QUARTERLY OF APPLIED MATHEMATICS
VOLUME LII, NUMBER 4
DECEMBER 1994, PAGES 759-775

JUSTIFICATION OF THE LINEAR LONG-WAVE APPROXIMATION TO VISCOUS FLUID FLOW DOWN AN INCLINED PLANE

By

S. M. SUN AND M. C. SHEN

University of Wisconsin-Madison, Madison, Wisconsin

Abstract. The objective of this paper is to justify the linear long-wave approximation used in the derivation of approximate equations for long waves on the free surface of a two-dimensional viscous fluid flow down an inclined plane. To the first order of a small parameter, the approximate equation is a heat equation, which becomes ill-posed if a Reynolds number R is greater than some critical value R_c. To overcome this difficulty we consider a higher-order approximate equation, which is well-posed even if $R > R_c$, and show that the solution of the higher-order equation is an approximation to the solution of the linearized Navier-Stokes equations. The justification is based upon a set of long-wave initial conditions, and the error bounds can also be expressed in terms of pointwise estimates.

1. Introduction. The problem of two-dimensional viscous fluid flow down an inclined plane has been investigated extensively in the past. The linear stability of a solution to this problem on the basis of a long-wave approximation was studied by Benjamin [1] and Yih [2] using the normal mode analysis, and a critical value R_c of the Reynolds number R defined for this problem was found. The long waves on the viscous fluid down the plane are stable if $R < R_c$ and unstable if $R > R_c$. Later Mei [3], Benney [4], and others developed nonlinear theories for the evolution of long surface waves in this problem. In [3] R was assumed small, and in [4] a two-parameter asymptotic expansion was developed to derive an evolution equation with high-order derivatives for long surface waves with an R of order one and small amplitude. Up to now a mathematical justification of the asymptotic method used in the nonlinear theories remains an unsolved problem. If we choose a moving coordinate system at a speed $(R\sin\theta)/2$ parallel to the plane where θ is the angle of inclination of the inclined plane and neglect nonlinear terms and all higher-order terms of a small parameter ε in the evolution equation, a heat equation is obtained. The equation subject to an initial condition is ill-posed if $R > R_c$ and well-posed if $R < R_c$. Therefore, R_c is also associated with the so-called Hadamard instability of an ill-posed problem as recently coined by Joseph [5]. For $R < R_c$, it has been rigorously justified by Shih and Shen [6] that a solution of the heat equation indeed

Received July 22, 1992.
1991 *Mathematics Subject Classification.* Primary 76D05, 35Q30.

is an approximation to the solution of the linear Navier-Stokes equations subject to the same initial condition of the free surface. For $R > R_c$ Carasso and Shen [7] used a regularization method to study the ill-posed heat equation. However, a uniform error bound cannot be achieved in the justification of the method. Note that the linear Navier-Stokes equations are well-posed as will be shown later.

The objective of this paper is to justify rigorously an asymptotic method for the formal derivation of an evolution equation with a fourth-order derivative for the free surface, which is well-posed for $R > R_c$ under less restrictive conditions. In this case we need not deal with an ill-posed problem and can establish uniform bounds for a solution of the equation with higher-order derivatives. Our main contribution is to show that a solution of this evolution equation is an approximation to the solution of the linear Navier-Stokes equations even if $R > R_c$. The method of justification can be extended in a straightforward manner to approximate equations beyond the fourth-order equation. Here instead of using the formal long-wave stretching at the beginning, we impose a set of long-wave initial conditions and show that the long-wave behavior persists in the solution of the linearized equations. Thus the validity of the long-wave approximation is clarified at least for the linear case.

We organize the paper as follows. In Sec. 2 the problem is formulated in a weak sense and several Hilbert spaces are defined for later use. In Sec. 3 we use a formal method [4] to derive the evolution equation with a fourth-order derivative. In Sec. 4 we prove our main theorem that under a set of assumptions the solution of the evolution equation indeed is an approximation to the solution of the linear Navier-Stokes equations and prove the error terms can be bounded by pointwise estimates.

2. Formulation. The fluid domain Ω that we are considering is a two-dimensional strip Ω: $-1 < z < 0, -\infty < x < +\infty$, bounded by the linearized free surface Γ: $z = 0$, and a lower rigid plane S: $z = -1$. The x-axis is in the direction down the plane. Let θ be the angle of inclination, $0 < \theta < \pi/2$, and choose a coordinate system moving at the constant speed λ_0 in the direction of the x-axis.

The equilibrium flow under gravity is one dimensional in the x-direction with a velocity given by

$$u_0(z) = (R\sin\theta/2)(1 - z^2) - \lambda_0,$$

and the linearized Navier-Stokes equations governing the flow are

$$V_t + u_0 V_x + V \cdot \nabla U = -\nabla p + R^{-1}\nabla^2 V \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot V = 0, \tag{2}$$

$$(R^{-1}\mathscr{D}(V) - p) \cdot \overline{n}_2 = \eta \sin\theta \overline{n}_1 + (\sigma\eta_{xx} - \eta\cos\theta)\overline{n}_2 \quad \text{on } \Gamma, \tag{3}$$

$$\eta_t + u_0\eta_x - w = 0 \quad \text{on } \Gamma, \tag{4}$$

$$V = 0 \quad \text{on } S, \tag{5}$$

$$V = V_0, \ \eta = \eta_0 \quad \text{at } t = 0. \tag{6}$$

Here $\overline{n}_i$ are the unit vectors of the coordinate system, a point is denoted by $(x, z) = (x_1, x_2)$, t is the time, $V = (v_1, v_2) = (u, w)$ is the velocity, $U = (u_0, 0)$, p is the pressure, $\eta(x, t)$ is the deviation of the free surface from $z = 0$, R is the Reynolds number, σ is the nondimensional surface tension coefficient, and

$$\mathscr{D}(V) = \left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]$$

denotes the deformation tensor. We note that all the variables have been nondimensionalized by appropriate units: x, z, and η are measured in units of the equilibrium depth h of the fluid, t is measured in units of $(h/g)^{1/2}$ where g is the constant gravitational acceleration, u and w are measured in units of $(gh)^{1/2}$, p is measured in units of ρhg where ρ is the density, $\sigma = s/(\rho g h^2)$ where s is the surface tension coefficient, and

$$R = \nu^{-1} h^{3/2} g^{1/2}$$

where ν is the kinematic viscosity.

We will define several function spaces and state some definitions and lemmas. Let $J_s^\infty(\Omega)$ be the class of all solenoidal C^∞-vector functions with compact support in $\Omega \cup \Gamma$ and $J_s(\Omega)$ be the Hilbert space obtained by completing $J_s^\infty(\Omega)$ with the scalar product

$$(V, \Phi) = \int_\Omega (v_1\varphi_1 + v_2\varphi_2)\, dA$$

where $V = (v_1, v_2)$ and $\Phi = (\varphi_1, \varphi_2)$. The L_2-norm of a vector function V on Ω is defined by

$$\|V\|^2 = \int_\Omega (v_1^2 + v_2^2)\, dA.$$

It is known [8] that

$$L_2(\Omega) = J_s(\Omega) \oplus G_\Gamma(\Omega)$$

where $G_\Gamma(\Omega)$ are all vector functions of the form ∇p and p is a single-valued locally square integrable scalar function on Ω and has first-order L_2-generalized derivatives with $p = 0$ on Γ. Let $H(\Omega)$ be the Hilbert space obtained by completion of $J_s^\infty(\Omega)$ with respect to the norm

$$\|V\|_H^2 = \int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial v_i}{\partial x_j}\right)^2 dA$$

and $E(\Omega)$ be the completion of $J_s^\infty(\Omega)$ with respect to the energy norm

$$\|V\|_E^2 = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2} \left(\left(\frac{\partial v_i}{\partial x_j}\right)^2 + \left(\frac{\partial v_j}{\partial x_i}\right)^2\right) dA$$

and the scalar product

$$E(V, \Phi) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i}\right) dA.$$

Also we use $W_2^m(\Omega)$ and $W_2^m(\Gamma)$ to denote the standard Sobolev spaces for $m = 1, 2, 3, \ldots$. We denote by $L_2(T; X)$ the class of functions $f\colon [0, T] \to X$ with the scalar product

$$\int_0^T (f(t), g(t))\,dt,$$

where T is a fixed positive constant and $0 \le t \le T$. $C(T; X)$ is the class of continuous functions with respect to the norm of X and

$$\|f(t)\|_{C(T;X)} = \sup_{t\in[0,T]} \|f(t)\|_X .$$

$C'(T; X)$ is the class of continuously strongly differentiable functions with

$$\|f(t)\|_{C'(T;X)} = \sup_{t\in[0,T]} \|f(t)\|_X + \sup_{t\in[0,T]} \|f'(t)\|_X ,$$

where f' is the strong derivative of f. We say that $f \in L_2[T; X]$ has a generalized derivative $f_t \in L_2[T; X]$ if

$$\int_0^T (f_t(t), g(t))\,dt = -\int_0^T (f(t), g_t(t))\,dt,$$

for all $g \in C'(T; X)$ with $g(0) = g(T) = 0$. Next we state two lemmas, the proofs of which can be found in [6].

Lemma 1. $\|\cdot\|_E$ and $\|\cdot\|_H$ are equivalent and

$$(1 - 2^{-1/2})\|V\|_H^2 \le \|V\|_E^2 \le 2\|V\|_H^2$$

if $V \in J_s^\infty(\Omega)$.

Lemma 2. For $V \in J_s^\infty(\Omega)$,

$$\|V\| \le 2^{3/2}\|V\|_E \quad \text{and} \quad \|V\|_\Gamma \le 2^{3/2}\|V\|_E ,$$

where

$$\|V\|_\Gamma = \int_\Gamma (v_1^2 + v_2^2)\,dx .$$

We call $\{V(t, x, z) = (v_1, v_2) = (u, w), \eta(t, x)\}$ a generalized solution of Eqs. (1)–(6) if $V \in L_2(T; E(\Omega))$, $V_t \in L_2(T; E(\Omega))$, $\eta \in L_2(T; L_2(\Gamma))$, $\eta_t \in L_2(T; L_2(\Gamma))$, $\eta_x \in L_2(T; L_2(\Gamma))$, $w_x \in L_2(T; L_2(\Gamma))$, and $\{V, \eta\}$ satisfies Eqs. (4) and (6) and

$$\begin{aligned}\int_0^T dt \int_\Omega (V_t + u_0 V_x + V\cdot\nabla U)\cdot\Phi\,dA &= \int_0^T dt \int_\Gamma (\eta \sin\theta\varphi_1 - \sigma\eta_x\varphi_{2x} - \eta\cos\theta\varphi_2)\,dx \\ &\quad - (2R)^{-1}\int_0^T dt \int_\Omega \mathscr{D}(V)\cdot\mathscr{D}(\Phi)\,dA,\end{aligned} \tag{7}$$

for all $\Phi = (\varphi_1, \varphi_2) \in L_2(T; E(\Omega))$ and $\varphi_{2x} \in L_2(T; L_2(\Gamma))$. We say the initial data V_0, η_0 in Eq. (6) are compatible with the free surface condition (3) if there exists a $p_0 \in L_2(\Gamma)$ such that V_0, η_0, and p_0 satisfy (3). We also use

$$D^\beta = \left(\frac{\partial^\beta}{\partial x^\beta}\right)$$

where β is a nonnegative integer. Sometimes we use D_x to denote the x-derivative more explicitly.

Now we state a theorem, the proof of which can also be found in [6] or [9].

THEOREM 1. If the initial data V_0 and η_0 are compatible with the free surface condition (3) and if V_0 and η_0 possess generalized derivatives with respect to x of all orders such that $D^\beta V_0 \in E(\Omega) \cap W_2^2(\Omega)$ and $D^\beta \eta_0 \in L_2(\Gamma)$ for all β in $0 \le \beta < \infty$, then there exists a unique generalized solution $\{V, \eta\}$ of Eqs. (1)–(6) such that

$$D^\beta V \in C(T; E(\Omega)), \qquad D^\beta V_t \in L_2(T; E(\Omega)),$$
$$D^\beta \eta \in C(T; L_2(\Gamma)), \qquad D^\beta \eta_t \in L_2(T; L_2(\Gamma)),$$

for all $T > 0$ and any $\beta \ge 0$. Moreover, for each β, $\{D^\beta V, D^\beta \eta\}$ is the generalized solution of Eqs. (1)–(6) corresponding to the initial data $\{D^\beta V_0, D^\beta \eta_0\}$.

Note here that since the equations (1)–(6) are linear equations with coefficients independent of x, by letting $\Phi = V$ in (7) and straightforward estimates, we have

$$\begin{aligned} &\|V(t)\|^2 + \cos\theta\|\eta(t)\|^2 + \sigma\|\eta(t)\|_H^2 \\ &\quad \le \|V(0)\|^2 + \cos\theta\|\eta(0)\|^2 + \sigma\|\eta(0)\|_H^2 \\ &\qquad + M\int_0^t (\|V(s)\|^2 + \cos\theta\|\eta(s)\|^2 + \sigma\|\eta(s)\|_H^2)\, ds, \end{aligned} \tag{8}$$

where M is a positive constant. By the Gronwall inequality

$$\|V(t)\|^2 + \cos\theta\|\eta(t)\|^2 + \sigma\|\eta(t)\|_H^2 \le e^{Mt}(\|V(0)\|^2 + \cos\theta\|\eta(0)\|^2 + \sigma\|\eta(0)\|_H^2).$$

Therefore, Eqs. (1)–(6) always pose a well-posed problem in $0 \le t \le T$.

3. Formal asymptotic expansion. Let ε be a small positive parameter, which measures the ratio of the length scale in the z-direction to that in the x-direction. Assume $\{V_0, \eta_0\}$ has the property that

$$D^k V_0 = O(\varepsilon^k), \quad D^k \eta_0 = O(\varepsilon^k) \qquad \text{for } k \ge 0,$$

and let $\{V, \eta, \rho\}$ be the solution of Eqs. (1)–(6) corresponding to the initial data $\{V_0, \eta_0\}$. We suppose $\{V, p\}$ has the property $\partial/\partial x = O(\varepsilon)$ and $\partial/\partial t = O(\varepsilon^3)$ and $\{V, p\}$ possesses an asymptotic expansion of the form

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \cdots, \tag{9}$$

where $\varphi_1 = O(1)$, $\varphi_2 = O(\varepsilon)$, ... when φ stands for u and p and $\varphi_1 = O(\varepsilon)$, $\varphi_2 = O(\varepsilon^2)$, ... when φ stands for w. For the time being, R and θ are fixed. Substituting Eq. (9) into Eqs. (1)–(6) and comparing orders of ε, we obtain the equations for the first-order approximations and solve for u_1, w_1, p_1, and λ_0 to have

$$\begin{aligned} u_1(t, x, z) &= \overline{u}_1(z)\eta(t, x) = (R\sin\theta)(1+z)\eta(t, x), \\ w_1 &= \overline{w}_1(z)\eta_x = (R\sin\theta/2)(1+z)^2\eta_x(t, x), \\ p_1 &= \overline{p}_1(z)\eta = \eta(t, x)\cos\theta. \end{aligned}$$

Here we have not used Eq. (4) yet. Then from the second-, third-, and fourth-order approximation, we have

$$u_2 = \overline{u}_2(z)\eta_x, \qquad p_2 = \overline{p}_2(z)\eta_x, \qquad w_2 = \overline{w}_2(z)\eta_{xx},$$
$$u_3 = \overline{u}_3(z)\eta_{xx}, \qquad p_3 = \overline{p}_3(z)\eta_{xx}, \qquad w_3 = \overline{w}_3(z)\eta_{xxx},$$
$$u_4 = \overline{u}_4(z)\eta_{xxx} + \overline{u}_{41}(z)\eta_t, \qquad p_4 = \overline{p}_4(z)\eta_{xxx} + \overline{p}_{41}(z)\eta_t,$$
$$w_4 = \overline{w}_4(z)\eta_{xxxx} + \overline{w}_{41}(z)\eta_{xt}.$$

The coefficients of η and its derivatives in the successive approximations are complicated polynomials of z distinguished by a bar with coefficients dependent only on R, θ, σ and will not be given. We substitute $w = w_1 + w_2 + w_3 + w_4 + \cdots$ into Eq. (4) and simplify it to obtain

$$\eta_t + (R\sin\theta - \lambda_0)\eta_x - ((R\sin\theta/2)\eta_x + \overline{w}_2(0)\eta_{xx} + \overline{w}_3(0)\eta_{xxx} + \overline{w}_4(0)\eta_{xxxx} + \overline{w}_{41}(0)\eta_{xt}) + w^*(t, x) = 0,$$

where $w^*(t, x)$ consists of higher-order terms of $w(t, x)$. We let $\lambda_0 = (R\sin\theta)/2$, omit w^*, remove η_{xt} by the space derivatives of η, and discard the higher-order terms to obtain

$$\eta_{1t} + A_1\eta_{1xx} + B_1\eta_{1xxx} + C_1\eta_{1xxxx} = 0, \tag{10}$$

where η_1 is formally an approximation of η and

$$A_1 = (R/3)((2/5)R^2\sin^2\theta - \cos\theta),$$
$$B_1 = R\sin\theta + (10/21)R^2A_1\sin\theta,$$
$$C_1 = (\sigma/3) + R[(75872/2027025)R^6\sin^4\theta - (17363/155925)R^4\sin^2\theta\cos\theta + (157/224)R^2\sin^2\theta - (8/15)\cos\theta + (2/45)R^2\cos\theta].$$

The detailed derivation related to Eq. (10) in the much more general case that includes all nonlinear terms and also a magnetic field in the fluid region was given in [10]. One can also check easily that (10) is equivalent to Eq. (46) in [4] except that in our formulation a different definition of R was used. Shih and Shen [6] assumed $A_1 < 0$ and proved that the solution of

$$\eta_{1t} + A_1\eta_{1xx} = 0 \tag{11}$$

is a long-wave approximation to a solution of Eqs. (1)–(6). But if $A_1 > 0$, (11) becomes an ill-posed problem. Therefore, (11) is no longer valid for $A_1 \geq 0$ and a regularization method was developed to study the ill-posed problem [7]. Here instead we include higher-order terms to get a meaningful equation (10). In the following we assume $A_1 \geq 0$ and $C_1 > 0$. Note that $C_1 > 0$ if σ or R is sufficiently large. Thus if $C_1 > 0$, then Eq. (10) coupled with the initial condition

$$\eta_1(0, x) = \eta_0(x) \quad \text{at } t = 0$$

is a well-posed problem and possesses a unique solution η_1. Therefore, we can obtain formal approximations of u, w, and p. Next we prove that these approximations are indeed approximate solutions of Eqs. (1)–(6) for the long-wave initial data.

4. Asymptotic approximate solution. We shall show that η_1 in Eq. (10) with the initial condition

$$\eta_1(0, x) = \eta_0(x) \quad \text{at } t = 0 \tag{12}$$

is an approximation of η and make the following assumptions:

A.1. $0 < \theta < (\pi/2)$, $C_1 > 0$, and $A_1 \geq 0$.

A.2. The initial data V_0 and η_0 possess generalized derivatives with respect to x of all orders such that $D^\beta V_0 \in E(\Omega) \cap W_2^2(\Omega)$, $D^\beta \eta_0 \in L_2(\Gamma)$ for all $\beta \geq 0$ and V_0, η_0 are compatible with the free surface conditions (3).

A.3. Long-wave condition. There exists a positive parameter ε such that for $0 < \varepsilon < 1$, the initial data $V(0)$, $\eta(0)$, $V_b(0)$ for V, η, V_b satisfy

$$\|D^\beta \eta(0)\| \leq \varepsilon^\beta, \qquad \|D^\beta (V(0) - V_b(0))\| \leq \varepsilon^{2+\beta}, \tag{13}$$

for all $\beta \geq 0$, where $V_b = (u_b, w_b) = (\overline{u}_1(z)\eta_1 + \overline{u}_2(z)\eta_{1x}, \overline{w}_1(z)\eta_{1x} + \overline{w}_2(z)\eta_{1xx})$.

Then we shall prove

THEOREM 2. There exists a positive number ε_0 which depends only on A_1, B_1, C_1, R, θ, and T. If the initial data V_0 and η_0 satisfy the assumptions A.1 to A.3 and $0 < \varepsilon < \varepsilon_0$, then the approximate solution $\{V_b, \eta_1\}$ which is defined in Eqs. (10) and (12) and A.3 is an asymptotic approximation to the generalized solution $\{V, \eta\}$ in the following sense:

$$\|u - u_b\| \leq \varepsilon^2 L_1 e^{\delta t}, \qquad \|w - w_b\| \leq \varepsilon^3 L_1 e^{\delta t}, \qquad \|\eta - \eta_1\| \leq \varepsilon^2 L_1 e^{\delta t} \tag{14}$$

for all $0 \leq t \leq T$, where L_1 and δ are positive constants depending on σ, R, θ and $\|\cdot\|$ is the usual L_2-norm on Ω or Γ.

We note here that $D^\beta V_b(0)$ is well defined and conditions in A.3 are not mutually independent. Since V_0 and η_0 satisfy the assumptions in Theorem 1, Eqs. (1)–(6) have a unique generalized solution $\{V, \eta\}$ corresponding to the initial data $\{V_0, \eta_0\}$ and $\{D^\beta V, D^\beta \eta\}$ is the generalized solution of Eqs. (1)–(6) corresponding to the initial data $\{D^\beta V_0, D^\beta \eta_0\}$. Let

$$\begin{aligned}
V &= (u, w) = V_a + V_i + V^*, \\
u &= u_a + \overline{u}_3(z)\eta_{xx} + \overline{u}_4(z)\eta_{xxx} - (A_1\eta_{xx} + B_1\eta_{xxx})\overline{u}_{41}(z) + u^*, \\
w &= w_a + \overline{w}_3(z)\eta_{xxx} + \overline{w}_4(z)\eta_{xxxx} - (A_1\eta_{xxx} + B_1\eta_{xxxx})\overline{w}_{41}(z) + w^*,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
V^* &= (u^*, w^*), \\
V_a &= (u_a, w_a) = (\overline{u}_1(z)\eta + \overline{u}_2(z)\eta_x, \overline{w}_1(z)\eta_x + \overline{w}_2(z)\eta_{xx}),
\end{aligned} \tag{16}$$

and V_i is the remainder. Since V and η are the generalized solutions of Eqs. (1)–(6), we substitute Eqs. (15) into Eq. (4) to get

$$\eta_t + A_1\eta_{xx} + B_1\eta_{xxx} + C_1\eta_{xxxx} = w^*, \tag{17}$$

$$\eta = \eta_0 \quad \text{at } t = 0. \tag{18}$$

By substituting Eqs. (15) into Eq. (7) and choosing Φ in such a form that $\Phi(s)=0$ for $t \le s \le T$, we have

$$\int_0^t ds \int_\Omega [V_{at} + V_{it} + V_t^* + u_0(V_{ax} + V_{ix} + V_x^*) + (V_a + V_i + V^*)\cdot \nabla U]\cdot \Phi\, dA$$
$$= \int_0^t ds \int_\Gamma (\eta\varphi_1 \sin\theta - \sigma\eta_x\varphi_{2x} - \eta\varphi_2\cos\theta)\, dx \qquad (19)$$
$$- (2R)^{-1}\int_0^t ds \int_\Omega [\mathscr{D}(V_a + V_i + V^*)]\cdot \mathscr{D}(\Phi)\, dA.$$

Since η possesses generalized derivatives with respect to x of all orders, we can use integration by parts to simplify Eq. (19) to obtain

$$\int_0^t ds \int_\Omega (V_t^* + u_0 V_x^* + V^*\cdot\nabla U)\cdot\Phi\, dA$$
$$= \int_0^t ds \int_\Gamma (G\cdot\phi - \sigma\eta_x\varphi_{2x})\, dx \qquad (20)$$
$$+ \int_0^t ds \int_\Omega F\cdot\Phi\, dA - (2R)^{-1}\int_0^t ds \int_\Omega \mathscr{D}(V^*)\cdot\mathscr{D}(\Phi)\, dA,$$

where F and G can be expressed as follows:

$$G = (G_0 + G_1 D_x + G_2 D_x^2)\eta_{xxx},$$
$$F = (F_0 + F_1 D_x + F_2 D_x^2 + F_3 D_x^3)\eta_{xxx}$$
$$+ (E_0 + E_1 D_x + E_2 D_x^2 + E_3 D_x^3 + E_4 D_x^4)\eta_t.$$

Note that $F_0, \dots, F_3$, $G_0, \dots, G_2$, and $E_0, \dots, E_4$ are vector functions of z only related to $\overline{u}_i$, $\overline{w}_i$, and $\overline{p}_i$.

Before proving Theorem 2, we begin by proving a series of lemmas. In the following, C_i for $i \neq 1$ and K_i will denote positive constants depending upon R, θ, and σ only.

Lemma 3. There exist positive constants C_0, C_2 such that

$$\int_0^t \|D^\beta V^*\|_E^2\, ds \le C_0\|D^\beta V^*(0)\|^2$$
$$+ C_2\exp(\nu t)\left(\sum_{i=2}^{5}\|D^\beta D_x^i\eta(0)\|^2 + \int_0^t\left(\sum_{i=1}^{4}\|D^\beta D_x^i V^*\|_E^2\right) ds\right) \qquad (21)$$

for all $\beta \ge 0$ and any $t \ge 0$ and ν is a fixed positive constant.

Proof. Replacing Φ in Eq. (20) by V^* and using $\int u_0 V_x^* V\, d\Omega = 0$, we have

$$\frac{1}{2}(\|V^*(t)\|^2 - \|V^*(0)\|^2) = -\int_0^t ds\int_\Omega u_{0z} w^* u^*\, dA + \int_0^t ds \int_\Omega F\cdot V^*\, dA \qquad (22)$$
$$+ \int_0^t ds \int_\Gamma G\cdot V^*\, dx - R^{-1}\int_0^t \|V^*\|_E\, ds.$$

By Lemmas 1 and 2, integration by parts, and noting that $\int_\Omega |w^*|^2\,dA \le K\|V_x^*\|_E^2$ since $w^* = \int_0^z w_z^*\,dz = -\int_0^z u_x^*\,dz$, we have the estimates

$$\left|\int_\Omega u_{0z} w^* u^*\,dA\right| \le K_1 \|V^*\|_E \|V_x^*\|_E,$$

$$\left|\int_\Omega F\cdot V^*\,dA\right| \le K_2\left(\sum_{i=3}^{5}\|D_x^i\eta\|_E^2 + \sum_{i=0}^{3}\|D_t D_x^i\eta\|^2\right)^{1/2}\|V^*\|_E,$$

$$\left|\int_\Gamma G\cdot V^*\,dA\right| \le K_3\left(\sum_{i=3}^{5}\|D_x^i\eta\|^2\right)^{1/2}\|V^*\|_E.$$

Therefore,

$$\begin{aligned}
&\|V^*(t)\|^2 - \|V^*(0)\|^2\\
&\quad\le 2K_1\int_0^t \|V^*\|_E\|V_x^*\|_E\,ds\\
&\qquad + K_4\int_0^t\left(\sum_{i=3}^{5}\|D_x^i\eta\|^2 + \sum_{i=0}^{3}\|D_t D_x^i\eta\|^2\right)^{1/2}\|V\|_E\,ds - 2R^{-1}\int_0^t\|V^*\|_E^2\,ds\\
&\quad\le K_5\int_0^t\|V_x^*\|_E\,ds + K_6\int_0^t\left(\sum_{i=3}^{5}\|D_x^i\eta\|^2 + \sum_{i=0}^{3}\|D_t D_x^i\eta\|^2\right)^{1/2}ds\\
&\qquad - R^{-1}\int_0^t\|V^*\|_E^2\,ds,
\end{aligned}$$

where $D_t = \partial/\partial t$. Thus

$$\begin{aligned}
\int_0^t\|V^*\|_E^2\,ds &\le R\|V^*(0)\|^2 + RK_5\int_0^t\|V_x^*\|_E^2\,ds\\
&\quad + RK_6\int_0^t\left(\sum_{i=3}^{5}\|D_x^i\eta\|^2 + \sum_{i=0}^{3}\|D_t D_x^i\eta\|^2\right)^{1/2}ds.
\end{aligned}\tag{23}$$

But from Eq. (17), we have

$$\begin{aligned}
&\int_\Gamma(\eta_t + A_1\eta_{xx} + B_1\eta_{xxx} + C_1\eta_{xxxx})^2\,dx\\
&\quad = \int_\Gamma |w^*|^2\,dx\\
&\quad = \|\eta_t\|^2 + A_1^2\|\eta_{xx}\|^2 + B_1^2\|\eta_{xxx}\|^2 + C_1^2\|\eta_{xxxx}\|^2\\
&\qquad - \int_\Gamma(A_1(\eta_x^2)_t - C_1(\eta_{xx}^2)_t - 2B_1\eta_t\eta_{xxx})\,dx - 2A_1C_1\|\eta_{xxx}\|^2
\end{aligned}\tag{24}$$

and

$$\int_0^t (\|\eta_t\|^2 + A_1^2\|\eta_{xx}\|^2 + (B_1^2 - 2A_1C_1)\|\eta_{xxx}\|^2 + C_1^2\|\eta_{xxxx}\|^2)\,ds \\ - A_1\|\eta_x(t)\|^2 + A_1\|\eta_x(0)\|^2 + C_1\|\eta_{xx}(t)\|^2 - C_1\|\eta_{xx}(0)\| \\ + 2B_1\int_0^t ds\int_\Gamma \eta_t\eta_{xxx}\,dx = \int_0^t ds\int_\Gamma |w^*|^2\,dx. \tag{25}$$

Multiply Eq. (17) by η_{xxx} to obtain

$$\int_0^t ds\int_\Gamma (\eta_t\eta_{xxx} + B_1\eta_{xxx}^2)\,dx = \int_0^t ds\int_\Gamma \eta_{xxx}w^*\,dx. \tag{26}$$

Then multiply Eq. (17) by η_{xx} to obtain

$$\|\eta_x(t)\|^2 - \|\eta_x(0)\|^2 \\ = A_1\int_0^t \|\eta_{xx}\|^2\,ds - C_1\int_0^t \|\eta_{xxx}\|^2\,ds - \int_0^t ds\int_\Gamma w^*\eta_{xx}\,dx. \tag{27}$$

From Eqs. (25)–(27) we have

$$\int_0^t (\|\eta_t\|^2 - (B^2 + A_1C_1)\|\eta_{xxx}\|^2 + C_1^2\|\eta_{xxxx}\|^2)\,ds \\ + C_1\|\eta_{xx}(t)\|^2 - C_1\|\eta_{xx}(0)\| + 2B_1\int_0^t ds\int_\Gamma \eta_{xxx}w^*\,dx \\ = \int_0^t ds\int_\Gamma |w^*|^2\,dx - A_1\int_0^t ds\int_\Gamma w^*\eta_{xx}\,dx.$$

By using $w^* = \int_0^z w_z^*\,dz = -\int_0^z u_x^*\,dz$,

$$\int_0^t (\|\eta_t\|^2 + C_1^2\|\eta_{xxxx}\|^2)\,ds + C_1\|\eta_{xx}(t)\|^2 - C_1\|\eta_{xx}(0)\|^2 \\ \le K_7\int_0^t (\|\eta_{xx}\|^2 + \|\eta_{xxx}\|^2)\,ds + K_8\int_0^t \|V_x^*\|_E^2\,ds.$$

But

$$\int_\Gamma \eta_{xxx}^2\,dx \le \frac{1}{2}\left(\rho\int_\Gamma \eta_{xxxx}^2\,dx + \frac{1}{\rho}\int_\Gamma \eta_{xx}^2\,dx\right)$$

by the usual interpolation inequalities where ρ is any small constant. Hence,

$$\int_0^t \left(\|\eta_t\|^2 + \left(C_1^2 - \frac{\rho}{2}\right)\|\eta_{xxxx}\|^2\right)ds + C_1\|\eta_{xx}(t)\|^2 - C_1\|\eta_{xx}(0)\|^2 \\ \le K_7\left(1 + \frac{1}{2\rho}\right)\int_0^t \|\eta_{xx}\|^2\,ds + K_8\int_0^t \|V_x^*\|_E^2\,ds. \tag{28}$$

Let $\rho = C_1^2$, and by the Gronwall inequality

$$\int_0^t \|\eta_{xx}\|^2\,ds \le C_1^{-1}\exp\left(K_7\left(1 + \frac{1}{2\rho}\right)tC_1^{-1}\right)\left(C_1^2(K_7(1 + (2\rho)^{-1}))^{-1}\|\eta_{xx}(0)\|^2\right. \\ \left. + \int_0^t \exp\left(-K_7\left(1 + \frac{1}{2\rho}\right)\tau C_1^{-1}\right)K_8\left(\int_0^\tau \|V_x^*\|_E^2\,ds\right)d\tau\right) \\ \le K_9\exp(K_{10}t)\left(\int_0^t \|V_x^*\|_E^2\,ds + \|\eta_{xx}(0)\|^2\right). \tag{29}$$

Therefore, by Eqs. (28) and (29)

$$\int_0^t (\|\eta_t\|^2 + \|\eta_{xx}\|^2 + \|\eta_{xxx}\|^2 + \|\eta_{xxxx}\|^2)\, ds \le K_{11}(\exp(K_{10}t) + 1)\left(\int_0^t \|V_x^*\|_E^2\, ds + \|\eta_{xx}(0)\|^2\right). \tag{30}$$

We note here that K_i, $i = 1, 2, \ldots, 11$, are independent of t. Since Eqs. (17) and (22) hold for $D^\beta \eta$, $D^\beta V^*$ with $\beta \ge 0$, by the same derivation,

$$\int_0^t (\|D^\beta \eta_t\|^2 + \|D^\beta \eta_{xx}\|^2 + \|D^\beta \eta_{xxx}\|^2 + \|D^\beta \eta_{xxxx}\|^2)\, ds \le K_{11}(\exp(K_{10}t) + 1)\left(\|D^\beta \eta_{xx}(0)\|^2 + \int_0^t \|D^\beta V_x^*\|_E^2\, ds\right), \tag{31}$$

$$\int_0^t \|D^\beta V^*\|_E\, ds \le R\|D^\beta V^*(0)\|^2 + RK_5 \int_0^t \|D^\beta V_x^*\|_E^2\, ds + RK_6 \int_0^t \left(\sum_{i=3}^{5} \|D^\beta D_x^i \eta\|^2 + \sum_{i=0}^{3} \|D_t D_x^i \eta\|^2\right)^{1/2} ds. \tag{32}$$

Thus combining Eqs. (31) and (32), we have

$$\begin{aligned}\int_0^t \|D^\beta V^*\|_E^2\, ds &\le C_0\|D^\beta V^*(0)\|^2 + C_3 \int_0^t \|D^\beta V_x^*\|_E^2\, ds \\ &\quad + C_4 \exp(\nu t)\left(\sum_{i=0}^{3}\left(\|D^\beta D_x^{i+2}\eta(0)\|^2 + \int_0^t \|D^\beta D_x^{i+1} V^*\|_E^2\, ds\right)\right) \\ &\le C_0\|D^\beta V^*(0)\|^2 \\ &\quad + C_2 \exp(\nu t)\left(\sum_{i=1}^{4} \int_0^t \|D^\beta D_x^i V^*\|_E^2\, ds + \sum_{i=2}^{5} \|D^\beta D_x^i \eta(0)\|^2\right),\end{aligned}$$

for all $\beta \ge 0$. This proves the lemma.

Lemma 4. There exist two positive numbers C_2 and ν defined in Lemma 3 such that if $0 < \varepsilon < \varepsilon_0 = ((2C_2)^{-1} \exp(-\nu t))^{1/2}$, we have

$$\int_0^t \|D^\beta V^*\|_E^2\, ds \le C_5 \varepsilon^{4+2\beta} \exp(\nu t)(1 - r)^{-1}, \tag{33}$$

where $r = (\varepsilon/\varepsilon_0)^2$ and C_5 depends on R, θ, σ only.

Proof. By A.3 and (15),

$$\begin{aligned}\|D^\beta V^*(0)\| &\le \|D^\beta (V(0) - V_a(0))\| + \|D^\beta V_i(0)\| \\ &\le \varepsilon^{2+\beta} + K_1 \varepsilon^{2+\beta} \le (K_1 + 1)\varepsilon^{2+\beta}.\end{aligned}$$

Without loss of generality, let $C_2 \geq 1$. Then Eq. (21) becomes

$$\begin{aligned}\int_0^t \|D^\beta V^*\|_E^2\, ds &\leq (C_0(K_1+1) + C_2\exp(\nu t))\varepsilon^6 \\ &\quad + C_2\exp(\nu t)\int_0^t \left(\sum_{i=1}^4 \|D^\beta D_x^i V^*\|_E^2\right) ds \\ &\leq K_2\exp(\nu t)\varepsilon^{1+2\beta} + C_2\exp(\nu t)\int_0^t \left(\sum_{i=1}^4 \|D^{\beta+i} V^*\|_E^2\right) ds.\end{aligned} \tag{34}$$

We claim that

$$\begin{aligned}\int_0^t \|D^\beta V^*\|_E^2\, ds &\leq K_2\exp(\nu t)\varepsilon^{4+2\beta}\left(\sum_{k=0}^n C_2^k \exp(k\nu t)\varepsilon^{2k}2^k\right) \\ &\quad + 2^n C_2^{n+1}\exp((n+1)\nu t)\int_0^t \left(\sum_{i=1}^4 \|D^{\beta+n+i} V^*\|_E^2\right) ds.\end{aligned} \tag{35}$$

We prove the claim by mathematical induction. When $n = 0$, this is Eq. (34). Assume that Eq. (35) is true for $n = m$. By Eq. (34)

$$\begin{aligned}&\int_0^t \|D^{\beta+m+i} V^*\|_E^2\, ds \\ &\quad \leq \left(K_2\exp(\nu t)\varepsilon^{4+2\beta+2m+2i} + C_2\exp(\nu t)\int_0^t \sum_{j=1}^4 \|D^{\beta+m+i+j} V^*\|_E^2\, ds\right).\end{aligned}$$

Then take $i = 1$ to get

$$\begin{aligned}&\int_0^t \sum_{i=1}^4 \|D^{\beta+m+i} V^*\|_E^2\, ds \\ &\quad \leq K_2\exp(\nu t)\varepsilon^{4+2\beta+2(m+1)} \\ &\qquad + C_2\exp(\nu t)\int_0^t \sum_{j=1}^4 \|D^{\beta+m+j+1} V^*\|_E^2\, ds + \sum_{i=2}^4 \int_0^t \|D^{\beta+m+i} V^*\|_E^2\, ds \\ &\quad \leq K_2\exp(\nu t)\varepsilon^{2\beta+2(m+3)} + (C_2\exp(\nu t)+1)\sum_{i=1}^4 \int_0^t \|D^{\beta+(m+1)+i} V^*\|_E^2\, ds.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^t \|D^\beta V^*\|_E^2\,ds &\le K_2\exp(\nu t)\varepsilon^{4+2\beta}\left(\sum_{k=0}^m C_2^k\exp(k\nu t)\varepsilon^{2k}2^k\right)\\ &\quad + 2^m C_2^{m+1}\exp((m+1)\nu t)\int_0^t\left(\sum_{i=1}^4 \|D^{\beta+m+i}V^*\|_E^2\right)ds\\ &\le K_2\exp(\nu t)\varepsilon^{4+2\beta}\\ &\quad\cdot\left(\sum_{k=0}^m C_2^k\exp(k\nu t)\varepsilon^{2k}2^k + 2^m C_2^{m+1}\exp((m+1)\nu t)\varepsilon^{2(m+1)}\right)\\ &\quad + 2^m C_2^{m+1}\exp((m+1)\nu t)\cdot 2C_2\exp(\nu t)\sum_{i=1}^4\int_0^t \|D^{\beta+(m+1)+i}V^*\|_E^2\,ds\\ &\le K_2\exp(\nu t)\varepsilon^{4+2\beta}\left(\sum_{k=0}^{m+1} C_2^k\exp(k\nu t)\varepsilon^{2k}2^k\right)\\ &\quad + 2^{m+1}C_2^{m+2}\exp((m+2)\nu t)\sum_{i=1}^4\int_0^t \|D^{\beta+(m+1)+i}V^*\|_E^2\,ds.\end{aligned}$$

So Eq. (35) is true for $n = m+1$. Thus we have proved the claim.

From Eq. (8) we have

$$\begin{aligned}&\|V(t)\|^2 + \cos\theta\|\eta(t)\|^2 + \sigma\|\eta(t)\|_H^2\\ &\qquad\le \exp(Mt)(\|V(0)\|^2 + \cos\theta\|\eta(0)\|^2 + \sigma\|\eta(0)\|_H^2)\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\int_0^t (\|V(s)\|^2 + \cos\theta\|\eta(s)\|^2 + \sigma\|\eta(s)\|_H^2)\,ds\\ &\qquad\le M^{-1}\exp(Mt)(\|V(0)\|^2 + \cos\theta\|\eta(0)\|^2 + \sigma\|\eta(0)\|_H^2).\end{aligned} \tag{37}$$

From Eq. (7)

$$\int_0^t \|V(s)\|_E^2\,ds \le K_3\|V(0)\|^2 + K_4\int_0^t (\|V(s)\|^2 + \|\eta(s)\|^2 + \|\eta_{xx}(s)\|^2)\,ds.$$

Therefore, by Eqs. (36) and (37)

$$\int_0^t \|V(s)\|_E\,ds \le K_5\exp(Mt)\left(\sum_{i=0}^2(\|D_x^iV(0)\|^2 + \|D_x^i\eta(0)\|^2 + \|D_x^i\eta(0)\|_H)\right). \tag{38}$$

By the same argument, for $\beta \ge 0$,

$$\begin{aligned}&\int_0^t \|D^\beta V(t)\|_E\,ds\\ &\qquad\le K_5\exp(Mt)\left(\sum_{i=0}^2(\|D^\beta D_x^iV(0)\|^2 + \|D^\beta D_x^i\eta(0)\|^2 + \|D^\beta D_x^i\eta(0)\|_H)\right).\end{aligned} \tag{39}$$

From the definition of V^*, we have

$$\|D^\beta V^*\|_E \le \|D^\beta V_a\|_E + \|D^\beta V_i\|_E + \|D^\beta V\|_E$$
$$\le \|D^\beta V\|_E + K_6 \left(\sum_{i=0}^{5} \|D^\beta D_x^i \eta\|\right).$$

By Eqs. (37) and (39)

$$\int_0^t \|D^\beta V(t)\|_E \, ds \le K_7 \exp(Mt) \left(\sum_{i=0}^{6} (\|D^\beta D_x^i V(0)\|^2 + \|D^\beta D_x^i \eta(0)\|^2)\right)$$
$$\le K_8 \exp(Mt) \varepsilon^{2(\beta-2)}. \tag{40}$$

Let

$$\varepsilon_0 = ((2C_2)^{-1} \exp(-\nu t))^{1/2},$$

where C_2 is the constant in Eq. (21). Then if $0 < \varepsilon < \varepsilon_0$ and letting $r = (\varepsilon/\varepsilon_0)^2$, by Eqs. (35) and (39) we have

$$\int_0^t \|D^\beta V^*\|_E^2 \, ds \le K_2 \exp(\nu t) \varepsilon^{4+2\beta} (1 - r^{n+1})(1 - r)^{-1}$$
$$+ 2^n C_2^{n+1} \exp((n+1)\nu t) K_8 \exp(Mt) \left(\sum_{i=1}^{4} \varepsilon^{2(\beta+n+i-2)}\right).$$

Since $r^n \to 0$ as $n \to \infty$,

$$\int_0^t \|D^\beta V^*\|_E^2 \, ds \le K_2 \varepsilon^{4+2\beta} \exp(\nu t)(1 - r)^{-1}.$$

This proves the lemma.

LEMMA 5. If the condition in Lemma 4 is true, then

$$\|D^i \eta^*(t)\| \le \varepsilon^{2+i} L_0 \exp(\delta t),$$
$$\|D^i \eta_{xx}(t)\| \le \varepsilon^{2+i} L_0 \exp(\delta t) \quad \text{for } i = 0, 1, 2,$$
$$\|V^*(t)\| \le \varepsilon^2 L_0 \exp(\delta t),$$
$$\|w^*(t)\| \le \varepsilon^3 L_0 \exp(\delta t),$$

where $\eta^* = \eta - \eta_1$ and L_0, $\delta > 0$ are constants depending only on θ, R, σ.

Proof. By Eqs. (10) and (17), η^* satisfies

$$\eta_t^* + A_1 \eta_{xx}^* + B_1 \eta_{xxx}^* + C_1 \eta_{xxxx}^* = w^*, \tag{41}$$
$$\eta^* = 0 \quad \text{at } t = 0. \tag{42}$$

Multiply Eq. (41) by η^*, integrate over $[0, t] \times \Gamma$, and perform integration by parts to obtain

$$\begin{aligned}\|\eta^*(t)\|^2 - 2A_1\int_0^t \|\eta_x^*\|^2\,ds + 2C_1\int_0^t \|\eta_{xx}^*\|^2\,ds \\ = 2\int_0^t ds\int_\Gamma w^*\eta^*\,d\Gamma \\ \le 2\int_0^t ds\int_\Gamma\left(\int_0^1 w_z^*\,dz\right)\eta^*\,d\Gamma \\ \le (a)\int_0^t \|\eta_x^*\|^2\,ds + \frac{1}{a}\int_0^t \|V^*\|^2\,ds,\end{aligned}$$

where a is a constant, and choose $a = 1$ so that

$$\|\eta^*(t)\|^2 + 2C_1\int_0^t \|\eta_{xx}^*\|^2\,ds \le (2A_1+1)\int_0^t \|\eta_x^*\|^2\,ds + \int_0^t \|V^*\|^2\,ds.$$

By $\|\eta_x^*\|^2 \le (a_1/2)\|\eta_{xx}^*\|^2 + (1/2a_1)\|\eta^*\|^2$, we have

$$\|\eta^*(t)\|^2 \le (1+2A_1)^2(8C_1)^{-1}\int_0^t \|\eta^*(s)\|^2\,ds + \int_0^t \|V^*\|^2\,ds.$$

Then by Lemma 4,

$$\begin{aligned}\|\eta^*(t)\|^2 &\le (8C_1)(1+2A_1)^{-2}\exp((8C_1)^{-1}(1+2A_1)^2t)\int_0^t \|V^*(s)\|_E^2\,ds \\ &\le L_0\varepsilon^4\exp((\nu+(8C_1)^{-1}(1+2A_1)^2)t)(1-r)^{-1},\end{aligned}\tag{43}$$

for all $t \ge 0$. Multiply Eq. (17) by η_{xxxx} and follow the same derivation as before to get

$$\begin{aligned}\|\eta_{xx}(t)\|^2 - \|\eta_{xx}(0)\|^2 - 2A_1\int_0^t \|\eta_{xxx}(s)\|^2\,ds + 2C_1\int_0^t \|\eta_{xxxx}(s)\|^2\,ds \\ = 2\int_0^t ds\int_\Gamma w^*\eta_{xxxx}\,dx\end{aligned}$$

and

$$\|\eta_{xx}(t)\|^2 \le \|\eta_{xx}(0)\|^2 + \int_0^t (A_1^2/C_1)\|\eta_{xx}\|^2\,ds + (1/C_1)\int_0^t \|V^*\|_E^2\,ds,$$

which implies

$$\begin{aligned}\|\eta_{xx}(t)\|^2 &\le \exp\left(\frac{A_1^2t}{C_1}\right)\left(\|\eta_{xx}(0)\|^2 + \int_0^t \frac{1}{C_1}\|V^*\|_E^2\,ds\right) \\ &\quad + \frac{1}{C_1}\int_0^t \|V^*\|_E^2\,ds \\ &\le L_0\varepsilon^4\exp((A_1^2C_1^{-1}+\nu)t)(1-r)^{-1}.\end{aligned}\tag{44}$$

Furthermore when Eq. (33) was derived, we used $D_x^iV^*$, $i = 1, 2$, instead of V^* to obtain

$$\int_0^t \|D^\beta D_x^t V^*\|_E^2\,ds \le C_6\varepsilon^{4+2(i+\beta)}\exp(\nu t)(1-r)^{-1} \quad \text{for all } t \ge 0. \tag{45}$$

Then similar to the derivations of Eqs. (43) and (44), we have

$$\|D_x^i\eta^*(t)\|^2 \le L_1\varepsilon^{2(2+i)}\exp((\nu+(8C_2)^{-1}(1+2A_1)^2)t)(1-r)^{-1}, \tag{46}$$

$$\|D_x^i\eta_{xx}(t)\|^2 \le L_1\varepsilon^{2(2+i)}\exp((A_1^2C_1^{-1}+\nu)t)(1-r)^{-1}, \tag{47}$$

for $i = 1, 2$ by A.3. From Eq. (22) and the derivation of Eq. (23), we have

$$\|V^*(t)\|^2 - \|V^*(0)\|^2 + R^{-1}\int_0^t \|V^*\|_E^2\, ds$$
$$\le K_5\int_0^t \|V_x^*\|_E^2\, ds + K_6\int_0^t \left(\sum_{i=3}^{5}\|D_x^i\eta\|^2 + \sum_{i=0}^{3}\|D_tD_x^i\eta\|^2\right)^{1/2} ds.$$

By Eqs. (31) and (45), we have

$$\|V^*(t)\|^2 - \|V^*(0)\|^2 \le C_{11}\varepsilon^6\exp(\nu t)(1-r)^{-1} + C_{12}\varepsilon^6\exp(2\nu t)(1-r)^{-1}$$

and, by A.3,

$$\begin{aligned}\|V^*(t)\|^2 &\le \|V^*(0)\|^2 + C_{11}\varepsilon^6\exp(\nu t)(1-r)^{-1} + C_{12}\varepsilon^6\exp(2\nu t)(1-r)^{-1}\\ &\le C_{13}\varepsilon^4 + C_{11}\varepsilon^6\exp(\nu t)(1-r)^{-1} + C_{12}\varepsilon^6\exp(2\nu t)(1-r)^{-1}\\ &\le C_{14}\varepsilon^4\exp(2\nu t)(1-r)^{-1}.\end{aligned} \tag{48}$$

In exactly the same manner, we also have

$$\|V_x^*(t)\|^2 \le C_{15}\varepsilon^6(2\nu t)(1-r)^{-1}. \tag{49}$$

Since

$$\|w^*(t)\|^2 = \int_{-\infty}^{\infty} dx\int_{-1}^{0} dz\left(\int_{-1}^{z} w_z^*\, dz\right)^2 \le \int_\Omega (u_z^*)^2\, dA,$$

by (49),

$$\|w^*(t)\|^2 \le C_{15}\varepsilon^6\exp(2\nu t)(1-r)^{-1}.$$

This completes the proof of Lemma 5.

Now we are ready to prove Theorem 2.

Proof of Theorem 2. By Lemma 4, there exists an $\varepsilon_0 = ((2C_2)^{-1}\exp(-\nu T))^{1/2} > 0$ where C_2 and ν depend on R, θ, and σ only. If the initial data V_0 and η_0 satisfy (A.3) with $\varepsilon < \varepsilon_0$, then all the estimates in Lemma 5 hold. If we denote $(\int_\Omega u^2\, dA)^{1/2}$ by $\|u\|$, then by Lemma 5 for $0 \le t \le T$,

$$\begin{aligned}\|u-u_b\| &\le \|\overline{u}_1(z)\eta^*\| + \|\overline{u}_2(z)\eta_x^*\| + \|u_i\| + \|u^*\|\\ &\le L_1\varepsilon^2\exp(\delta t),\end{aligned}$$

$$\begin{aligned}\|w-w_b\| &\le \|\overline{w}_1(z)\eta_x^*\| + \|\overline{w}_2(z)\eta_{xx}^*\| + \|w_i\| + \|w^*\|\\ &\le L_1\varepsilon^3\exp(\delta t),\end{aligned}$$

and

$$\|\eta-\eta_1\| = \|\eta^*\| \le L_1\varepsilon^2\exp(\delta t),$$

where L_1 and δ only depend on θ, R, σ. Therefore, $\{V_b, \eta_1\}$ is an asymptotic approximation for $0 \le t \le T$ in the L_2 norm to the exact solution $\{V, \eta\}$, and for $0 \le t \le T$,

$$u = u_b + O(\varepsilon^2), \qquad w = \varepsilon w_b + O(\varepsilon^3), \qquad \eta = \eta_1 + O(\varepsilon^2),$$

where ε is a small positive parameter defined by the initial data. This proves the theorem.

Since all the estimates in Theorem 2 still hold if we replace V, η, V_b, and η_1 by $D^\beta V$, $D^\beta \eta$, $D^\beta V_b$, and $D^\beta \eta_1$ respectively, then by the usual Sobolev embedding theorem, we have the following corollary.

COROLLARY. Theorem 2 also implies the following pointwise estimates for error bounds:

$$\max_{-\infty<x<+\infty} |D^\beta \eta(t, x) - D^\beta \eta_1(t, x)| \le M_0 \varepsilon^2 \exp(\delta t)$$

for $0 \le t \le T$ and all β in $0 \le \beta \le m$ where m is a nonnegative integer and δ, $M_0 > 0$ depending only on R, θ, σ, and m.

Acknowledgment. The research reported here was supported in part by the National Science Foundation under grant CMS 8903083.

REFERENCES

[1] T. B. Benjamin, *Wave formulation in laminar flow down an inclined plane*, J. Fluid Mech. **2**, 554–574 (1957)

[2] C. S. Yih, *Stability of liquid flow down an inclined plane*, Phys. Fluids **6**, 321–334 (1963)

[3] C. C. Mei, *Nonlinear gravity waves in a thin sheet of viscous fluid*, J. Math. Phys. **45**, 266–288 (1966)

[4] D. J. Benney, *Long waves on liquid films*, J. Math. Phys. **45**, 150–155 (1966)

[5] D. D. Joseph, *Fluid Dynamics of Viscoelastic Liquids*, Springer-Verlag, New York, 1990

[6] S. M. Shih and M. C. Shen, *Uniform asymptotic approximation for viscous fluid flow down an inclined plane*, SIAM J. Math. Anal. **6**, 560–582 (1975)

[7] A. Carasso and M. C. Shen, *On viscous fluid flow down an inclined plane and the development of roll waves*, SIAM J. Appl. Math. **33**, 399–426 (1977)

[8] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969

[9] S. M. Shih, *Contributions to the theory of surface waves on a viscous fluid*, Ph.D. Thesis, University of Wisconsin-Madison, 1973

[10] M. C. Shen, S. M. Sun, and R. E. Meyer, *Surface waves on viscous magnetic fluid flow down an inclined plane*, Phys. Fluids A **3**, 439–445 (1991)