EXISTENCE OF CLASSICAL SOLUTIONS
FOR SINGULAR PARABOLIC PROBLEMS

By
C. Y. CHAN and BENEDICT M. WONG
University of Southwestern Louisiana

Abstract. Let \( Lu = u_{xx} + bu_x/x - u_t \) with \( b \) a constant less than 1. Its Green's function corresponding to first boundary conditions is constructed by eigenfunction expansion. With this, a representation formula is established to obtain existence of a classical solution for the linear first initial-boundary value problem. Uniqueness of a solution follows from the strong maximum principle. Properties of Green's function and of the solution are also investigated.

1. Introduction. Let

\[
    L = \frac{\partial^2}{\partial x^2} + \frac{b}{x} \frac{\partial}{\partial x} - \frac{\partial}{\partial t}.
\]

We are interested in studying existence and uniqueness of classical solutions for linear initial-boundary value problems involving \( L \). This operator arises in many situations, such as degenerate elliptic-parabolic operators (cf. Brezis, Rosenkrantz, and Singer with an appendix by Lax [2]), stochastic processes (cf. Lamperti [14]), and phase change processes (cf. Solomon [18]). When \( b = 0 \), it is the heat operator. For further discussions of the study and the significance of \( L \), we refer to Chan and Chen [5, 6], Chan and Cobb [7], Chan and Kaper [8], and the references cited there.

Without loss of generality and for simplicity, we take the spatial interval to be \([0, 1]\). Let \( b \) (< 1) and \( \Gamma \) (> 0) be constants, \( \Omega_\Gamma = (0, 1) \times (0, \Gamma) \), \( Q_\Gamma = (0, 1) \times (0, \Gamma) \), \( \tilde{Q}_\Gamma = (0, 1) \times [0, \Gamma] \), and \( \overline{Q}_\Gamma \) denote the closure of \( Q_\Gamma \). We study the linear singular problem,

\[
    Lu = -\Psi(x, t) \quad \text{in } Q_\Gamma,
\]

\[
    u(x, 0) = g(x) \quad \text{for } 0 < x < 1, \quad u(0, t) = 0 = u(1, t) \quad \text{for } 0 < t \leq \Gamma.
\]

More general linear problems with \( b \) a real constant were investigated by Alexiades [1]. Hence, an existence result for the above problem can be deduced from his work [1, Sec. 11]. For \( b < 1 \), he assumed that \( x^{b-1}\Psi(x, t) \) is in \( C(\overline{Q}_\Gamma) \); we note that if the solution \( u \) were known, the function \( x^{b-1}[1-u(x, t)]^{-1} \) would be discontinuous at \( x = 0 \), and thus would not satisfy his assumption in the case \( b < 1 \). Hence, his
(linear) result cannot be used through methods of successive approximations to study semilinear singular problems of the type,

\[ v_{xx} - v_t = - (1 - v)^{-1} \text{ in } \Omega_T, \]

\[ v(x, 0) = g(x) \quad \text{for } 0 \leq x \leq 1, \quad v(0, t) = 0 = v(1, t) \quad \text{for } 0 < t < T \leq \infty. \]

This problem with \( g(x) \equiv 0 \) was studied by Kawarada [12], through which he introduced the concept of quenching. Since then, many scientists have studied quenching problems (cf. Chan [4]).

In Sec. 2, we construct explicitly Green's function corresponding to the problem (1.1) and (1.2). Under appropriate conditions on \( g(x) \) and \( \Psi(x, t) \) (without assuming \( x^{b-1} \Psi(x, t) \) is in \( C(\overline{Q}_T) \)), we prove existence of a unique classical solution by establishing its representation formula. We also establish properties of Green's function and of the solution. In Sec. 3, we extend existence of a unique classical solution to nonhomogeneous boundary conditions.

2. Linear problem. Using separation of variables on the homogeneous problem corresponding to the problem (1.1) and (1.2), we obtain the singular Sturm-Liouville problem,

\[ (x^b X')' + \lambda x^b X = 0, \quad X(0) = 0 = X(1), \]

where \( \lambda \) is an eigenvalue. Let \( \nu = (1 - b)/2 \). Since \( \nu > 0 \), it follows from McLachlan [15, pp. 26 and 116] that the eigenvalues \( \lambda \) are positive and satisfy the equation \( J_{\nu}(\lambda^{1/2}) = 0 \), where \( J_{\nu}(z) \) is the Bessel function of the first kind of order \( \nu \). For \( z > 0 \), \( J_{\nu}(z) \) has infinitely many countable zeros; hence, there are infinitely many countable eigenvalues \( \lambda_n \), which can be arranged as \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) with \( \lambda_n \to \infty \) as \( n \to \infty \) (cf. Watson [19, pp. 490–492]). The corresponding eigenfunctions,

\[ \phi_n(x) = 2^{1/2} x^{\nu} J_{\nu}(\lambda_n^{1/2} x)/(|J_{\nu+1}(\lambda_n^{1/2})|), \]

form an orthonormal set with weight function \( x^b \) (cf. McLachlan [15, pp. 102–104]).

In the sequel, we let \( k_j \) \((j = 1, 2, 3, \ldots, 8)\) denote appropriate constants. For simplicity, we introduce the following notations:

\[ E_n(y) \equiv \exp(-\lambda_n y), \]

\[ I_n(h) \equiv \int_0^1 x^b h(x) \phi_n(x) dx. \]

If instead of \( h(x) \), we have \( h(x, t) \), then we use the notation \( I_n(h)(t) \). Similarly, let

\[ I(h) \equiv \int_0^1 x^{b/2} h(x) dx, \]

\[ I^2(h) \equiv \int_0^1 x^b h^2(x) dx, \]

and define \( I(h)(t) \) and \( I^2(h)(t) \) accordingly.

For convenience, we state the following results.
Lemma 1.
(a) \( |\phi_n(x)| \leq k_1 x^{-b/2} \) for \( x \) in \((0, 1]\).
(b) \( |\phi_n(x)| \leq k_2 x^{1/4} \) for \( x \) in \([0, 1]\).
(c) If \( I^2(h_1)(t) \leq k_3 \) for \( t \) in \([0, \Gamma_1]\), then for \( t \) in \([0, \Gamma_1]\),
\[
\sum_{n=1}^{\infty} [I_n(h_1)(t)]^2 \leq I^2(h_1)(t).
\]
(d) \( |\phi_n'(x)| \leq k_4 x^{1/2} \) for \( x \) in \([x_0, 1]\) where \( x_0 > 0 \) and \( k_4 \) depends on \( x_0 \).
(e) If \( I(h_2) \) exists (and is absolutely convergent in case the integral is improper), and if \( h_2(x) \) is continuous and of bounded variation on \([x_1, x_2]\), where \( 0 < x_1 < x_2 < 1 \), then \( \sum_{n=1}^{\infty} I_n(h_2) \phi_n(x) \) converges uniformly to \( h_2(x) \) on \((x_1 + \varepsilon, x_2 - \varepsilon)\) where \( \varepsilon \) is any positive number.

For the proofs of Lemma 1(a), (b), (d), and (e), we refer to Lemma 1(i) and (ii), (2.15), and Lemma 3 of Chan and Wong [9]. Lemma 1(c) follows directly from the Bessel inequality (cf. Weinberger [20, p. 73]).

Let us construct Green's function \( G(x, t; \xi, \tau) \) corresponding to the problem (1.1) and (1.2). It is determined by the following system: for \( x \) and \( \xi \) in \((0, 1]\), and \( t \) and \( \tau \) in \((-\infty, \infty)\),
\[
LG(x, t; \xi, \tau) = -\delta(x - \xi)\delta(t - \tau),
\]
\[
G(x, t; \xi, \tau) = 0, \quad t < \tau,
\]
\[
G(0, t; \xi, \tau) = 0 = G(1, t; \xi, \tau),
\]
where \( \delta(x) \) is the Dirac delta function. By the eigenfunction expansion,
\[
G(x, t; \xi, \tau) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x).
\]
Since
\[
\phi_n''(x) + \frac{b}{x} \phi_n'(x) + \lambda_n \phi_n(x) = 0,
\]
it follows that
\[
\sum_{n=1}^{\infty} [a'_n(t) + \lambda_n a_n(t)] \phi_n(x) = \delta(x - \xi)\delta(t - \tau).
\]
Multiplying both sides by \( x^b \phi_n(x) \), and integrating from 0 to 1 with respect to \( x \), we obtain
\[
\frac{d}{dt} \{\exp(\lambda_n t) a_n(t)\} = \xi^b \phi_n(\xi) [\exp(\lambda_n t)] \delta(t - \tau).
\]
By integrating from \( \tau^- \) to \( t \),
\[
[\exp(\lambda_n t)] a_n(t) - [\exp(\lambda_n \tau^-)] a_n(\tau^-) = \xi^b \phi_n(\xi) \exp(\lambda_n \tau).
\]
Since \( G(x, t; \xi, \tau) = 0 \) for \( t < \tau \), it follows that \( a_n(\tau^-) = 0 \) for all \( n \). Thus,
\[
a_n(t) = \xi^b \phi_n(\xi) E_n(t - \tau),
\]
and hence
\[ G(x, t; \xi, \tau) = \sum_{n=1}^{\infty} \xi^n \phi_n(\xi)\phi_n(x)E_n(t - \tau). \]  

Let \( D = \{(x, t; \xi, \tau) : x \text{ and } \xi \text{ are in } (0, 1), \text{ and } t > \tau \} \). By Lemma 1(b) and the fact that \( O(\lambda_n) = O(n^2) \) for large \( n \) (cf. Watson [19, p. 506]), it follows that the series in (2.1) converges in \( D \). Hence, \( G(x, t; \xi, \tau) \) exists.

A function \( u \) is said to be a classical solution of the problem (1.1) and (1.2) if

(a) \( u \) is in \( C(\overline{Q}_\Gamma) \),
(b) \( u_x, u_{xx}, \) and \( u_t \) are in \( C(Q_\Gamma) \),
(c) \( u \) satisfies (1.1) and (1.2).

Throughout this paper, by a solution of the problem (1.1) and (1.2), we refer to its classical solution.

Let \( \Psi(x, t) \) be defined in \( Q^- \). We need the following conditions:

(A) \( I^2(\Psi)(t) \leq k_5 \) for \( t \) in \([0, \Gamma]\),
(B) \( I(\|\Psi\|)(t) \leq k_6 \) a.e. for \( t \) in \([0, \Gamma]\).

**Theorem 2.** The problem (1.1) and (1.2) has at most one solution. Suppose \( \Psi(x, t) \) is in \( C(\overline{Q}_\Gamma) \), absolutely continuous on the interval \( 0 \leq t \leq \Gamma \) for each \( x \) in \((0, 1)\), and of bounded variation with respect to \( x \) on every given closed subinterval of \((0, 1)\). If Conditions (A) and (B) hold, then the problem (1.1) and (1.2) with \( g \equiv 0 \) has a unique solution \( u \) given by
\[ u(x, t) = \int_0^t \int_0^1 G(x, t; \xi, \tau)\Psi(\xi, \tau) \, d\xi \, d\tau. \]  

**Proof.** Uniqueness of a solution follows from the strong maximum principle (cf. Protter and Weinberger [16, pp. 168–170]).

From (2.1) and (2.2),
\[ u(x, t) = \int_0^t \int_0^1 \sum_{n=1}^{\infty} \xi^n \phi_n(\xi)\phi_n(x)E_n(t - \tau)\Psi(\xi, \tau) \, d\xi \, d\tau. \]

By Lemma 1(a) and (b), we have for \( x \) in \([0, 1]\) and \( \xi \) in \((0, 1)\),
\[ |\xi^n \phi_n(\xi)\phi_n(x)\Psi(\xi, \tau)| \leq k_1 k_2 \lambda_n^{1/4} \xi^{b/2} |\Psi(\xi, \tau)|. \]  

For any fixed \((x, t)\) in \( \overline{Q}_\Gamma \), let
\[ G_m(\xi, \tau) = \begin{cases} \sum_{n=1}^{m} \xi^n \phi_n(\xi)\phi_n(x)E_n(t - \tau) & \text{for } t - \tau > 0, \\ 0, & \text{otherwise.} \end{cases} \]

Then, \( G_m(\xi, \tau)\Psi(\xi, \tau) \) converges to \( G(x, t; \xi, \tau)\Psi(\xi, \tau) \) a.e. on \( \overline{Q}_\Gamma \). From (2.3),
\[ |G_m(\xi, \tau)\Psi(\xi, \tau)| \leq \rho(\xi, \tau) \]
for all positive integers \( m \) where
\[ \rho(\xi, \tau) = \begin{cases} k_1 k_2 \xi^{b/2} |\Psi(\xi, \tau)| \sum_{n=1}^{\infty} \lambda_n^{1/4} E_n(t - \tau) & \text{for } t - \tau > 0, \\ 0, & \text{otherwise.} \end{cases} \]
Let \( \rho_m(\xi, \tau) \) be the \( m \)th partial sum of \( \rho(\xi, \tau) \). Then, \( \{\rho_m\} \) is a sequence of nonnegative measurable functions that converge monotonically to \( \rho \) on \( \overline{Q}_T \), and \( \rho_m \leq \rho \) for all positive integers \( m \). By the Monotone Convergence Theorem and the Fubini Theorem (cf. Royden [17, pp. 84 and 269]),

\[
\int_{Q_T} \rho(\xi, \tau) \, d\xi \, d\tau = \lim_{m \to \infty} \int_0^T \int_0^1 \rho_m(\xi, \tau) \, d\xi \, d\tau
\]

\[
= \lim_{m \to \infty} k_1k_2 \sum_{n=1}^m \left[ \int_0^1 I(|\Psi|)(\tau)\lambda_n^{1/4} E_n(t-\tau) \, d\tau \right].
\]

By the Schwarz inequality and Condition (A),

\[
\int_{Q_T} \rho(\xi, \tau) \, d\xi \, d\tau \leq k_1k_2k_5^{1/2} \lim_{m \to \infty} \sum_{n=1}^m \lambda_n^{-3/4}.
\]

Since \( O(\lambda_n) = O(n^2) \) for large \( n \), it follows that \( \sum_{n=1}^m \lambda_n^{-3/4} \) converges. Hence, \( \rho(\xi, \tau) \) is integrable, and for each fixed \( (x, t) \) in \( \overline{Q}_T \), the integral in (2.2) exists. By the Lebesgue Convergence Theorem (cf. Royden [17, p. 88]) and the Fubini Theorem,

\[
u(x, t) = \sum_{n=1}^\infty \int_0^t I_n(\Psi)(\tau) E_n(t-\tau) \, d\tau \phi_n(x).
\]

By Lemma 1(c) and Condition (A),

\[
\left| \int_0^t I_n(\Psi)(\tau) E_n(t-\tau) \, d\tau \right| \leq k_5^{1/2}\lambda_n^{-1}.
\]

It follows from Lemma 1(b) that the series representing \( u(x, t) \) converges absolutely and uniformly on \( \overline{Q}_T \). Thus, \( u(x, t) \) is in \( C(\overline{Q}_T) \), and hence \( u(x, t) \) satisfies the homogeneous initial and boundary conditions.

Next, we would like to show the differentiability of the solution \( u(x, t) \). Let

\[
S_m(x, t) = \sum_{n=1}^m \int_0^t I_n(\Psi)(\tau) E_n(t-\tau) \, d\tau \phi_n(x)
\]

\[
= \sum_{n=1}^m \int_0^1 \xi^b \phi_n(\xi) \left[ \int_0^t \Psi(\xi, \tau) E_n(t-\tau) \, d\tau \right] d\xi \phi_n(x).
\]

Since \( \Psi(\xi, \tau) \) is absolutely continuous on the interval \( 0 \leq \tau \leq \Gamma \) for each \( \xi \) in \( (0, 1) \), it follows from integration by parts with respect to \( \tau \) (cf. Chae [3, pp. 227–228]) that

\[
S_m(x, t) = \sum_{n=1}^m \lambda_n^{-1} \left[ I_n(\Psi)(t) - I_n(\Psi)(0) E_n(t) \right.
\]

\[
- \int_0^1 \xi^b \phi_n(\xi) \int_0^t \Psi(\xi, \tau) E_n(t-\tau) \, d\tau \, d\xi \phi_n(x).
\]
For \( x \) in \([x_0, 1]\) where \( x_0 \) is any positive number in \((0, 1)\), it follows from Lemma 1(d) that for any positive integers \( p \) and \( m \) with \( p > m \),

\[
\left| \frac{\partial S_p}{\partial x} - \frac{\partial S_m}{\partial x} \right| \leq k_4 \sum_{n=m+1}^{p} \lambda_n^{-1/2} |I_n(\Psi)(t)| + k_4 \sum_{n=m+1}^{p} \lambda_n^{-1/2} |I_n(\Psi)(0)|
\]

\[
+ k_4 \sum_{n=m+1}^{p} \lambda_n^{-1/2} \left| \int_{0}^{1} \xi^b \phi_n(\xi) \int_{0}^{t} \Psi_{\tau}(\xi, \tau) E_n(t-\tau) \, d\tau \, d\xi \right|.
\]

(2.5)

From Condition (A) and Lemma 1(c),

\[
\left( \sum_{n=m+1}^{p} |I_n(\Psi)(t)|^2 \right)^{1/2} \leq k_5^{1/2}.
\]

By the Schwarz inequality, the first term on the right-hand side of the inequality (2.5) is bounded by

\[
k_4 k_5^{1/2} \left( \sum_{n=m+1}^{p} \lambda_n^{-1} \right)^{1/2},
\]

which converges to 0 as \( p \) and \( m \) tend to infinity since \( O(\lambda_n) = O(n^2) \) for large \( n \). Similarly, the second term converges to 0 as \( p \) and \( m \) tend to infinity. By Lemma 1(a) and Condition (B),

\[
\left| \int_{0}^{t} I_n(\Psi_{\tau})(\tau) E_n(t-\tau) \, d\tau \right| \leq k_1 \int_{0}^{t} |I(\Psi_{\tau})| E_n(t-\tau) \, d\tau
\]

\[
\leq k_1 k_6 \lambda_n^{-1} [1 - E_n(t)]
\]

\[
\leq k_1 k_6 \lambda_n^{-1}.
\]

(2.6)

It follows from the Tonelli Theorem (cf. Royden [17, p. 270]) that

\[
\xi^b \phi_n(\xi) \Psi_{\tau}(\xi, \tau) E_n(t-\tau)
\]

is integrable on \( \overline{Q}_T \). By the Fubini Theorem,

\[
\left| \int_{0}^{1} \xi^b \phi_n(\xi) \int_{0}^{t} \Psi_{\tau}(\xi, \tau) E_n(t-\tau) \, d\tau \, d\xi \right| = \left| \int_{0}^{t} I_n(\Psi_{\tau})(\tau) E_n(t-\tau) \, d\tau \right|
\]

\[
\leq k_1 k_6 \lambda_n^{-1}.
\]

Thus, the third term on the right-hand side of (2.5) is bounded by

\[
k_1 k_4 k_6 \sum_{n=m+1}^{p} \lambda_n^{-3/2},
\]

which converges to 0 as \( p \) and \( m \) tend to infinity. Therefore on \([x_0, 1] \times [0, \Gamma]\), \(|\partial S_p/\partial x - \partial S_m/\partial x|\) converges to 0 uniformly as \( p \) and \( m \) tend to infinity. Hence, \( \partial S_m/\partial x \) converges uniformly. Since \( x_0 \) (> 0) is arbitrarily chosen and each term in the series representing \( \partial S_m/\partial x \) is continuous, it follows that \( \partial S_m/\partial x \) converges uniformly on every given closed subset of \((0, 1] \times [0, \Gamma]\) to

\[
u_x(x, t) = \sum_{n=1}^{\infty} \int_{0}^{t} I_n(\Psi)(\tau) E_n(t-\tau) \, d\tau \phi_n(x),
\]
and \( u_x(x, t) \) is in \( C(\overline{Q}_\Gamma \setminus P_1) \) where \( P_1 \equiv \{(0, t) : 0 \leq t \leq \Gamma\} \).

Let the \( m \)th partial sum of \( u_x(x, t) \) be denoted by \( S_{xm}(x, t) \). Since

\[ \phi_n''(x) + \frac{b}{x} \phi_n'(x) + \lambda_n \phi_n(x) = 0, \]

we have from (2.4) that

\[
\frac{\partial S_{xm}(x, t)}{\partial x} = -\frac{b}{x} S_{xm}(x, t) - \sum_{n=1}^{m} I_n(\Psi)(t)\phi_n(x) \\
+ \sum_{n=1}^{m} I_n(\Psi)(0)E_n(t)\phi_n(x) + \sum_{n=1}^{m} \int_{0}^{t} I_n(\Psi_{\tau})(\tau)E_n(t-\tau)\,d\tau\phi_n(x). 
\]

(2.7)

Since \( S_{xm}(x, t) \) converges uniformly on \([x_0, 1] \times [0, \Gamma]\) for arbitrarily fixed \( x_0 > 0 \), we have \((b/x)S_{xm}(x, t)\) converges uniformly there. For each fixed \( t \geq 0 \), it follows from Condition (A) and Lemma 1(e) that the second term on the right-hand side of (2.7) converges uniformly to \(-\Psi(x, t)\) on every given closed subinterval of \((0, 1)\). By Lemma 1(e) and the Abel test (cf. Knopp [13, p. 346]), the third term converges uniformly on every given closed subset of \( Q_\Gamma \); because of the term \( E_n(t) \), it converges absolutely and uniformly on every given closed subset of \([0, 1] \times (0, \Gamma]\). Hence, the third term converges uniformly on every given closed subset of \( \overline{Q}_\Gamma \) where \( P_2 \equiv \{(0, 0)\} \cup \{(1, 0)\} \). From (2.6), the absolute value of the last term is bounded by \( \sum_{n=1}^{m} k_1k_6\lambda_n^{-1}|\phi_n(x)| \), and hence converges absolutely and uniformly on \( \overline{Q}_\Gamma \). Therefore, for each fixed \( t \geq 0 \), \( \partial S_{xm}(x, t)/\partial x \) converges uniformly on every given closed subinterval of \((0, 1)\). Thus from (2.7),

\[
\begin{align*}
  u_{xx}(x, t) &= \sum_{n=1}^{\infty} \int_{0}^{t} I_n(\Psi)(\tau)E_n(t-\tau)\,d\tau\phi_n''(x) \\
  &= -\frac{b}{x} u_x(x, t) - \Psi(x, t) + \sum_{n=1}^{\infty} I_n(\Psi)(0)E_n(t)\phi_n(x) \\
  &\quad + \sum_{n=1}^{\infty} \int_{0}^{t} I_n(\Psi_{\tau})(\tau)E_n(t-\tau)\,d\tau\phi_n(x). 
\end{align*}
\]

(2.8)

Since each term on the right-hand side of (2.8) is continuous in \( Q_\Gamma^- \), it follows that \( u_{xx}(x, t) \) is in \( C(Q_\Gamma^-) \).

To show that \( u(x, t) \) is differentiable with respect to \( t \), it follows from the Leibnitz rule on differentiation that

\[
\frac{\partial S_m(x, t)}{\partial t} = \sum_{n=1}^{m} I_n(\Psi)(t)\phi_n(x) - \sum_{n=1}^{m} \lambda_n \int_{0}^{t} I_n(\Psi)(\tau)E_n(t-\tau)\,d\tau\phi_n(x).
\]
By using integration by parts on $\int_0^t \Psi(\xi, \tau) E_n(t - \tau) d\tau$ of the last term, we have

$$\frac{\partial S_m(x, t)}{\partial t} = \sum_{n=1}^m I_n(\Psi)(0) E_n(t) \phi_n(x) + \sum_{n=1}^m \int_0^t I_n(\Psi(\tau))(t) E_n(t - \tau) d\tau \phi_n(x),$$

which are equal to the last two terms on the right-hand side of (2.7). Thus, $\frac{\partial S_m(x, t)}{\partial t}$ converges uniformly on every given closed subset of $Q_\Gamma \setminus P_2$. Hence,

$$u_t(x, t) = \sum_{n=1}^\infty I_n(\Psi)(0) E_n(t) \phi_n(x) + \sum_{n=1}^\infty \int_0^t I_n(\Psi(\tau))(t) E_n(t - \tau) d\tau \phi_n(x); \quad (2.9)$$

that is,

$$u_t(x, t) = \int_0^t G(x, t; \xi, 0) \Psi(\xi, 0) d\xi + \int_0^t \int_0^t G(x, t; \xi, \tau) \Psi(\xi, \tau) d\xi d\tau. \quad (2.10)$$

Also, we have $u_t(x, t)$ is in $C(Q_\Gamma \setminus P_2)$.

From (2.8) and (2.9), we have

$$Lu(x, t) = -\Psi(x, t) \quad \text{in} \quad Q_\Gamma^-.$$
Proof. (a) By Lemma 1(b),
\[ \sum_{n=1}^{\infty} |\xi^b \phi_n(\xi) \phi_n(x) E_n(t - \tau)| \leq \xi^b k_2^2 \sum_{n=1}^{\infty} \lambda_n^{1/2} E_n(t - \tau). \]
Since \( O(\lambda_n) = O(n^2) \) for large \( n \), \( \sum_{n=1}^{\infty} \lambda_n^{1/2} E_n(t - \tau) \) converges uniformly for \( t - \tau \geq \varepsilon \) where \( \varepsilon \) is any positive number. Hence, \( G(x, t; \xi, \tau) \) is continuous for \( t - \tau \geq \varepsilon \). Since \( \varepsilon \) is arbitrarily chosen, our assertion follows.

(b) From Lemma 6 of Chan and Wong [10], the \( m \)th derivative of \( \phi_n(x) \) satisfies the inequality,
\[ |\phi_n^{(m)}(x)| \leq K_m \lambda_n^{m/2} x^{\nu-m} |J_{\nu+1}(\lambda_n^{1/2})|, \quad n = 1, 2, 3, \ldots, \]
where \( K_m \) is a constant depending on \( m \). From Lemma 1(a),
\[ \sum_{n=1}^{\infty} |\xi^b \phi_n(\xi) \phi_n^{(m)}(x) E_n(t - \tau)| \leq k_1 K_m \xi^{b/2} x^{\nu-m} \sum_{n=1}^{\infty} \lambda_n^{m/2} E_n(t - \tau) / |J_{\nu+1}(\lambda_n^{1/2})|. \]
It follows from (2.10) of Chan and Wong [9], and \( O(\lambda_n) = O(n^2) \) for large \( n \) that
\[ \sum_{n=1}^{\infty} \lambda_n^{m/2} E_n(t - \tau) / |J_{\nu+1}(\lambda_n^{1/2})| \]
converges uniformly for \( t - \tau \geq \varepsilon \), and hence \( \partial^m G / \partial x^m \) is continuous for \( t - \tau > 0 \) since \( \varepsilon \) is arbitrarily chosen. Now,
\[ \frac{\partial^m}{\partial t^m} E_n(t - \tau) = (-1)^m \lambda_n^m E_n(t - \tau). \]
An argument similar to the above shows that \( \partial^m G / \partial t^m \) is continuous for \( t - \tau > 0 \).
Since \( m \) is any positive integer, our assertion follows.

(c) Suppose \( G(x, t; \xi, \tau) < 0 \) at some point \( (x_1, t_1; \xi_1, \tau_1) \) in \( D_1 \). Since \( G(x, t; \xi, \tau) \) is continuous in \( D_1 \), we may assume \( \tau_1 > 0 \). Hence, there exists a positive number \( \varepsilon \) such that \( G(x, t; \xi, \tau) < 0 \) in the set
\[ W_0 = (x_1 - \varepsilon, x_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon) \times (\xi_1 - \varepsilon, \xi_1 + \varepsilon) \times (\tau_1 - \varepsilon, \tau_1 + \varepsilon) \]
contained in \( D_1 \). Let
\[ W_1 = (\xi_1 - \varepsilon, \xi_1 + \varepsilon) \times (\tau_1 - \varepsilon, \tau_1 + \varepsilon), \]
\[ W_2 = \left( \frac{\xi_1 - \varepsilon}{2}, \frac{\xi_1 + \varepsilon}{2} \right) \times \left( \frac{\tau_1 - \varepsilon}{2}, \frac{\tau_1 + \varepsilon}{2} \right). \]
There exists (cf. Dunford and Schwartz [11, pp. 1640–1641]) a function \( h_3(x, t) \) in \( C^\infty(\mathbb{R}^2) \) such that \( h_3 \equiv 1 \) on \( W_2 \), \( h_3 \equiv 0 \) outside \( W_1 \), and \( 0 \leq h_3 \leq 1 \) in \( W_1 \setminus W_2 \). It is clear that \( h_3(x, t) \) satisfies the conditions for \( \Psi \) in Theorem 2. Hence, the solution of the problem,
\[ Lw(x, t) = -h_3(x, t) \quad \text{in} \quad Q_\alpha, \quad t_1 < \alpha, \]
with \( w \) satisfying zero initial and boundary conditions, is given by
\[ w(x, t) = \int_{t_1 - \varepsilon}^{\tau_1 + \varepsilon} \int_{\xi_1 - \varepsilon}^{\xi_1 + \varepsilon} G(x, t; \xi, \tau) h_3(\xi, \tau) \, d\xi \, d\tau. \]
Since \( G(x, t; \xi, \tau) < 0 \) in \( W_0 \), \( h_3(\xi, \tau) \geq 0 \) in \( W_1 \), and \( h_3 = 1 \) on \( \overline{W}_2 \), it follows that
\[
w(x, t) < 0 \quad \text{for } (x, t) \in (x_1 - \epsilon, x_1 + \epsilon) \times (t_1 - \epsilon, t_1 + \epsilon).
\]
On the other hand, \( h_3(x, t) \geq 0 \) in \( Q_\alpha \) implies \( w(x, t) \geq 0 \) by the weak maximum principle. We have a contradiction. Therefore, \( G(x, t; \xi, \tau) \geq 0 \) in \( D_1 \).

Suppose \( G(x, t; \xi, \tau) = 0 \) at some point \((x_2, t_2; \xi_2, \tau_2)\) in \( D_1 \). Then by the strong maximum principle,
\[
G(x, t; \xi_2, \tau_2) = 0 \quad \text{in } D_1 \cap \{(x, t; \xi_2, \tau_2): 0 < x < 1, \ t < t_2\}.
\]
On the other hand,
\[
G(\xi_2, t_2; \xi_2, \tau_2) = \sum_{n=1}^{\infty} \xi_2^b \phi_n^2(\xi_2) E_n(t_2 - \tau_2),
\]
which is positive. This contradiction implies \( G > 0 \) in \( D_1 \).

We would like to establish some properties of the solution \( u(x, t) \). Let
\[
el = \frac{\partial^2}{\partial x^2} + \frac{b}{x} \frac{\partial}{\partial x}.
\]

**Theorem 5.** Under the hypotheses of Theorem 3, if \( I^2(g) \) exists, then the solution \( u(x, t) \) of the problem (1.1) and (1.2) has the following properties:

(a) \( u_x \) is in \( C(\overline{Q}_\Gamma \setminus P_1) \), \( u_{xx} \) is in \( C(Q_\Gamma^-) \), and \( u_t \) is in \( C(\overline{Q}_\Gamma \setminus P_2) \);

(b) \( u(x, t) \) is absolutely continuous on the interval \( 0 \leq t \leq \Gamma \) for each \( x \) in \([0, 1]\); furthermore, \( I^2(u)(t) \leq k_7 \) and \( I^2(u_t)(t) \leq k_8 \) for \( t \) in \([0, \Gamma]\);

(c) \( I^2(\ell u)(t) < \infty \) for \( t \) in \([0, \Gamma]\).

**Proof.** (a) This property follows from the hypotheses on \( \Psi(x, t) \) and \( g(x) \), and a proof as in that of Theorem 2 (with \( \Psi \) replaced by \( \Psi + Lg \)).

(b) It follows from Theorem 5(a) and \( u(0, t) = 0 = u(1, t) \) that \( u(x, t) \) is absolutely continuous on the interval \( 0 \leq t \leq \Gamma \) for each \( x \) in \([0, 1]\).

By the Schwarz inequality,
\[
I^2(u)(t) = I^2(u - g)(t) + I^2(g) + 2 \int_0^1 x^b \left[u(x, t) - g(x)\right]g(x) \, dx \tag{2.12}
\]
\[
\leq I^2(u - g)(t) + I^2(g) + 2 [I^2(u - g)(t)]^{1/2} [I^2(g)]^{1/2}.
\]
From (2.11),
\[
u(x, t) = \sum_{n=1}^{\infty} \int_0^t I_n(\Psi + Lg)(\tau) E_n(t - \tau) \, d\tau \phi_n(x) + g(x).
\]
From the proof of Theorem 2 (on \( u \) with \( \Psi \) replaced by \( \Psi + Lg \)), the above series (on the right-hand side) representing \( u(x, t) - g(x) \) is absolutely and uniformly convergent on \( \overline{Q}_\Gamma \). By Lemma 1(a) and (c), this is also true for the series representing \( x^{b/2}[u(x, t) - g(x)] \). Hence, the series representing \( x^b[u(x, t) - g(x)]^2 \) is also absolutely and uniformly convergent on \( \overline{Q}_\Gamma \) (cf. Knopp [13, pp. 146 and 337]). Since
\{\phi_n(x)\} is an orthonormal set with weight function \(x^b\), it follows that
\[
I^2(u - g)(t) = \sum_{n=1}^{\infty} \left[ \int_0^t I_n(\Psi + Lg)(\tau)E_n(t - \tau) d\tau \right]^2.
\]

By Lemma 1(c),
\[
I^2(u - g)(t) \leq \left[ \sup_{0 \leq \tau \leq \Gamma} I^2(\Psi + Lg)(\tau) \right] \sum_{n=1}^{\infty} \left[ \int_0^t E_n(t - \tau) d\tau \right]^2
\]
\[
\leq \left[ \sup_{0 \leq \tau \leq \Gamma} I^2(\Psi + Lg)(\tau) \right] \sum_{n=1}^{\infty} \lambda_n^{-2}.
\]

From (2.12),
\[
I^2(u)(t) \leq \left[ \sup_{0 \leq \tau \leq \Gamma} I^2(\Psi + Lg)(\tau) \right] \sum_{n=1}^{\infty} \lambda_n^{-2} + I^2(g)
\]
\[
+ 2 \left[ \left( \sum_{n=1}^{\infty} \lambda_n^{-2} \right)^{1/2} \right]^2 \left[ I^2(g) \right]^{1/2}.
\]

It follows from the hypotheses on \(\Psi\) and \(Lg\) that
\[
\sup_{0 \leq \tau \leq \Gamma} I^2(\Psi + Lg)(\tau) < \infty.
\]

Because \(O(\lambda_n) = O(n^2)\) for large \(n\), we have from (2.13) that \(I^2(u)(t) \leq k_7\) for \(t\) in \([0, \Gamma]\).

By (2.9) (with \(\Psi(x, t)\) replaced by \(\Psi(x, t) + Lg(x)\)),
\[
u_t(x, t) = \sum_{n=1}^{\infty} I_n(\Psi + Lg)(0)E_n(t)\phi_n(x) + \sum_{n=1}^{\infty} \int_0^t I_n(\Psi_\tau)(\tau)E_n(t - \tau) d\tau \phi_n(x).
\]

Let \(t_0\) in \((0, \Gamma]\) be fixed. By Lemma 1(a) and (c), the right-hand side of (2.14) multiplied by \(x^{b/2}\) converges absolutely and uniformly on \([0, 1]\) to \(x^{b/2}u_t(x, t_0)\).

Hence, the series representing \(x^b u_t^2(x, t_0)\) is absolutely and uniformly convergent on \([0, 1]\). Integrating this series representing \(x^b u_t^2(x, t_0)\) with respect to \(x\) and using the orthonormality of the sequence \{\phi_n(x)\} with weight function \(x^b\), we have
\[
I^2(u_t)(t_0) = \sum_{n=1}^{\infty} \left[ I_n(\Psi + Lg)(0)E_n(t_0) \right]^2
\]
\[
+ \sum_{n=1}^{\infty} \left[ \int_0^{t_0} I_n(\Psi_\tau)(\tau)E_n(t_0 - \tau) d\tau \right]^2
\]
\[
+ 2 \sum_{n=1}^{\infty} \left[ \int_0^{t_0} I_n(\Psi_\tau)(\tau)E_n(t_0 - \tau) d\tau \right] \left[ I_n(\Psi + Lg)(0)E_n(t_0) \right].
\]

From Lemma 1(c) and \(E_n(t_0) \leq 1\) for all positive integers \(n\), the first term on the right-hand side is bounded by \(I^2(\Psi + Lg)(0)\). From Lemma 1(a) and Condition (B),
the second term is bounded by
\[ k_1 k_6 \sum_{n=1}^{\infty} \left[ \int_0^{t_0} E_n(t_0 - \tau) d\tau \right]^2 \leq k_1 k_6 \sum_{n=1}^{\infty} \lambda_n^{-2}. \]

By using the Schwarz inequality on the third term, we obtain
\[ I^2(u_t)(t_0) \leq I^2(\Psi + Lg)(0) + k_1 k_6 \sum_{n=1}^{\infty} \lambda_n^{-2} + 2[I^2(\Psi + Lg)(0)]^{1/2} \left[ k_1 k_6 \sum_{n=1}^{\infty} \lambda_n^{-2} \right]^{1/2}. \]

We note that the right-hand side is independent of \( t_0 \). Hence, \( I^2(u_t)(t) \) is bounded on \((0, \Gamma]\). As for \( I^2(u_t)(0) \), it follows from Lemma 1(e) that for \( x \) in \((0, 1)\),
\[ u_t(x, 0) = \sum_{n=1}^{\infty} I_n(\Psi + Lg)(0) \phi_n(x) = \Psi(x, 0) + Lg(x), \]
from which,
\[ I^2(u_t)(0) = I^2(\Psi + Lg)(0). \]

Thus, \( I^2(u_t)(t) \leq k_8 \) on \([0, \Gamma]\) for some constant \( k_8 \).

(c) Since \( \ell u = u_t - \Psi \), it follows from the Schwarz inequality that
\[ I^2(\ell u)(t) = I^2(u_t - \Psi)(t) = I^2(u_t)(t) + I^2(\Psi)(t) - 2 \int_0^1 x^b u_t(x, t) \Psi(x, t) dx \leq I^2(u_t)(t) + I^2(\Psi)(t) + 2[I^2(u_t)(t)I^2(\Psi)(t)]^{1/2}. \]

Then from Theorem 5(b) and Condition (A), \( I^2(\ell u)(t) < \infty \) on \([0, \Gamma]\).

3. Nonhomogeneous boundary conditions. In this section, we assume \( |b| < 1 \); we also assume as in Sec. 2 that \( g(x) \), \( Lg(x) \), and \( \Psi(x, t) \) satisfy the hypotheses of Theorem 3, except that \( g(0) = 0 = g(1) \). Let us consider the linear problem, (1.1), subject to
\[ u(x, 0) = g(x) \text{ for } 0 \leq x \leq 1, \]
\[ u(0, t) = r_1(t) \text{ and } u(1, t) = r_2(t) \text{ for } 0 < t \leq \Gamma < \infty, \]
where \( r_1(t) \) and \( r_2(t) \) are in \( C^2[0, \infty) \) such that \( r_1(0) = g(0) \) and \( r_2(0) = g(1) \).

**Theorem 6.** The problem (1.1) and (3.1) has a unique solution.

**Proof.** Let us consider the problem,
\[ Lw(x, t) = -[\Psi(x, t) + Ls(x, t)] \text{ in } Q_\Gamma, \]
\[ w(x, 0) = g(x) - s(x, 0) \text{ for } 0 \leq x \leq 1, \quad w(0, t) = 0 = w(1, t) \text{ for } 0 < t \leq \Gamma, \]
where
\[ s(x, t) = (1 - x^{2\nu})r_1(t) + x^{2\nu} r_2(t). \]
It follows from the assumptions on \( \Psi(x, t) \), \( g(x) \), \( r_1(t) \), and \( r_2(t) \) that \( \Psi(x, t) + Ls(x, t) \) and \( g(x) - s(x, 0) \) satisfy the conditions for \( \Psi(x, t) \) and \( g(x) \), respectively, in Theorem 3. Hence, \( w(x, t) \) exists and is unique. It follows that \( u \) given by \( u = w + s \) is the unique solution of the problem (1.1) and (3.1).

We note that the solution \( u \) in Theorem 6 has the properties stated in Theorem 5.

REFERENCES