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SOME ESTIMATES FOR THE MAXIMUM SHEAR STRESS IN PLANE, ISOTROPIC ELASTICITY

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Abstract. Upper and lower bounds for the maximum shear stress in a configuration corresponding to a purely distortional deformation originating from a given undistorted (ground) state are obtained in the framework of plane, isotropic, nonlinear elasticity. The bounds are shown to be expressible in terms of the deformation and the boundary traction that is required for maintaining the purely dilatational deformation in the ground state.

1. Introduction. Consider a homogeneous, isotropic, elastic body which, in a reference configuration $C(\mathbf{1})$, occupies the open subset Ω of the Euclidean space $\mathbb{R}^2$. In a typical deformation the particle $\mathbf{X} \in \Omega$ is displaced to $\mathbf{x}(\mathbf{X}) \in \mathbb{R}^2$. Given any deformation gradient $\mathbf{F} \equiv \nabla \mathbf{x}$ with $J \equiv$ determinant $\mathbf{F} > 0$, we can define (cf. [1, 2]) a unimodular tensor $\mathbf{A}$ by $\mathbf{A} \equiv J^{-1/2}\mathbf{F}$ so that $\mathbf{F} = \sqrt{J}\mathbf{A}$ is composed of an isochoric deformation $\mathbf{A}$ and a pure dilatation $\sqrt{J}\mathbf{1}$, where $\mathbf{1}$ denotes the identity tensor.

In a recent paper [3] it was shown that in plane, isotropic elasticity, a certain version of the Baker-Ericksen inequalities [4] implies that the stored energy corresponding to any purely distortional deformation originating from a given undistorted (ground) state $C(\sqrt{J}\mathbf{1})$ must be greater than the stored energy of the ground state. If $\mathbf{F}^{\mathrm{T}}$ denotes the transpose of $\mathbf{F}$, this minimum property can be formally written as

$$\frac{\partial W}{\partial I}(I, J) \neq 0, \qquad \frac{\partial W}{\partial I}(2J, J) > 0 \Rightarrow W(I, J) > W(2J, J) \quad \text{for } I \neq 2J, \quad (1.1)$$

where $W = W(I, J)$ is the stored-energy function and $I \equiv \mathrm{trace}(\mathbf{F}\mathbf{F}^{\mathrm{T}})$. Trivial modifications of the proof given in [3] for Eq. (1.1) show that we also have

$$\frac{\partial^2 W}{\partial I^2}(I, J) \neq 0, \qquad \frac{\partial^2 W}{\partial I^2}(2J, J) > 0 \Rightarrow \frac{\partial W}{\partial I}(I, J) > \frac{\partial W}{\partial I}(2J, J) \quad \text{for } I \neq 2J, \quad (1.2)$$

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and

$$\frac{\partial^2 W}{\partial I^2}(I, J) \neq 0, \qquad \frac{\partial^2 W}{\partial I^2}(2J, J) < 0 \Rightarrow \frac{\partial W}{\partial I}(I, J) < \frac{\partial W}{\partial I}(2J, J) \quad \text{for } I \neq 2J. \tag{1.3}$$

In the next section we show that Eq. (1.2) leads to a lower bound for the maximum shear stress in certain distorted configurations $C(\mathbf{F})$ and that this bound is expressible in terms of the deformation $\mathbf{F}$ and the boundary traction required for maintaining the deformation $\sqrt{J}\mathbf{1}$, $J =$ determinant $\mathbf{F}$. In Sec. 3 we make use of both Eqs. (1.2) and (1.3) in order to obtain similar upper and lower bounds for the maximum shear stress in the case when, under dead-load boundary tractions, branching of solutions may occur from a bifurcation point on a deformation path corresponding to pure dilatation (see [5]). In the last section we discuss the application of the theory to the class of harmonic materials introduced in [6].

2. A lower bound for the maximum shear stress. For isotropic bodies there exists a reference configuration such that the response function relating the Cauchy stress tensor $\mathbf{T}$ to the deformation gradient $\mathbf{F}$ is given by

$$\mathbf{T} = \frac{\partial W}{\partial J}\mathbf{1} + \frac{2}{J}\frac{\partial W}{\partial I}\mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{2.1}$$

From Eq. (2.1) we deduce that the principal stresses t_i, $i = 1, 2,$, which are defined to be the eigenvalues of $\mathbf{T}$, are delivered by

$$t_i = \frac{\partial W}{\partial J} + \frac{2\lambda_i^2}{J}\frac{\partial W}{\partial I} = \frac{\lambda_i}{J}\frac{\partial \widehat{W}}{\partial \lambda_i}, \qquad i = 1, 2 \text{ (no sum)}, \tag{2.2}$$

where λ_i are the eigenvalues of $(\mathbf{F}\mathbf{F}^{\mathrm{T}})^{1/2}$ (called principal stretches) and

$$\widehat{W}(\lambda_1, \lambda_2) \equiv W(\lambda_1^2 + \lambda_2^2, \lambda_1\lambda_2). \tag{2.3}$$

From Eq. (2.2), we find that when $\lambda_1 \neq \lambda_2$

$$(t_1 - t_2)\Big/\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right) = 2\frac{\partial W}{\partial I}(I, J). \tag{2.4}$$

If $\lambda_1 = \lambda_2 \equiv \lambda$, Eq. (2.2) gives

$$t_1 = t_2 \equiv t = t(\lambda) = \frac{\partial W}{\partial J}(2J, J) + 2\frac{\partial W}{\partial I}(2J, J) = \frac{1}{\lambda}\frac{\partial \widehat{W}}{\partial \lambda_1}(\lambda, \lambda). \tag{2.5}$$

We note that, because $\widehat{W}$ is a symmetric function of its arguments, we have

$$\frac{\partial^\alpha \widehat{W}}{\partial \lambda_1^\alpha}(\lambda, \lambda) = \frac{\partial^\alpha \widehat{W}}{\partial \lambda_2^\alpha}(\lambda, \lambda), \qquad \alpha = 1, 2. \tag{2.6}$$

Using Eqs. (1.2) and (2.4) we now infer that the assumptions

$$\frac{\partial^2 W}{\partial I^2}(I, J) \neq 0, \qquad \frac{\partial^2 W}{\partial I^2}(2J, J) > 0 \tag{2.7}$$

imply

$$\frac{1}{2}|t_1 - t_2| > |\lambda_1^2 - \lambda_2^2|(\lambda_1\lambda_2)^{-1}\frac{\partial W}{\partial I}(2\lambda_1\lambda_2, \lambda_1\lambda_2), \qquad \lambda_1 \neq \lambda_2. \tag{2.8}$$

The expression occurring in the left-hand side of Eq. (2.8) represents the maximum shear stress according to orientation (see, e.g., [7, Chapter 4]). Since

$$\frac{\partial \widehat{W}}{\partial \lambda_i} = 2\lambda_i \frac{\partial W}{\partial I} + \frac{J}{\lambda_i}\frac{\partial W}{\partial J}, \qquad i = 1, 2, \tag{2.9}$$

$$\begin{aligned}\frac{\partial^2 \widehat{W}}{\partial \lambda_i \partial \lambda_j} = 4\lambda_i\lambda_j \frac{\partial^2 W}{\partial I^2} + 2J\left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i}\right)\frac{\partial^2 W}{\partial I \partial J} + \frac{J^2}{\lambda_i\lambda_j}\frac{\partial^2 W}{\partial J^2} \\ + 2\delta_{ij}\frac{\partial W}{\partial I} + (1-\delta_{ij})\frac{\partial W}{\partial J}, \qquad i, j = 1, 2,\end{aligned} \tag{2.10}$$

where δ_{ij} denotes the Kronecker symbol, we have

$$4\frac{\partial W}{\partial I}(2J, J) = \frac{\partial^2 \widehat{W}}{\partial \lambda_1^2}(\sqrt{J}, \sqrt{J}) - \frac{\partial^2 \widehat{W}}{\partial \lambda_1 \partial \lambda_2}(\sqrt{J}, \sqrt{J}) + \frac{1}{\sqrt{J}}\frac{\partial \widehat{W}}{\partial \lambda_1}(\sqrt{J}, \sqrt{J}). \tag{2.11}$$

We now appeal to the exclusion conditions (cf. [5; 8, Sec. 6.2]) which, under dead-load boundary tractions, are necessary and sufficient for the infinitesimal stability of homogeneous deformations:

$$\left\| \begin{matrix} \frac{\partial^2 \widehat{W}}{\partial \lambda_1^2} & \frac{\partial^2 \widehat{W}}{\partial \lambda_1 \partial \lambda_2} \\ \frac{\partial^2 \widehat{W}}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \widehat{W}}{\partial \lambda_2^2} \end{matrix} \right\| \text{ is positive definite}, \tag{2.12}$$

$$\frac{\partial \widehat{W}}{\partial \lambda_1} + \frac{\partial \widehat{W}}{\partial \lambda_2} > 0, \tag{2.13}$$

$$\left(\frac{\partial \widehat{W}}{\partial \lambda_1} - \frac{\partial \widehat{W}}{\partial \lambda_2}\right) \Big/ (\lambda_1 - \lambda_2) > 0, \tag{2.14}$$

where Eq. (2.14) incorporates the case $\lambda_1 = \lambda_2$ by an appropriate limiting procedure. According to [9] nonhomogeneous conformal deformations $\sqrt{J}\mathbf{Q}$ ($J \neq$ constant, $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{1}$) are possible in the absence of body forces if and only if the corresponding (hydrostatic) stress field is independent of the stretch parameter $\sqrt{J}$ in which case the classical pressure-compression condition (that requires that the volume of an isotropic elastic solid should be decreased by pressure but increased by tension; see [10, Sec. 51]) is violated. For this reason we consider in the following only deformations whose associated principal stretches λ_1, λ_2 satisfy the condition

$$\lambda_1\lambda_2 = \text{constant}. \tag{2.15}$$

Besides homogeneous deformations (which possess constant principal stretches), this class of deformations contains nonhomogeneous deformations with constant principal stretches (see [11], for instance) as well as other nonhomogeneous deformations (for an example see [12]). We note, however, that in the absence of body forces the only deformations possible in all unconstrained isotropic elastic solids are the homogeneous deformations [13]. Nevertheless, nonhomogeneous deformations can also be achieved subject to boundary tractions alone but only for special types of these unconstrained materials (see [3], for instance, where a necessary and sufficient condition for equilibrium in terms of the stored-energy function only is found for the deformation considered in [12]). Assuming that the homogeneous purely dilatational deformation $\sqrt{\lambda_1\lambda_2}\mathbf{1}$ is infinitesimally stable under dead-load boundary tractions

$$\frac{\partial\widehat{W}}{\partial\lambda_1}(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}) = \frac{\partial\widehat{W}}{\partial\lambda_2}(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}), \tag{2.16}$$

we find from Eqs. (2.6) and (2.12)–(2.14) that the following two conditions must be satisfied:

$$\frac{\partial^2\widehat{W}}{\partial\lambda_1^2}(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}) - \frac{\partial^2\widehat{W}}{\partial\lambda_1\partial\lambda_2}(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}) > 0 \tag{2.17}$$

and

$$\frac{\partial\widehat{W}}{\partial\lambda_1}(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}) > 0. \tag{2.18}$$

The inequalities (2.17) and (2.18) imply (see Eq. (2.11))

$$\frac{\partial W}{\partial I}(2J, J) > 0. \tag{2.19}$$

According to the stability criterion established in [14], the latter together with (see Eq. (2.5))

$$\frac{dt}{d(\sqrt{J})} > 0, \tag{2.20}$$

which is slightly stronger than the classical pressure-compression inequality, are necessary and sufficient for the (infinitesimal) stability of the configuration $C(\sqrt{J}\mathbf{1})$ in the case when the Cauchy stress is prescribed on the boundary. Thus, when Eq. (2.20) holds, if $C(\sqrt{J}\mathbf{1})$ is stable for prescribed dead-load boundary tractions, it is stable also in the framework of the Cauchy traction boundary-value problem.

Combining Eqs. (2.8), (2.11), (2.17), and (2.18) we arrive at the lower bound

$$\frac{1}{2}|t_1 - t_2| > \frac{1}{4}|\lambda_1^2 - \lambda_2^2|(\lambda_1\lambda_2)^{-3/2}\frac{\partial\widehat{W}}{\partial\lambda_1}(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}) > 0, \qquad \lambda_1 \neq \lambda_2. \tag{2.21}$$

Finally we remark here that in the case when the body is incompressible the dependence of W upon J may be suppressed (as only volume-preserving deformations, with $J = 1$, are possible) and $\partial W/\partial J$ in Eq. (2.1) has to be replaced by an arbitrary function (to be determined from the equilibrium equations). Then, under the assumptions (2.7), Eq. (2.8) still holds (with $\lambda_1\lambda_2 = 1$) and $\partial W/\partial I(2, 1)$ is simply one-half of the shear modulus of the linearized theory.

3. The case of branching. Following [5] we prescribe

$$\frac{\partial\widehat{W}}{\partial\lambda_1} = \frac{\partial\widehat{W}}{\partial\lambda_2} \equiv \tau > 0 \tag{3.1}$$

and assume that bifurcation into configurations with $\lambda_1 \neq \lambda_2$ occurs on the path of deformation characterized by $\lambda_1 = \lambda_2 = \lambda$. Two (homogeneous) branches with principal stretches $\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}$, $\alpha = 1, 2$, satisfying $\lambda_1^{(\alpha)} \neq \lambda_2^{(\alpha)}$ and $\lambda_1^{(\alpha)}\lambda_2^{(\alpha)} = \lambda_c^2$, then originate from the bifurcation point $\lambda = \lambda_c$ (which depends on the material), and these are shown in [5] to be at best neutrally stable. Neutral stability occurs when [5]

$$\tau > \tau_c \equiv \frac{\partial \widehat{W}}{\partial \lambda_1}(\lambda_c, \lambda_c). \tag{3.2}$$

Since [5]

$$\frac{\partial^2 \widehat{W}}{\partial \lambda_1^2}(\lambda_c, \lambda_c) - \frac{\partial^2 \widehat{W}}{\partial \lambda_1 \lambda_2}(\lambda_c, \lambda_c) = 0, \tag{3.3}$$

Eq. (2.11) gives

$$4\frac{\partial W}{\partial I}(2\lambda_c^2, \lambda_c^2) = \frac{1}{\lambda_c}\frac{\partial \widehat{W}}{\partial \lambda_1}(\lambda_c, \lambda_c), \tag{3.4}$$

so that, on making use of Eq. (2.4), we deduce from Eq. (1.2) that the assumptions (2.7) imply

$$\frac{1}{2}|t_1^{(\alpha)} - t_2^{(\alpha)}| > \frac{1}{4}\lambda_c^{-3}|(\lambda_1^{(\alpha)})^2 - (\lambda_2^{(\alpha)})^2|\tau_c, \qquad \alpha = 1, 2. \tag{3.5}$$

On the other hand, making use of Eq. (1.3), we find from Eqs. (2.4) and (3.4) that the assumptions

$$\frac{\partial^2 W}{\partial I^2}(I, J) \neq 0, \qquad \frac{\partial^2 W}{\partial I^2}(2J, J) < 0 \tag{3.6}$$

yield

$$(t_1^{(\alpha)} - t_2^{(\alpha)}) \Big/ \left(\frac{\lambda_1^{(\alpha)}}{\lambda_2^{(\alpha)}} - \frac{\lambda_2^{(\alpha)}}{\lambda_1^{(\alpha)}}\right) < \frac{1}{2\lambda_c}\frac{\partial \widehat{W}}{\partial \lambda_1}(\lambda_c, \lambda_c). \tag{3.7}$$

The Baker-Ericksen inequality [4] is the requirement that in a deformation process the greater principal stress should occur in the direction of the greater principal stretch:

$$(t_1 - t_2)/(\lambda_1 - \lambda_2) > 0 \quad \text{for } \lambda_1 \neq \lambda_2. \tag{3.8}$$

Assuming this we obtain from Eq. (3.7) that

$$\frac{1}{2}|t_1^{(\alpha)} - t_2^{(\alpha)}| < \frac{1}{4}\lambda_c^{-3}|(\lambda_1^{(\alpha)})^2 - (\lambda_2^{(\alpha)})^2|\tau_c, \qquad \alpha = 1, 2. \tag{3.9}$$

4. Application to harmonic materials. Harmonic materials are elastic solids with a stored-energy density function in plane strain given by

$$W(I, J) = 2\mu[H(Q) - J], \qquad Q \equiv (I + 2J)^{1/2} = \lambda_1 + \lambda_2, \tag{4.1}$$

where μ is a constant and H is a twice continuously differentiable function of its argument. From Eq. (2.1) we find

$$\mathbf{T} = 2\mu\left\{\left[\frac{H'(Q)}{Q} - 1\right]\mathbf{1} + \frac{H'(Q)}{QJ}\mathbf{F}\mathbf{F}^{\mathrm{T}}\right\} \tag{4.2}$$

and the requirement that both the stored energy and the stress vanish in the reference configuration (i.e., for $\lambda_1 = \lambda_2 = 1$) yields

$$H(2) = H'(2) = 1, \tag{4.3}$$

where $'$ denotes differentiation with respect to the argument.

We shall suppose that strong ellipticity holds for infinitesimal deformations of harmonic materials. As shown in [15], this is the case if and only if

$$\mu > 0, \qquad H''(2) > 0. \tag{4.4}$$

It then follows [16] that the stored-energy function (4.1) is positive, except in the undeformed state, if and only if

$$H(Q) > Q^2/4, \qquad Q \in (0, \infty) - \{2\}. \tag{4.5}$$

The pressure-compression inequality (2.20) requires that

$$QH''(Q) - H'(Q) > 0, \qquad Q \in (0, \infty), \tag{4.6}$$

and we see from (4.1) that the latter is equivalent to

$$\frac{\partial^2 W}{\partial I^2}(I, J) > 0. \tag{4.7}$$

When (4.6) is satisfied, the condition (2.18) holds (see Eqs. (2.5) and (2.20)) provided

$$\sqrt{\lambda_1 \lambda_2} > 1, \tag{4.8}$$

whereas Eq. $(4.4)_1$ ensures that Eq. (2.17) is fulfilled at all states of deformation. We then deduce that for harmonic materials the estimate (2.21) holds at all deformations that satisfy Eqs. (2.15) and (4.8). Thus, if in a laboratory experiment, it is found that the inequality (2.21) is violated at a relevant deformation, it can be concluded that the material involved in the experiment is not of harmonic type.

Additional material on harmonic elastic media is provided in [8] and [17–19]. An example of a harmonic stored-energy function which fulfills the above requirements is given by [18]

$$H(Q) = \frac{1}{2}Q^2 + \frac{Q}{m-1}\left(\frac{2}{Q}\right)^m + \frac{1+m}{1-m}, \qquad m \neq 1, \ m \geq 0, \tag{4.9}$$

where m is a material constant.

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