

A PRINCIPAL STRESS CONTOUR FOR NONLINEAR ELASTIC DEFORMATIONS

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Abstract. We seek special principal stress contours, for a nonlinear elastic plane deformation, on which both the principal stresses are constant and the principal axes do not change directions. It is found that they must be straight lines and that the lines must be parallel to one another or intersect at one point. It is also shown that the principal stress contours in the deformation for the straightening of a circular arc are such straight lines.

1. Introduction. The principal stress and strain trajectories for a nonlinear elastic plane deformation have been discussed by Ogden [1]. He obtained a system of four first-order partial differential equations governing the principal stretches of the plane deformation and the two angles that define the orientation of the Lagrangean and Eulerian principal axes of the deformation.

Assume in a nonlinear plane deformation of isotropic elastic material, the two in-plane principal stresses (or strains) and the angle representing the direction of the Eulerian principal axes can be expressed as functions of a single parameter, which in turn is a function of the position coordinates of the points. Letting the parameter be equal to some constant, we can obtain a curve in the plane of the deformation, and then along the curve the two principal stresses (and strains) are constant and the Eulerian principal axes do not change direction. Thus the curve is a special principal stress contour. Different values of the parameter produce different contours. In this paper we will discuss the properties of the contours.

To do so, we first in Sec. 2 express the deformation gradient in terms of principal stretches and the angles that define the orientations of the Lagrangean and Eulerian principal axes of the deformation. Then we write the constitutive equations for compressible isotropic elastic materials, the equilibrium equations, and compatibility equations. In Sec. 3 we will search for the special principal stress contours, in the deformed configuration, on which the principal stresses (and strains) are constant and the Eulerian axes do not change direction. We find that the contours must be straight lines and that the lines must be parallel to one another or intersect at one point. At the same time, we establish the governing equations in terms of the parameter for the

Received July 2, 1991.

1991 *Mathematics Subject Classification.* Primary 73G05.

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deformation that has this property. Finally in Sec. 4 we show that the straightening of a circular arc is an inhomogeneous deformation that has these principal stress contours. The principal stress contours lie along the straight lines deformed from the circumferential curves. This result is true for any isotropic material, although the precise expression for the deformation depends on the material properties.

2. Governing equations for nonlinear elastic deformations. We consider a cylinder of homogeneous, isotropically elastic material and establish a Cartesian coordinate system (X_1, X_2, X_3) such that the X_3 -axis is parallel to the generators of the cylinder and the X_1, X_2 -axes lie in the plane perpendicular to the generators. We assume that the applied forces are such that the cylinder undergoes a deformation

$$x_1 = x_1(X_1, X_2), \quad x_2 = x_2(X_1, X_2), \quad x_3 = \lambda X_3, \quad (2.1)$$

where X_i and x_i ($i = 1, 2, 3$) are the initial and final coordinates of the same point respectively and λ is constant. Then the X_3 -axis is one of the principal axes of the deformation, and the corresponding principal stretch is λ . The remaining two principal axes lie in the (X_1, X_2) -plane. These latter axes and the corresponding principal stretches can be completely determined from the in-plane deformation gradient tensor [2]

$$\mathbf{A} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{pmatrix}. \quad (2.2)$$

It can be shown [1, 3, 4] that this tensor can be expressed as

$$\mathbf{A} = \begin{pmatrix} \lambda_1 \cos \varphi \cos \psi + \lambda_2 \sin \varphi \sin \psi & \lambda_1 \sin \varphi \cos \psi - \lambda_2 \cos \varphi \sin \psi \\ \lambda_1 \cos \varphi \sin \psi - \lambda_2 \sin \varphi \cos \psi & \lambda_1 \sin \varphi \sin \psi + \lambda_2 \cos \varphi \cos \psi \end{pmatrix}. \quad (2.3)$$

Here λ_1 and λ_2 are the principal stretches in the (X_1, X_2) -plane, and φ and ψ describe the orientations of the Lagrangean and Eulerian principal axes in the plane. Then the inverse of \mathbf{A} has the form

$$\mathbf{A}^{-1} = \begin{pmatrix} \lambda_1^{-1} \cos \varphi \cos \psi + \lambda_2^{-1} \sin \varphi \sin \psi & \lambda_1^{-1} \cos \varphi \sin \psi - \lambda_2^{-1} \sin \varphi \cos \psi \\ \lambda_1^{-1} \sin \varphi \cos \psi - \lambda_2^{-1} \cos \varphi \sin \psi & \lambda_1^{-1} \sin \varphi \sin \psi + \lambda_2^{-1} \cos \varphi \cos \psi \end{pmatrix}. \quad (2.4)$$

Since the material is isotropic, the principal axes of the stress coincide with those of the strain. The Piola-Kirchhoff (or nominal) stress tensor in the plane has the form

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} s_1 \cos \varphi \cos \psi + s_2 \sin \varphi \sin \psi & s_1 \sin \varphi \cos \psi - s_2 \cos \varphi \sin \psi \\ s_1 \cos \varphi \sin \psi - s_2 \sin \varphi \cos \psi & s_1 \sin \varphi \sin \psi + s_2 \cos \varphi \cos \psi \end{pmatrix}, \quad (2.5)$$

where s_1 and s_2 are the Piola-Kirchhoff (or nominal) principal stresses in the plane of the deformation. From the relation between the Cauchy stress tensor \mathbf{T} and the Piola-Kirchhoff stress tensor \mathbf{S} , i.e.,

$$\mathbf{T} = (\det \mathbf{A})^{-1} \mathbf{A} \mathbf{S}$$

(see Spencer [5]), the Cauchy stress tensor has the form

$$\mathbf{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_1 \cos^2 \psi + t_2 \sin^2 \psi & (t_1 - t_2) \cos \psi \sin \psi \\ (t_1 - t_2) \cos \psi \sin \psi & t_1 \sin^2 \psi + t_2 \cos^2 \psi \end{pmatrix}, \quad (2.6)$$

in which t_1 and t_2 are the Cauchy principal stresses in the plane of the deformation.

The stress-strain relations for an isotropically elastic compressible material with the plane deformation are (see Spencer [5])

$$\begin{aligned} t_{\alpha\beta} &= a_0 \delta_{\alpha\beta} + a_1 B_{\alpha\beta} + a_2 B_{\alpha\gamma} B_{\gamma\beta}, \\ t_{3\alpha} &= t_{\alpha 3} = 0 \quad (\alpha, \beta = 1, 2), \\ t_{33} &= a_0 + a_1 \lambda^2 + a_2 \lambda^4, \end{aligned} \quad (2.7)$$

where

$$B_{\alpha\beta} = \frac{\partial x_\alpha}{\partial X_\gamma} \frac{\partial x_\beta}{\partial X_\gamma}, \quad (2.8)$$

$$\begin{cases} a_0 = 2I_3^{1/2} \partial W / \partial I_3, \\ a_1 = 2I_3^{-1/2} (\partial W / \partial I_1 + I_1 \partial W / \partial I_2), \\ a_2 = -2I_3^{-1/2} \partial W / \partial I_2, \end{cases} \quad (2.9)$$

$$\begin{cases} I_1 = B_{\alpha\alpha} + \lambda^2, \\ I_2 = [(B_{\alpha\alpha} + \lambda^2)^2 - (B_{11}^2 + B_{22}^2 + 2B_{12}^2 + \lambda^4)]/2, \\ I_3 = \lambda^2 (B_{11} B_{22} - B_{12}^2), \end{cases} \quad (2.10)$$

and $W = W(I_1, I_2, I_3)$ is the strain-energy function for the material.

If body forces are absent, then the equilibrium equations in the deformed configuration are

$$\frac{\partial t_{11}}{\partial x_1} + \frac{\partial t_{12}}{\partial x_2} = 0, \quad \frac{\partial t_{21}}{\partial x_1} + \frac{\partial t_{22}}{\partial x_2} = 0. \quad (2.11)$$

For the continuity of the deformation field x_1, x_2 , the stress and strain must satisfy certain compatibility conditions. These conditions are obtained by substituting (2.3) into

$$\frac{\partial^2 x_1}{\partial X_1 \partial X_2} = \frac{\partial^2 x_1}{\partial X_2 \partial X_1}, \quad \frac{\partial^2 x_2}{\partial X_1 \partial X_2} = \frac{\partial^2 x_2}{\partial X_2 \partial X_1}, \quad (2.12)$$

or (2.4) into

$$\frac{\partial^2 X_1}{\partial x_1 \partial x_2} = \frac{\partial^2 X_1}{\partial x_2 \partial x_1}, \quad \frac{\partial^2 X_2}{\partial x_1 \partial x_2} = \frac{\partial^2 X_2}{\partial x_2 \partial x_1}. \quad (2.13)$$

3. Special principal stress contours and governing equations for associated deformations. If we assume that in a plane deformation, t_1, t_2 , and ψ can be expressed as continuous functions:

$$t_1 = t_1(\alpha), \quad t_2 = t_2(\alpha), \quad \psi = \psi(\alpha) \quad (3.1)$$

of a parameter α , which in turn is a continuous function of x_1 and x_2 :

$$\alpha = \alpha(x_1, x_2), \quad (3.2)$$

then along the curve $\alpha(x_1, x_2) = \text{constant}$, in the deformed configuration, the principal stresses t_1 and t_2 are constant and the Eulerian principal axes do not change direction. The question is whether or not there exists such a deformation and what is the shape of the curve. It is well known that, in the case of a pure homogeneous deformation, the principal stresses are constant and the principal axes do not change direction throughout the plane of the deformation; therefore the expressions (3.1) and (3.2) are valid for any α . In the following we will show that if there is such an inhomogeneous deformation, then the curve must be a straight line. Giving different values to α will yield different straight lines, so we have a family of straight lines. It will further be shown that the lines must be parallel or all intersect at a single point. Then in the next section, we will give an example of such an inhomogeneous deformation.

Substituting (2.7) with (3.1) into the equilibrium equations (2.11), we have

$$\begin{aligned} \frac{d}{d\alpha}[t_1 \cos^2 \psi + t_2 \sin^2 \psi] \frac{\partial \alpha}{\partial x_1} + \frac{d}{d\alpha}[(t_1 - t_2) \cos \psi \sin \psi] \frac{\partial \alpha}{\partial x_2} &= 0, \\ \frac{d}{d\alpha}[(t_1 - t_2) \cos \psi \sin \psi] \frac{\partial \alpha}{\partial x_1} + \frac{d}{d\alpha}[t_1 \sin^2 \psi + t_2 \cos^2 \psi] \frac{\partial \alpha}{\partial x_2} &= 0. \end{aligned} \quad (3.3)$$

If there is a nonconstant continuous function $\alpha = \alpha(x_1, x_2)$, i.e., $\partial\alpha/\partial x_1 \neq 0$ and/or $\partial\alpha/\partial x_2 \neq 0$, then the determinant of the coefficients of $\partial\alpha/\partial x_1$ and $\partial\alpha/\partial x_2$ must vanish; i.e.,

$$\frac{d}{d\alpha}[t_1 \cos^2 \psi + t_2 \sin^2 \psi] \frac{d}{d\alpha}[t_1 \sin^2 \psi + t_2 \cos^2 \psi] - \left\{ \frac{d}{d\alpha}[(t_1 - t_2) \cos \psi \sin \psi] \right\}^2 = 0,$$

which yields, after some manipulation,

$$t_1' t_2' - (t_1 - t_2) \psi' = 0. \quad (3.4)$$

Here the prime denotes the derivatives with respect to α . This is one governing equation for the state of stress in the inhomogeneous deformation in terms of α . When (3.4) is satisfied, we can solve from (3.3) for $\partial\alpha/\partial x_1$ and $\partial\alpha/\partial x_2$. The solutions are

$$\begin{aligned} \frac{\partial \alpha}{\partial x_1} &= h(x_1, x_2) f_1(\alpha), \\ \frac{\partial \alpha}{\partial x_2} &= -h(x_1, x_2) f_2(\alpha), \end{aligned} \quad (3.5)$$

where $h(x_1, x_2)$ is an arbitrary function of x_1 and x_2 , and $f_1(\alpha)$ and $f_2(\alpha)$ are defined as

$$\begin{aligned} f_1(\alpha) &= \frac{d}{d\alpha}[(t_1 - t_2) \cos \psi \sin \psi], \\ f_2(\alpha) &= \frac{d}{d\alpha}[t_1 \cos^2 \psi + t_2 \sin^2 \psi]. \end{aligned} \quad (3.6)$$

If a value for α is specified, say C , then $\alpha(x_1, x_2) = C$ represents a curve in the (x_1, x_2) -plane, and the slope of its tangent is given by

$$\frac{dx_2}{dx_1} = -\frac{\partial\alpha/\partial x_1}{\partial\alpha/\partial x_2}. \quad (3.7)$$

From (3.5) we can see that the slope of the curve we want is

$$\frac{dx_2}{dx_1} = \frac{f_1(C)}{f_2(C)} \equiv M(C). \quad (3.8)$$

Along the curve, C is a constant, and hence the slope $M(C)$ at any point of the curve is constant too. This implies that the curve must be a straight line. The straight line may be expressed as

$$x_2 = x_1 M(C) + D(C), \quad (3.9)$$

where $D(C)$ is a constant, representing the position of the intersecting point of the straight line and the x_2 -axis. Different constants C correspond to different lines; different lines may have different slopes and different intersecting points with the coordinate axis.

The constant C is the value chosen for α ; thus Eq. (3.9) can be written in terms of α :

$$x_2 = x_1 M(\alpha) + D(\alpha). \quad (3.10)$$

We can see from (3.6) and (3.8) that the slope $M(\alpha)$ of the straight line is related to the state of stress by

$$\frac{d}{d\alpha}[(t_1 - t_2) \cos \psi \sin \psi] = M(\alpha) \frac{d}{d\alpha}[t_1 \cos^2 \psi + t_2 \sin^2 \psi], \quad (3.11)$$

but $D(\alpha)$ is arbitrary. Actually, arbitrariness of $D(\alpha)$ is equivalent to that of $h(x_1, x_2)$. To see this, differentiating both sides of (3.10) with respect to x_1 and x_2 respectively, we find

$$\begin{aligned} \frac{\partial \alpha}{\partial x_1} &= -\frac{M(\alpha)}{D'(\alpha) + x_1 M'(\alpha)}, \\ \frac{\partial \alpha}{\partial x_2} &= \frac{1}{D'(\alpha) + x_1 M'(\alpha)}. \end{aligned} \quad (3.12)$$

Then comparing these with (3.5), we have

$$h(x_1, x_2) = -[(D' + x_1 M')f_2]^{-1}.$$

It can also be shown that

$$\frac{\partial^2 \alpha}{\partial x_1 \partial x_2} = \frac{\partial^2 \alpha}{\partial x_2 \partial x_1} \quad (3.13)$$

for any $D(\alpha)$.

We have, from the equilibrium consideration, two equations that relate five parameters t_1 , t_2 , ψ , M , and D . More equations can be obtained from the compatibility conditions. It follows, from the isotropy of the material, that the principal stretches can also be expressed as functions of α :

$$\lambda_1 = \lambda_1(\alpha), \quad \lambda_2 = \lambda_2(\alpha). \quad (3.14)$$

Substituting (2.4) with (3.14) into (2.13), we have two equations from which we can solve for $\partial \varphi / \partial x_1$ and $\partial \varphi / \partial x_2$ in terms of λ_1 , λ_2 , ψ , α , and their derivatives.

These solutions are

$$\frac{\partial \varphi}{\partial x_1} = -\lambda_1 \lambda_2 \left\{ \left[-\frac{\cos \psi}{\lambda_1} \frac{d}{d\alpha} \left(\frac{\sin \psi}{\lambda_1} \right) + \frac{\sin \psi}{\lambda_2} \frac{d}{d\alpha} \left(\frac{\cos \psi}{\lambda_2} \right) \right] \frac{\partial \alpha}{\partial x_1} + \left[\frac{d}{d\alpha} \left(\frac{\cos^2 \psi}{2\lambda_1^2} \right) + \frac{d}{d\alpha} \left(\frac{\sin^2 \psi}{2\lambda_2^2} \right) \right] \frac{\partial \alpha}{\partial x_2} \right\}, \quad (3.15a)$$

$$\frac{\partial \varphi}{\partial x_2} = -\lambda_1 \lambda_2 \left\{ \left[-\frac{d}{d\alpha} \left(\frac{\sin^2 \psi}{2\lambda_1^2} \right) - \frac{d}{d\alpha} \left(\frac{\cos^2 \psi}{2\lambda_2^2} \right) \right] \frac{\partial \alpha}{\partial x_1} + \left[\frac{\sin \psi}{\lambda_1} \frac{d}{d\alpha} \left(\frac{\cos \psi}{\lambda_1} \right) - \frac{\cos \psi}{\lambda_2} \frac{d}{d\alpha} \left(\frac{\sin \psi}{\lambda_2} \right) \right] \frac{\partial \alpha}{\partial x_2} \right\}. \quad (3.15b)$$

It follows from (3.12) that

$$\frac{\partial \alpha}{\partial x_1} = -M(\alpha) \frac{\partial \alpha}{\partial x_2}. \quad (3.16)$$

Introducing (3.16) into (3.15), we have

$$\frac{\partial \varphi}{\partial x_1} = g_1(\alpha) \frac{\partial \alpha}{\partial x_2}, \quad \frac{\partial \varphi}{\partial x_2} = g_2(\alpha) \frac{\partial \alpha}{\partial x_2}, \quad (3.17)$$

where

$$g_1(\alpha) = \lambda_1 \lambda_2 \left\{ \left[-\frac{\cos \psi}{\lambda_1} \frac{d}{d\alpha} \left(\frac{\sin \psi}{\lambda_1} \right) + \frac{\sin \psi}{\lambda_2} \frac{d}{d\alpha} \left(\frac{\cos \psi}{\lambda_2} \right) \right] M(\alpha) - \left[\frac{d}{d\alpha} \left(\frac{\cos^2 \psi}{2\lambda_1^2} \right) + \frac{d}{d\alpha} \left(\frac{\sin^2 \psi}{2\lambda_2^2} \right) \right] \right\}, \quad (3.18a)$$

$$g_2(\alpha) = \lambda_1 \lambda_2 \left\{ \left[-\frac{d}{d\alpha} \left(\frac{\sin^2 \psi}{2\lambda_1^2} \right) - \frac{d}{d\alpha} \left(\frac{\cos^2 \psi}{2\lambda_2^2} \right) \right] M(\alpha) - \left[\frac{\sin \psi}{\lambda_1} \frac{d}{d\alpha} \left(\frac{\cos \psi}{\lambda_1} \right) - \frac{\cos \psi}{\lambda_2} \frac{d}{d\alpha} \left(\frac{\sin \psi}{\lambda_2} \right) \right] \right\} \quad (3.18b)$$

are functions of α . A continuous function $\varphi(x_1, x_2)$ can be obtained by integrating (3.17) when the following condition is satisfied:

$$\frac{\partial}{\partial x_2} \left(g_1(\alpha) \frac{\partial \alpha}{\partial x_2} \right) = \frac{\partial}{\partial x_1} \left(g_2(\alpha) \frac{\partial \alpha}{\partial x_2} \right). \quad (3.19)$$

The condition (3.19) places a further restriction on the stresses. Expanding (3.19), we have

$$g_1(\alpha) \frac{\partial^2 \alpha}{\partial x_2^2} + g_1'(\alpha) \left(\frac{\partial \alpha}{\partial x_2} \right)^2 = g_2(\alpha) \frac{\partial^2 \alpha}{\partial x_1 \partial x_2} + g_2'(\alpha) \frac{\partial \alpha}{\partial x_1} \frac{\partial \alpha}{\partial x_2},$$

and then making use of (3.13) and (3.16) yields

$$[g_1(\alpha) + M(\alpha)g_2(\alpha)] \frac{\partial^2 \alpha}{\partial x_2^2} + [g_1'(\alpha) + M(\alpha)g_2'(\alpha) + M'(\alpha)g_2(\alpha)] \left(\frac{\partial \alpha}{\partial x_2} \right)^2 = 0,$$

which can be rewritten as

$$\frac{g_1' + (Mg_2)'}{g_1 + Mg_2} = -\frac{\partial^2 \alpha / \partial x_2^2}{(\partial \alpha / \partial x_2)^2}.$$

Introducing the second equation of (3.12) into this, we obtain

$$\frac{g_1' + (Mg_2)'}{g_1 + Mg_2} = \frac{D''(\alpha) + x_1 M''(\alpha)}{D'(\alpha) + x_1 M'(\alpha)}. \quad (3.20)$$

Since the left-hand side is the function of α alone, so too is the right-hand side. Thus one of the following three must hold:

$$\begin{aligned} (1) \quad & M'(\alpha) = 0 \quad \text{while } D'(\alpha) \neq 0, \\ (2) \quad & M'(\alpha) \neq 0 \quad \text{while } D' = 0, \\ (3) \quad & M'(\alpha) \neq 0, \quad D'(\alpha) \neq 0 \quad \text{while } \frac{D''(\alpha)}{D'(\alpha)} = \frac{M''(\alpha)}{M'(\alpha)}. \end{aligned} \quad (3.21)$$

The first one implies that the straight lines are parallel. The second one means that the straight lines intersect at a point on the x_2 -axis. From the third condition, we have

$$D(\alpha) = C_0 M(\alpha) + C_1.$$

Here C_0 and C_1 are constants. Introducing this into (3.10) we obtain

$$x_2 = M(\alpha)(x_1 + C_0) + C_1.$$

This implies that the lines intersect at one point $(-C_0, C_1)$ too, but not on the x_2 -axis. So the straight lines must be parallel or intersect at one point. Accordingly the condition (3.20) becomes

$$\frac{g_1' + (Mg_2)'}{g_1 + Mg_2} = \frac{D''(\alpha)}{D'(\alpha)} \quad \text{if } M'(\alpha) = 0, \quad (3.22)$$

$$\frac{g_1' + (Mg_2)'}{g_1 + Mg_2} = \frac{M''(\alpha)}{M'(\alpha)} \quad \text{if } M'(\alpha) \neq 0. \quad (3.23)$$

Integrating both sides with respect to α , we have

$$g_1(\alpha) + M(\alpha)g_2(\alpha) = C_2 D'(\alpha) \quad \text{if } M'(\alpha) = 0, \quad (3.24)$$

$$g_1(\alpha) + M(\alpha)g_2(\alpha) = C_3 M'(\alpha) \quad \text{if } M'(\alpha) \neq 0. \quad (3.25)$$

Here C_2 and C_3 are arbitrary constants.

We note here that when $M'(\alpha) = D'(\alpha) = 0$, all the straight lines collapse onto a single line since this is the only possibility if the lines are to be parallel ($M'(\alpha) = 0$) and intersect at one point ($D'(\alpha) = 0$). This singular case, which we will not discuss further, indicates the material would collapse onto a plane.

With the aid of the constitutive relations, Eqs. (3.24) and (3.25) can be transformed into the equations for the principal stresses and ψ , and Eqs. (3.4) and (3.11) can be transformed into the equations for the principal stretches. The five quantities $t_1, t_2, \psi, M(\alpha)$, and $D(\alpha)$ satisfy one of (3.24) or (3.25) and Eqs. (3.4) and (3.11);

these were derived from the equations of equilibrium and the compatibility conditions. At the same time φ may be found by integrating (3.15) or (3.17). Thus we can choose two of the five variables independently and then determine the others from the equations.

Before concluding this section, we note the following special cases.

(1) First suppose $\psi'(\alpha) = 0$, so that $\psi = \text{constant}$; then the principal axes do not change directions. For, without loss of generality, we may assume $\psi = 0$. Then Eq. (3.4) requires $t'_2 = 0$ but $t'_1 \neq 0$ (for inhomogeneous deformation) or $t'_1 = 0$ but $t'_2 \neq 0$. Assume the former holds; then (3.11) yields

$$M(\alpha) = 0. \quad (3.26)$$

Therefore the lines are parallel, and further they are parallel to the x_1 -axis. Introducing (3.26) into (3.24) we obtain

$$g_1(\alpha) = C_2 D'(\alpha),$$

with $g_1(\alpha)$ being given by

$$g_1(\alpha) = -\lambda_1 \lambda_2 \frac{d}{d\alpha} \left(\frac{1}{2\lambda_1^2} \right) = \frac{\lambda_2 \lambda_1'}{\lambda_1^2}. \quad (3.27)$$

If we further specify $D'(\alpha) \neq 0$ and the response relation $t_2 = t_2(\lambda_1, \lambda_2)$, then λ_1, λ_2 may be determined from

$$t_2(\lambda_1, \lambda_2) = t_{20}, \quad \frac{\lambda_2 \lambda_1'}{\lambda_1^2} = C_2 D'(\alpha), \quad (3.28)$$

with t_{20} being a constant.

(2) The second case is that $M(\alpha) = 0$, i.e., $M'(\alpha) = 0$. Then Eqs. (3.11) and (3.24) yield

$$(t_1 - t_2) \cos \psi \sin \psi = C_4, \quad (3.29)$$

$$\frac{d}{d\alpha} \left[\frac{\cos^2 \psi}{\lambda_1^2} + \frac{\sin^2 \psi}{\lambda_2^2} \right] = -\frac{2C_2 D'(\alpha)}{\lambda_1 \lambda_2}, \quad (3.30)$$

with C_4 being a constant. The principal stresses t_1, t_2 , and the angle ψ should be determined from (3.4), (3.29), and (3.30) when $D'(\alpha)$ and the response relations are given.

For these first two cases where $M(\alpha) = 0$, we have from (3.10) that $x_2 = D(\alpha)$, so that $\alpha = \alpha(x_2)$; i.e., α does not depend on x_1 .

(3) In the third case take $M(\alpha) = \tan \alpha$ and $D(\alpha) \equiv D_0 = \text{constant}$; then Eq. (3.10) gives

$$\alpha = \arctan \left(\frac{x_2 - D_0}{x_1} \right).$$

Thus the straight lines intersect at a point on the x_2 -axis, and Eqs. (3.11) and (3.25)

become

$$\begin{aligned}
 & t'_1 \cos \psi \sin(\psi - \alpha) - t'_2 \sin \psi \cos(\psi - \alpha) + (t_1 - t_2) \psi' \cos(2\psi - \alpha) = 0, \\
 & \frac{\lambda'_1}{\lambda_1^3} \cos^2(\psi - \alpha) + \frac{\lambda'_2}{\lambda_2^3} \sin^2(\psi - \alpha) + \left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} \right) \psi' \cos(\psi - \alpha) \sin(\psi - \alpha) = \frac{C_2}{\lambda_1 \lambda_2}.
 \end{aligned}
 \tag{3.31}$$

The principal stresses and the angle may be determined from (3.4) and (3.31) when the response relations are given.

4. Example—straightening of a circular arc. In this section we will show, by example, that there is an inhomogeneous deformation in which the principal stresses (or strains) are constant and the Eulerian principal axes do not change direction along a straight line. The example considers the straightening of a circular arc. Introducing the cylindrical polar coordinate system (R, Θ, X_3) in the undeformed configuration with

$$R = (X_1^2 + X_2^2)^{1/2}, \quad \Theta = \arctan \left(\frac{X_2}{X_1} \right), \tag{4.1}$$

we can describe the deformation as

$$\begin{aligned}
 x_1 &= f(\Theta) = f \left(\arctan \left(\frac{X_2}{X_1} \right) \right), \\
 x_2 &= g(R) = g((X_1^2 + X_2^2)^{1/2}), \\
 x_3 &= \lambda X_3.
 \end{aligned}
 \tag{4.2}$$

Therefore the in-plane deformation gradient tensor and the left Cauchy-Green deformation tensor are

$$\mathbf{A} = \begin{pmatrix} -f'(\Theta) \frac{X_2}{X_1^2 + X_2^2} & f'(\Theta) \frac{X_1}{X_1^2 + X_2^2} \\ g'(R) \frac{X_1}{(X_1^2 + X_2^2)^{1/2}} & g'(R) \frac{X_2}{(X_1^2 + X_2^2)^{1/2}} \end{pmatrix}, \tag{4.3}$$

$$\mathbf{B} = \mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} \frac{[f'(\Theta)]^2}{R^2} & 0 \\ 0 & [g'(R)]^2 \end{pmatrix}. \tag{4.4}$$

It follows that the Cauchy principal axes coincide with the coordinate axes; so $\psi = 0$ everywhere, and the in-plane principal stretches are

$$\lambda_1 = \frac{f'(\Theta)}{R} = \frac{f'(f^{-1}(x_1))}{g^{-1}(x_2)}, \quad \lambda_2 = g'(R) = g'(g^{-1}(x_2)). \tag{4.5}$$

Here we have assumed that the inverses of the functions $f(\Theta)$ and $g(R)$ exist.

The stretch λ_2 is a function of x_2 alone; therefore the parameter function α must depend on x_2 only, i.e., $\alpha = \alpha(x_2)$, in order to express λ_2 in terms of α . Furthermore $f'(\Theta)$ must be constant so that λ_1 can be expressed in terms of α too. Thus we have

$$x_1 = f(\Theta) = C_4 \Theta + x_{10}, \quad x_2 = g(R), \tag{4.6}$$

and

$$\lambda_1 = C_4/R = C_4/g^{-1}(x_2) = \lambda_1(\alpha), \quad \lambda_2 = g'(g^{-1}(x_2)) = \lambda_2(\alpha), \quad (4.7)$$

with C_4 and x_{10} being constants. The angle ψ is zero everywhere, and along the straight lines $x_2 = \text{a constant}$, λ_1 and λ_2 do not change value. Thus the class of deformations (4.6), which is not homogeneous, satisfies the assumptions embodied in (3.1) and (3.2); and the straight lines $x_2 = \text{a constant}$ are the special principal stress contours we are seeking. Note that this conclusion holds for any general isotropic material.

Since in this class of deformations $\psi \equiv 0$, we have the special case (1) discussed at the end of the last section. Therefore the principal stretches λ_1 and λ_2 are to be determined from (3.28) when the response relations are given. We have

$$\lambda_1' \equiv \frac{d\lambda_1}{d\alpha} = \frac{d\lambda_1}{dx_2} \frac{dx_2}{d\alpha} = D'(\alpha) \frac{d\lambda_1}{dR} \bigg/ \frac{dx_2}{dR} = -\frac{D'(\alpha)}{C_4} \lambda_1^2 / \lambda_2;$$

thus the second equation of (3.28) becomes $C_2 C_4 = -1$, which just relates two constants and determines the constant C_4 for the principal stretch λ_1 . The principal stretch λ_2 or the function $g(R)$ may then be determined from the first equation of (3.28).

5. Conclusion. In this paper it has been shown that in nonlinear plane deformation the straight line is the only possible principal stress contour on which the principal stretches (stresses) are of constant value and the principal axes do not change direction. It has also been shown that the principal stress contours in the deformation for the straightening of a circular arc are such straight lines.

Acknowledgment. The work in this paper was initially carried out at the University of Nottingham and modified at the University of Queensland. The author is very grateful to Dr. W. A. Green for his suggestion of the problem and constructive guide on the initial research work. Thanks are also to Dr. V. G. Hart and Dr. J. A. Belward for their helpful discussion and careful check. The financial supports of the Agricultural Ministry of the People's Republic of China and the Department of Mathematics of the University of Queensland are acknowledged.

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