \in Q$

$$\det\left\|\sum_{r=2}^{3} e_{ksr}\Delta_r - \Delta e_{ks\eta}\right\| \neq 0,$$

where

$$e_{ksr} = \frac{\partial e_{ks}}{\partial p_r}, \quad r = 2, 3, \qquad e_{ks\eta} = \frac{\partial e_{ks}}{\partial \eta}.$$

Then, all conservation laws of the system (1.8) corresponding to its solution $U = (v, P, \eta)$ for which $(P, \eta) \in \Omega$ are the following:

$$D = -c_{22}e - c_{23}\int T_3\,dp_2 - \sum_{i=2}^{3}(\rho_0 M_k x + L_k)p_k - \tfrac{1}{2}\rho_0 \sum_{i,j=2}^{3} c_{ij}v_i v_j$$
$$- \sum_{i=2}^{3}(\rho_0 M_i t + N_i)v_i + \varphi(x, \eta) + \int \psi(x, t)\,dt + \mu(x),$$

$$F = \sum_{i,j=2}^{3} c_{ij} v_i T_j + \sum_{i=2}^{3} (\rho_0 M_i x + L_i) v_i + \sum_{i=2}^{3} (M_i t + N_i) e_i + \int \psi(x, t)\, dt + \nu(t),$$

where φ, ψ, μ, ν are arbitrary functions, M_i, N_i, L_i are arbitrary constants, and the constants c_{ij} satisfy the conditions: $c_{ij} = c_{ji}$ and

(iii) $$(c_{22} - c_{33}) e_{23} + c_{23}(e_{22} - e_{33}) = 0.$$

This theorem is not a corollary of the previous one, but the proof is similar.

We shall discuss the conditions (i) and (iii) of Theorems 2.1, 3.1 in the following example. We consider a transversally-isotropic elastic material with energy function [15]

$$\begin{aligned} e = e_0 &+ \tfrac{1}{2}\zeta_2 p_2^2 + \tfrac{1}{2}\zeta_3 p_3^3 + \Theta\eta + \tfrac{1}{4}\mu_{22} p_2^4 \\ &+ \tfrac{1}{2}\mu_{23} p_2^2 p_3^2 + \tfrac{1}{4}\mu_{33} p_3^4 + \beta\eta^2 + \tfrac{1}{2}\xi_2 p_2^2 \eta + \tfrac{1}{2}\xi_3 p_3^2 \eta. \end{aligned} \tag{3.1}$$

At the restriction of the material constants

$$\zeta_2 = \zeta_3 = \zeta, \qquad \mu_{22} = \mu_{33} = \mu, \qquad \xi_2 = \xi_3 = \xi,$$

the formula (3.4) describes the state of isotropic material. In this case the determinant Δ in Theorem 3.1 is

$$\Delta = \zeta^2 + 2\zeta\xi\eta + \xi^2\eta^2 + 4\mu(\zeta + 3\eta)(p_2^2 + p_3^2) + 3\mu^2(p_2^2 + p_3^2)^2.$$

For fixed value of the entropy η, Δ may be zero on as many as two circles $p_2^2 + p_3^2 =$ const. in the plane (p_2, p_3). Thus the condition (i) may be invalidated on those points of Q that lie on two cylindrical surfaces of the space (p_2, p_3, η). Therefore, the region Q can be divided into a few subregions in each of which the condition (i) is fulfilled and formulas for D, F are valid with proper coefficients c_{ij}, M_i, L_i, N_i and function $\varphi(x, \eta)$. The conservation laws thus obtained can be continued on the boundaries, i.e., everywhere in Q. But now the coefficients c_{ij}, M_i, L_i, N_i and function φ according to Definition 1.1 must be the same everywhere in Q. For an isotropic material the condition (iii) of Theorem 3.1 takes the form

$$(c_{22} - c_{33}) p_2 p_3 + c_{23}(p_3^2 - p_3^2) = 0,$$

which is fulfilled if and only if $c_{22} = 0$, $c_{22} = c_{33} = c$ (c an arbitrary constant). When the material is only transversally isotropic, (iii) has the form

$$\begin{aligned} 2\mu_{23}(c_{22} - c_{33}) p_2 p_3 + c_{23}(\zeta_2 - \zeta_3) + (3\mu_{22} - \mu_{23}) p_2^2 \\ + (\mu_{23} - 3\mu_{33}) p_3^2 + (\zeta_2 - \zeta_3)\eta_3 = 0. \end{aligned}$$

Now let $\mu_{23} \neq 0$, $\zeta_2 = \zeta_3$, $\mu_{23} = 3\mu_{22} = 3\mu_{33}$, $\zeta_2 = \zeta_3$. Then $c_{22} = c_{33} = c_1$, $c_{23} = c_2$, c_1, c_2 are arbitrary constants.

The assumption (i) in Theorem 2 (and the similar assumption in Theorem 3.1) arises by studying the system (2.5). If one considers shear deformation, i.e., $d_1 = d_2 = 0$, $d_3 = d(x, t)$, the governing equations of the process take the form of the gas dynamics equations in Lagrange variables, the conservation laws for which are studied by Rozdestevenski [13], p. 39. The analogue of the system (2.5) now is the equation

$$e_{pp}(p\,,\,\eta)\frac{\partial^2 F}{\partial p^2}=\frac{\partial^2 F(p\,,\,v)}{\partial^2 v^2}\,,\qquad p=\frac{\partial d}{\partial x}\,,\qquad v=\frac{\partial d}{\partial t}.$$

The condition (ii) takes the form

$$e_{pp}e_{pp\eta}-e_{p\eta}e_{ppp}\neq 0\,,$$

which excludes the nonphysical relation $e_p=g(p+f(\eta))$ where g, f are arbitrary smooth functions. The internal energy functions which do not satisfy (ii) in Theorems 2.1, 3.1 are also solutions of a third-order partial differential equation, namely $Y(e)=0$, but are much more complicated.

It is easy to verify that the examples above satisfy the assumption in Theorems 2.1, 3.1.

Let us discuss the role of entropy for the variety of conservation laws in elasticity in the equations of an incompressible elastic solid (1.8). Let us consider the system (1.8) at $\eta=\text{const}$. Multiplying the first equation of (1.8) by p_2, the second by p_3, the third by v_2, the fourth by v_3, and summing the results, we get

$$\rho_0\frac{\partial}{\partial t}(v_2p_2+v_3p_3)-\frac{\partial}{\partial x}\left(\rho_0\frac{v_2^2+v_3^2}{2}+p_2\frac{\partial e}{\partial p_2}+p_3\frac{\partial e}{\partial p_3}-e\right)=0. \tag{3.2}$$

Such a conservation law does not exist in adiabatic incompressible elasticity according to Theorem 3.1. Now for determination of the density D and flow F we have to solve the system

$$\begin{aligned}\frac{\partial F}{\partial p_2}&=-e_{22}\frac{\partial D}{\partial v_2}-e_{23}\frac{\partial D}{\partial v_2}\,, &\qquad \frac{\partial F}{\partial p_3}&=-e_{32}\frac{\partial D}{\partial p_2}-e_{33}\frac{\partial D}{\partial p_3}\,,\\ \frac{\partial F}{\partial p_2}&=-\frac{\partial D}{\partial p_2}\,, & \frac{\partial F}{\partial v_3}&=-\frac{\partial D}{\partial p_3}.\end{aligned} \tag{3.3}$$

From the system (3.3) one gets the following equation:

$$e_{23}\left(\frac{\partial^2 D}{\partial v_2^2}-\frac{\partial^2 D}{\partial v_3}\right)+(e_{22}-e_{33})\frac{\partial^2 D}{\partial v_2\partial v_3}=0. \tag{3.4}$$

It can be solved explicitly for D and by substituting it in (3.3) we find F. But for our purpose it is sufficient to obtain those partial solutions of (3.4) for which

$$\frac{\partial^2 D}{\partial v_2^2}-\frac{\partial^2 D}{\partial v_3^2}=0\,,\qquad \frac{\partial^2 D}{\partial v_2^2\partial v_3^2}=0. \tag{3.5}$$

After tedious, but routine analysis one can prove the following proposition.

Proposition. Let the energy function $e=\overline{e}(p_2\,,\,p_3)$ have continuous partial derivatives up to third order in a region $Q\in\mathbf{R}^2$, and almost everywhere in Q the condition (1) is fulfilled. Then all conservation laws in incompressible isothermic elasticity, which do not depend explicitly on time and space, satisfying the condition (3.5) and corresponding to those solutions of the system (1.8) at $\eta = \text{const.}$, for which $(p_2, p_3) \in Q$, are

$$\begin{aligned} D = c\left(e + \rho_0 \frac{v_2^2 + v_3^2}{2}\right) &+ \rho_0(\alpha p_2 + \beta p_3 + \gamma)v_2 \\ &+ (\beta p_2 + \delta p_3 + \varepsilon)v_3 + \mu p_2 + \nu p_3, \\ F = -\tfrac{1}{2}\rho_0(\alpha v_2^2 + \delta v_3^2) &- \beta v_2 v_3 - c\left(\frac{\partial e}{\partial p_2}v_2 + \frac{\partial e}{\partial p_3}v_3\right) \\ -\mu v_2 - \nu p_3 + \delta e &- (\alpha p_2 + \beta p_3 + \gamma)\frac{\partial e}{\partial p_2}(\beta p_2 + \delta p_3 + \varepsilon)\frac{\partial e}{\partial p_3} \\ + \beta \int \frac{\partial e}{\partial p_2}\, dp_3, & \end{aligned} \tag{3.6}$$

where the constants α, δ, β satisfy the condition

$$(\alpha - \delta)e_{23} + \beta(e_{33} - e_{22}) = 0$$

and $c, \alpha, \beta, \gamma, \varepsilon, \mu, \nu$ are arbitrary constants.

The difference between the conservation laws in Theorem 3.1 and those in the Proposition is obvious. The conservation law (3.2) is obtained from (3.6) by putting $c = 0$, $\alpha = \delta = 1$, $\beta = \varepsilon = 0$, $\mu = \nu = 0$.

References

[1] C. Atkinson, *Stress singularities and fracture mechanics*, Appl. Mech. Rev. **32**, 123–135 (1979)

[2] J. Rice, *A path-independent integral and approximate analysis of strain concentration by notches and cracks*, J. Appl. Mech. **35**, 379–386 (1968)

[3] P. J. Olver, *Conservation laws in elasticity.* I, Arch. Rational. Mech. Anal. **85**, 131–160 (1984)

[4] J. K. Knowles and E. Sternberg, *On a class of conservation laws in linear and finite elasto-statics*, Arch. Rational. Mech. Anal. **44**, 187–211 (1972)

[5] D. C. Fletcher, *Conservation laws in linear elastodynamics*, Arch. Rational. Mech. Anal. **60**, 329–353 (1976)

[6] J. Qing, *Conservation laws in linear visco-elastodynamics*, J. of Elasticity **16**, 213–219 (1986)

[7] L. U. Ovsiannĭkov, *Group Analysis of Equations*, Academic Press, New York, 1982

[8] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986

[9] J. M. Ball, *Convexity conditions and existence theorems in non-linear elasticity*, Arch. Rational. Mech. Anal. **63**, 337–403 (1977)

[10] D. Terentev and J. D. Smyglevski, *Full set divergent equations of ideal gas dynamics*, Zh. Vychisl. Mat. i Mat. Fiz. **15** 1535–1544 (1976) (in Russian)

[11] L. G. Vulkov, *On the conservation laws in magnetohydrodynamics in infini co conductivity*, Zh. Vychisl. Mat. i Mat. Fiz. **25**, 1401–1408 (1985) (in Russian)

[12] C. A. Truesdell and W. Noll, *The Non-linear Field Theories of Mechanics*, in Handbuch der Physik, S. Fugge (Ed.), vol. III/3, Springer, Berlin (1965)

[13] B. L. Rozdestvenskii and N. N. Janenko, *Systems of Quasilinear Equations*, Nauka, Moscow, 1968 (in Russian)

[14] C. Dafermos, *The equations of elasticity are special*, Trends and Appl. of Pure Math. in Mech. **V**, 96–103 (1981)

[15] W. D. Collins, *One-dimensional non-linear wave propagation in incompressible elastic materials*, Quart. J. Mech. Appl. Math. **XIX**, Pt. 3, 259–328 (1965)