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ON THE SOLUTION OF THE EQUATION $u_t + u^n u_x + H(x, t, u) = 0$

By

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Abstract. We consider the equation $u_t + u^n u_x + H(x, t, u) = 0$ and derive a transformation relating it to $u_t + u^n u_x = 0$. Special cases of the equation appearing in applications are discussed. Initial value problems and asymptotic behaviour of the solution are studied.

1. Introduction. The scalar conservation law

$$u_t + u^n u_x = 0 \tag{1.1}$$

has been treated extensively. The basic work of Hopf [6] and Lax [7] for $n = 1$ led to considerable work for the system of conservation laws. Eq. (1.1) with $n = 1$ is the inviscid limit of the Burgers equation,

$$u_t + uu_x = \frac{\delta}{2} u_{xx}, \tag{1.2}$$

as $\delta \to 0$. Equation (1.2) can be linearised to the heat equation via a Hopf-Cole transformation (see Hopf [6]). The generalised Burgers equation

$$u_t + u^n u_x + H(x, t, u) = \frac{\delta}{2} u_{xx}, \tag{1.3}$$

whose special cases describe a large number of physical models (see Sachdev [12]), does not, in general, admit a Hopf-Cole transformation. The only exception is the inhomogeneous Burgers equation

$$u_t + uu_x + f(x, t) = \frac{\delta}{2} u_{xx} \tag{1.4}$$

(see Nimmo and Crighton [11]), for which a Hopf-Cole transformation exists changing it to a linear parabolic equation.

The inviscid form of (1.3), namely

$$u_t + u^n u_x + H(x, t, u) = 0, \tag{1.5}$$

is of importance in the sense that (1.1) is. The inhomogeneous term in (1.5) represents the effects of damping and/or geometrical spreading. Equation (1.5) also

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plays an important role in the analysis of (1.3). The singular perturbation analysis of Crighton and Scott [3] and generalised similarity analysis of Sachdev et al. [13, 14] show that the inviscid solution is of primary importance for N-wave initial conditions. In the former it is the outer solution correct to all orders, while in the latter it gives precise information near the node of the wave and helps build a uniformly valid solution for all time after a finite initial time.

The purpose of this paper is to study (1.5) in some detail. Section 2 derives a transformation relating Eq. (1.1) to Eq. (1.5); special cases of (1.5) appearing in applications are also discussed. Section 3 deals with the initial value problem for (1.5) over the real line for $t > 0$, while Sec. 4 gives its asymptotic behaviour. An interesting special case of (1.5) is considered in detail in Sec. 5 and related to the work of Murray [9].

2. The transformation. In this section we seek the most general equation of the type

$$u_t + u^n u_x + H(x, t, u) = 0, \tag{2.1}$$

which can be reduced to the homogeneous conservation form

$$v_\tau + v^n v_y = 0, \tag{2.2}$$

via the transformation

$$\tau = \tau(x, t), \tag{2.3}$$

$$y = y(x, t), \tag{2.4}$$

$$v(y, \tau) = f(x, t)u(x, t). \tag{2.5}$$

A more general form $v = F(x, t, u)$ is not considered, so that the Rankine-Hugoniot conditions for (2.1) and (2.2) are the same. Here we assume that $f(x, t) > 0$ and

$$J = \det \begin{pmatrix} y_t & y_x \\ \tau_t & \tau_x \end{pmatrix} \neq 0. \tag{2.6}$$

Differentiating (2.5) with respect to x and t, we get

$$\begin{aligned} v_y y_x + v_\tau \tau_x &= f u_x + f_x u, \\ v_y y_t + v_\tau \tau_t &= f u_t + f_t u. \end{aligned} \tag{2.7}$$

Solving for v_τ and v_y from (2.7) we obtain

$$\begin{aligned} v_\tau &= -\frac{1}{J}[(y_x f_t - y_t f_x)u + y_x f u_t - y_t f u_x], \\ v_y &= -\frac{1}{J}[\tau_t f_x - \tau_x f_t)u + \tau_t f u_x - \tau_x f u_t]. \end{aligned} \tag{2.8}$$

Substituting (2.8) in (2.2) we have

$$(y_x f)u_t + (y_x f_t - y_t f_x)u - y_t f u_x + f^n u^n[(\tau_t f_x - \tau_x f_t)u + (\tau_t f u_x - \tau_x f u_t)] = 0. \tag{2.9}$$

For (2.9) to be of the form (2.1), we must have

$$y_t = 0, \quad \tau_x = 0, \quad \text{and} \quad \frac{f^n \tau_t}{y_x} = 1. \tag{2.10}$$

Using (2.10) we get from (2.9)

$$u_t + u^n u_x + \frac{f_x}{f} u^{n+1} + \frac{f_t}{f} u = 0. \tag{2.11}$$

From (2.10) we find that y is a function of x and τ is a function of t alone so that

$$f(x, t) = \left(\frac{\frac{dy}{dx}}{\frac{d\tau}{dt}} \right)^{1/n}. \tag{2.12}$$

Eq. (2.11) becomes

$$u_t + u^n u_x + G(t)u + F(x)u^{n+1} = 0, \tag{2.13}$$

where

$$G(t) = -\frac{d}{dt} \log \left(\frac{d\tau}{dt} \right)^{1/n}, \tag{2.14}$$

and

$$F(x) = \frac{d}{dx} \log \left(\frac{dy}{dx} \right)^{1/n}. \tag{2.15}$$

Conversely for given $G(t)$ and $F(x)$, the differential equations (2.14) and (2.15) determine the transformation functions

$$\tau = \tau(t) = \int^t \left(\exp \left(\int^s G(s_1)\, ds_1 \right) \right)^{-n} ds, \tag{2.16}$$

$$y = y(x) = \int^x \left(\exp \left(\int^s F(s_1)\, ds_1 \right) \right)^{n} ds, \tag{2.17}$$

and

$$f(x, t) = \exp \left(\int^t G(s)\, ds \right) \cdot \exp \left(\int^x F(y)\, dy \right). \tag{2.18}$$

Now if y, τ, and f are given by (2.16)–(2.18), then (2.13) transforms to (2.2) via (2.3)–(2.5). Thus we arrive at the following result: the most general equation of the form (2.1) that can be reduced to (2.2) by the transformation (2.3)–(2.5) is (2.13); the transformation is given by (2.16)–(2.18).

Equations of the type (2.13) appear quite naturally in many physical applications and have been considered by several authors. Nimmo and Crighton [11] considered the case $n = 1$ with $F(x) \equiv 0$ and $G(t) = (\frac{j}{2t} + \alpha)$, $j = 0, 1, 2$. In this case (2.13) becomes

$$u_t + uu_x + \left(\frac{j}{2t} + \alpha \right) u = 0. \tag{2.19}$$

Using the transformation

$$y = x, \quad \tau = \int^t s^{-j/2} e^{-\alpha s}\, ds, \quad v(y, \tau) = f(x, t)u, \tag{2.20}$$

where $f(x, t) = e^{\alpha t} t^{j/2}$, in (2.19), we get $v_\tau + vv_y = 0$. Nimmo and Crighton analysed the periodic solution for (2.19) with sinusoidal initial conditions.

Lefloch [8] considered (2.13) when $n = 1$, $G(t) \equiv 0$, and $F(x) = \frac{\beta}{x}$, that is, the equation

$$u_t + uu_x + \frac{\beta}{x}u^2 = 0. \tag{2.21}$$

The transformation

$$y = x^{\beta+1}, \quad \tau = t, \quad V = (\beta+1)x^\beta u \tag{2.22}$$

was used to change (2.21) to $v_\tau + vv_y = 0$. Using the formulation of Bardos, Leroux, and Nedelec [1] for the initial boundary value problem for (2.2) with $n = 1$, Lefloch formulated an initial boundary value problem for (2.21) in the quadrant $x > 0$, $t > 0$.

When $n = 1$, $F(x) = \frac{1}{x}$, $G(t) = 2k$, $k > 0$, Eq. (2.13) assumes the form

$$u_t + uu_x + \frac{u^2}{x} + 2ku = 0. \tag{2.23}$$

Wedemeyer [16] considered (2.23) with boundary condition $u(0, t) = 0$ and obtained the solution

$$u(x, t) = \begin{cases} -kx, & \text{if } kx \le e^{-kt} \\ \frac{(kx)^2 e^{2kt} - 1}{(kx)(e^{2kt} - 1)} - kt, & \text{if } kx > e^{-kt}. \end{cases} \tag{2.24}$$

Recently, Eq. (2.23) was also considered by Dolzhanskii et al. [5], and an implicit relation was obtained for a special class of solutions, namely

$$e^{2kt} + \left(\frac{u + kx}{u}\right) T(kx(u + kx)) = 0, \tag{2.25}$$

where T is an arbitrary function to be determined by initial or boundary data.

Murray [9] considered the equation

$$u_t + g(u)u_x + \lambda u^\alpha = 0, \tag{2.26}$$

where $g'(u) > 0$, for $u > 0$ and λ is a positive constant. He analysed the asymptotic behaviour of solutions of (2.26) that satisfy some initial conditions that are compactly supported. We recover his results exactly for the special case $g(u) = u$ and $\alpha = 2$ in Sec. 5, using the transformation (2.3)–(2.5).

The equation

$$u_t + uu_x + u^2 - u = 0 \tag{2.27}$$

is the inviscid limit of the Burgers-Fisher equation

$$u_t + uu_x + u(u - 1) = \frac{\delta}{2}u_{xx},$$

which was proposed by Murray [10]. When we take $F(x) = 1$ and $G(t) = -1$ in Eq. (2.13) we get Eq. (2.27). The transformation

$$y = e^x, \quad \tau = e^t, \quad v(y, \tau) = e^{x-t}u(x, t)$$

can be used to change (2.27) to $v_\tau + vv_y = 0$.

3. Initial value problem. In this section we study the initial value problem for (2.13) with the initial condition

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty. \tag{3.1}$$

We take $n = 1$. Under the transformation (2.16)–(2.18), (2.13) and (3.1) become

$$v_\tau + v v_y = 0, \qquad v(y, 0) = v_0(y). \tag{3.2}$$

Here $v_0(y)$ is obtained from $u_0(x)$ through $v_0(y) = f(x, 0)u_0(x)$, where $y = \int^x (\exp \int^{x_1} F(y)\, dy)\, dx_1$ and $f(x, 0) = \exp(\int^x F(y)\, dy)$. We assume $y(\pm\infty) = \pm\infty$. Hopf [6] obtained an explicit formula for the solution of (3.2), namely

$$v(y, \tau) = \frac{y - z_0(y, \tau)}{\tau}, \tag{3.3}$$

where $z_0(y, \tau)$ is a minimiser for the problem

$$\operatorname*{Min}_{-\infty<z<\infty} \left[\int_0^z v_0(z_1)\, dz_1 + \frac{(y-z)^2}{2t} \right]. \tag{3.4}$$

From (3.3) and (3.4), we have the following explicit formula for the solution $u(x, t)$ of (2.13) and (3.1):

$$\exp\left(\int^t G(s)\, ds\right) \exp\left(\int^x F(y)\, dy\right) u(x, t) = \frac{y - z_0(y, \tau)}{\tau},$$

where y and τ are given by (2.16) and (2.17) and $z_0(y, \tau)$ is as given above.

Initial boundary value problems are well studied for the equation $v_\tau + v v_y = 0$ (see Bardos, Leroux, and Nedelec [1]). One can through the transformation (2.16)–(2.18) formulate an initial boundary value problem for (2.13) and hence analyse the solution.

4. Asymptotic behaviour. In this section we study the asymptotic behaviour of solutions of (2.13). First we recall the decay results for (2.2) obtained by Lax [7] and Dafermos [4]. Let $v(y, \tau)$ be the solution of

$$\begin{aligned} &v_\tau + v^n v_y = 0, \qquad -\infty < y < \infty, \\ &v(y, 0) = v_0(y) = \begin{cases} f(y) & \text{if } -\infty < a < y < b < \infty \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{4.1}$$

Then

$$|v(y, \tau)| \le \frac{c}{\tau^{1/(n+1)}}. \tag{4.2}$$

From this result we deduce the decay results for (2.13). Let the function $u_0(x)$ have compact support; that is,

$$u_0(x) = \begin{cases} g(x) & \text{if } -\infty < a < x < b < \infty, \\ 0 & \text{otherwise}. \end{cases} \tag{4.3}$$

Let us assume that the functions $G(t)$ and $F(x)$ are such that

$$\tau = \tau(t) = \int^t \left(\exp\left(\int^s G(s_1)\, ds_1\right)\right)^{-n} ds$$

and

$$y = y(x) = \int^{x} \left(\exp\left(\int^{x_1} F(y)\,dy\right)\right)^{n} dx_1 \tag{4.4}$$

satisfy the conditions

$$\tau(+\infty) = +\infty, \qquad y(\pm\infty) = \pm\infty. \tag{4.5}$$

Let $u(x, t)$ be the solution of (2.13) with initial condition

$$u(x, 0) = u_0(x) \tag{4.6}$$

with $u_0(x)$ given by (4.3); then

$$|\exp\left(\int^{x} F(y)\,dy\right) u(x, t)| \le \frac{C}{(\exp(\int^{t} G(s)\,ds))(\int^{t}(\exp(\int^{s} G(s_1)\,ds_1))^{-n}\,ds)^{1/(n+1)}}. \tag{4.7}$$

For the special case when $n = 1$, $G(t) = \frac{j}{2t} + \alpha$, $F(x) = \frac{\beta}{x} + \lambda$, $\beta > 0$, $\lambda > 0$, $\alpha < 0$, we obtain the following asymptotic law:

$$|u(x, t)| \le \frac{Cx^{-\beta}e^{-\lambda x}}{t^{j/2}e^{\alpha t}}. \tag{4.8}$$

This choice includes all the equations (2.19), (2.21), (2.23), and (2.27) as special cases. Lax [7] has shown that for $n = 1$, the solution of (3.2) goes to N-wave as $\tau \to \infty$ in the L^1-norm. More precisely, let

$$p = -2\min_{y} \int_{-\infty}^{y} v_0(z)\,dz, \qquad q = 2\max_{y} \int_{y}^{\infty} v_0(z)\,dz, \tag{4.9}$$

and let

$$\overline{v}(y, \tau) = \begin{cases} \frac{y}{\tau} & \text{if } -(p\tau)^{1/2} < y < (q\tau)^{1/2}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.10}$$

Let $v(y, \tau)$ be the solution of (3.2); then $v(y, \tau) \approx \overline{v}(y, \tau)$ in the L^1-norm w.r.t. x.

This result can be generalised to the case of Eq. (2.13) when τ and y satisfy (4.5):

$$\exp\left(\int^{x} F(y)\,dy\right) \exp\left(\int^{t} G(s)\,ds\right) u(x, t) \approx \begin{cases} \frac{y}{\tau} & \text{if } -(p\tau)^{1} < y < (q\tau)^{1/2}, \\ 0 & \text{otherwise} \end{cases}$$

in the L^1-norm, where $y = y(x)$ and $\tau = \tau(t)$ are given by (4.4) with $n = 1$. For the case $G(t) = \frac{j}{2t} + \alpha$, $F(x) = \frac{\beta}{x} + \lambda$ we have

$$x^{\beta}e^{\lambda(x-1)}t^{j/2}e^{\alpha(t-1)}u(x, t) \approx \begin{cases} \frac{y}{\tau} & \text{if } -(p\tau)^{1/2} < y < (q\tau)^{1/2}, \\ 0 & \text{otherwise}, \end{cases}$$

where $y = \int_0^x s^{\beta}e^{\lambda(s-1)}\,ds$, $\tau = \int_1^t s^{-j/2}e^{\alpha(s-1)}\,ds$, p and q are defined by (4.9), and $v_0(y)$ is obtained from $u_0(x)$ through the transformation

$$v_0(y) = x^{\beta}e^{\lambda(x-1)-\alpha}u_0(x).$$

5. An example. In connection with propagation of waves in tubes, Shih [15] proposed the equations

$$\left(\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial t}\right)\left(u \pm \frac{2a}{\gamma + 1}\right) = -\frac{f}{D}u|u|$$

in which a term proportional to u^2 with Chézy friction coefficient f is used to model wall losses; D is another constant. For right running waves alone we get the equation

$$u_t + \left(a_0 + \frac{\gamma + 1}{2}u\right)u_x + \frac{F}{4D}u^2 = 0 \tag{5.0}$$

(see Crighton [2]). By simple scaling, Eq. (5.0) can be written as

$$u_t + (a + u)u_x + u^2 = 0. \tag{5.1}$$

Set $x^* = x - at$; then (5.1) becomes

$$u_t + uu_{x^*} + u^2 = 0. \tag{5.2}$$

We consider the initial value problem for (5.1) with

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty. \tag{5.3}$$

Let

$$y(x^*) = e^{x^*}, \qquad V(y, t) = e^{x^*}u(x^*, t) = yu(\log y, t). \tag{5.4}$$

In terms of the variable $V(y, t) = yu(\log y, t)$, we write the problem (5.1) and (5.3) as

$$\begin{gathered} V_t + VV_y = 0, \qquad y > 0,\ t > 0, \\ V(0, t) = 0, \qquad V(y, 0) = V_0(y) = yu_0(\log y). \end{gathered} \tag{5.5}$$

Following Hopf [6] we solve (5.5) by the vanishing viscosity method. Let $V^\varepsilon(y, t)$ be the solution of

$$\begin{gathered} V_t^\varepsilon + V^\varepsilon V_y^\varepsilon = \frac{\varepsilon}{2}V_{yy}^\varepsilon, \qquad y > 0,\ t > 0, \\ V^\varepsilon(0, t) = 0, \qquad V(y, 0) = V_0(y). \end{gathered} \tag{5.6}$$

Using the Hopf-Cole transformation

$$z^\varepsilon(y, t) = -\varepsilon(\log V^\varepsilon)_y, \tag{5.7}$$

we get

$$\begin{gathered} z_t^\varepsilon = \frac{\varepsilon}{2}z_{yy}^\varepsilon, \qquad y > 0,\ t > 0, \\ z_y^\varepsilon(0, t) = 0, \\ z^\varepsilon(y, 0) = e^{-(1/\varepsilon)W_0(y)}, \qquad W_0(y) = \int_0^y V_0(y)\,dy. \end{gathered} \tag{5.8}$$

We can explicitly write the solution of (5.8) as

$$\begin{aligned} z^\varepsilon(y, t) = \frac{1}{(2\pi t\varepsilon)^{1/2}} \Bigg[&\int_0^\infty \exp\left(-\frac{1}{\varepsilon}\left[\frac{(y - z)^2}{2t} + W_0(z)\right]\right) dz \\ &+ \int_0^\infty \exp\left(-\frac{1}{\varepsilon}\left[\frac{(y + z)^2}{2t} + W_0(z)\right]\right) dz \Bigg] \end{aligned} \tag{5.9}$$

and hence from (5.7) and (5.9)

$$V^{\varepsilon}(y,t)=\frac{\int_0^\infty(\frac{y-z}{t})\cdot\exp(-\frac{1}{\varepsilon}[\frac{(y-z)^2}{2t}+W_0(z)]\,dz+\int_0^\infty(\frac{y+z}{t})\exp(-\frac{1}{\varepsilon}[\frac{(y+z)^2}{dt}+W_0(z)]\,dz}{\int_0^\infty\exp(-\frac{1}{\varepsilon}[\frac{(y-z)^2}{dt}+W_0(z)]\,dz+\int_0^\infty\exp(-\frac{1}{\varepsilon}[\frac{(y+z)^2}{2t}+W_0(z)])\,dz} \tag{5.10}$$

Assume $y>0$. It is clear that

$$-\left[\frac{(y+z)^2}{2t}+W_0(z)\right]<-\left[\frac{(y-z)^2}{2t}+W_0(z)\right].$$

It then follows from the analysis of Hopf [6] and the use of the method of stationary phase that, as $\varepsilon\to 0$,

$$V^{\varepsilon}(y,t)\approx\frac{(\frac{y-z_0(y,t)}{t})\cdot\exp(-\frac{1}{\varepsilon}[\frac{(y-z_0(y,t))^2}{2t}+W_0(z_0(y,t))])}{\exp(-\frac{1}{\varepsilon}[\frac{(y-z_0(y,t))^2}{2t}+W_0(z_0(y,t))])}, \tag{5.11}$$

where $z_0(y,t)$ is a minimizer for the problem

$$\min_{z\geq 0}\left[\frac{(y-z)^2}{2t}+W_0(z)\right]. \tag{5.12}$$

From (5.11) we have the following explicit formula:

$$V(y,t)=\frac{y-z_0(y,t)}{t}, \tag{5.13}$$

where $z_0(y,t)$ is as defined earlier.

Writing (5.13) in terms of u, we have

$$u(x,t)=y^{-1}\left[\frac{y-z_0(y,t)}{t}\right], \tag{5.14}$$

where $y=e^{X}=e^{x-at}$. That is,

$$u(x,t)=\frac{1-e^{-(x-at)}z_0(e^{x-at},t)}{t}, \tag{5.15}$$

where $z_0(y,t)$ has been defined earlier.

We can read off the decay result from (5.15). For this purpose consider the minimisation problem (5.12). Let

$$u_0(x)=\begin{cases} f(x) & \text{for } -\infty<c<x<d<\infty; \\ 0 & \text{otherwise}; \end{cases} \tag{5.16}$$

then $V_0(y)=yu_0(\log y)$, and $W_0(z)=\int_0^z y_1u_0(\log y_1)\,dy_1$. Let

$$H(z,y,t)=\frac{(x-y)^2}{2t}+W_0(z). \tag{5.17}$$

At the point of minimum $z_0=z_0(y,t)$ of $\mathrm{Min}_{z\geq 0}H(z,y,t)$,

$$\frac{\partial H}{\partial z}(z_0,y,t)=0.$$

This gives

$$v_0(z_0) = \frac{z_0 - y}{t}. \tag{5.18}$$

If z_0 is outside the interval $[e^c, e^d]$, then $v_0(z_0) = 0$. It follows from (5.18) that $z_0 = y$ in this case. If z_0 is inside the interval $[e^c, e^d]$, clearly $|z_0| \le e^d$. Combining the two cases we get the estimate for $z_0(y, t)$:

$$|z_0(y, t)| \le e^d + y. \tag{5.19}$$

Using the estimate (5.19) in (5.15) we obtain the following result: let $u(x, t)$ be the solution of (5.1) with initial condition (5.3), where $u_0(x)$ is given by (5.16); then

$$|u(x, t)| \le \frac{1 + e^{-(x-at)}[e^d + e^{(x-at)}]}{t} = \frac{2 + e^{d-x+at}}{t}. \tag{5.20}$$

When $a = 0$ in (5.20) we recover Murray's result [9].

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