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ON A METHOD TO EVALUATE FOURIER-BESSEL SERIES WITH POOR CONVERGENCE PROPERTIES AND ITS APPLICATION TO LINEARIZED SUPERSONIC FREE JET FLOW

By

A. DILLMANN (*Deutsche Forschungsanstalt für Luft- und Raumfahrt, Göttingen, Germany*)

AND

G. GRABITZ (*Max-Planck-Institut für Strömungsforschung, Göttingen, Germany*)

Abstract. An analytical method based on Kummer's series transformation is presented, which allows for the evaluation of Fourier-Bessel series with poor convergence properties and, in addition, yields the singularities of the series in closed form. This method is applied to the Fourier-Bessel series which arise as solutions of the linearized gas dynamic potential equation for the cylindrical flow field of a supersonic free jet. As an illustrative example, the presented method is applied to D. C. Pack's classical solution for the axisymmetric free jet with initial homogeneous pressure perturbation, which in its original form cannot be evaluated directly. It is shown that in contrast to the plane jet, the flow field of the axisymmetric jet does not exhibit a strictly periodic behaviour, whereas its singularities are distributed periodically.

1. Introduction. Many problems of mathematical physics are described by boundary value problems for linear partial differential equations, whose solutions arise as infinite series of the eigenfunctions of the differential operators involved. In many cases, such series converge rather rapidly and, thus, there is no serious problem in evaluating their sum numerically. However, there is also a number of cases in which an obtained series diverges or at least converges so badly that it is practically worthless because its sum cannot be evaluated. The mathematical reason for poor convergence properties is that the function described by such a series contains singularities in its domain of definition. These singularities arise mostly from some overidealized mathematical modeling of the physical problem such as linearization or discontinuous boundary conditions, which, however, is often unavoidable in order to bring the problem into a mathematical form which allows for an analytical treatment. Thus, one is sometimes in the situation of having solved a problem analytically but being unable to discuss the solution because its numerical evaluation is not feasible in spite of the enormous computational power available today.

The described situation occurs rather frequently in linearized supersonic poten-

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tial flow theory, since discontinuities of the physical flow variables (e.g., pressure) are definitely allowed by the fundamental laws of fluid mechanics for an inviscid compressible fluid [1]. A basic example for such a case is the supersonic free jet emerging from a circular nozzle into a medium at rest. The pressure on the cylindrical jet boundary is constant and equal to that of the surrounding medium, whereas throughout the nozzle exit cross section (i.e., the initial free jet cross section), the pressure is in general variable and differs from the ambient pressure. It will be shown in the following that a discontinuity of pressure (or other flow variables) at the nozzle edge leads to singularities in the flow field and, thus, to nonuniform convergence (or sometimes, even divergence) of the Fourier-Bessel series arising as solutions of the cylindrical free jet problem. In this case, the direct evaluation and discussion of the obtained solutions is impracticable.

In order to resolve this problem, a series transform based on Kummer's comparison method [2, 3] is presented, which allows for a complete evaluation of Fourier-Bessel series with singularities. Since the presented method yields the singularities of a given series in closed analytical form, it allows for their identification and convenient discussion. To demonstrate its capabilities, the presented method is applied to D. C. Pack's classical solution for the axisymmetric free jet with constant initial pressure perturbation [4], which was not evaluated by Pack himself because of the convergence problems mentioned above.

2. Linearized potential theory of supersonic free jet flow. In the following, we consider a supersonic jet of gas issuing from a nozzle into a medium at rest with the pressure difference between the jet and the surrounding medium being small. The nozzle exit cross section is assumed to deviate only slightly from a circle with radius R_0. In consequence of these two basic assumptions, the jet contour will be very close to a cylinder of radius R_0 and, thus, the flow can be thought of as being a uniform cylindrical parallel flow with superimposed small perturbations. Since the pressure difference between the jet and the medium is small, the strength of the shocks in the jet will be weak; thus, the vorticity introduced is negligible and linear potential flow theory becomes applicable.

Let the velocity in the undisturbed parallel flow be w_0 with the Mach number M_0. Since the existence conditions for a vortex sheet separating a flow from a medium at rest require the pressure to be continuous, the pressure in the uniform parallel flow is equal to the pressure p_0 of the surrounding medium. It is well established in linear potential flow theory [1] that in nondimensional cylindrical coordinates (r, φ, ζ), which are connected with the usual dimensional cartesian coordinates (x, y, z) via

$$\begin{aligned} x &= rR_0 \cos\varphi, \\ y &= rR_0 \sin\varphi, \\ z &= \zeta R_0 \sqrt{M_0^2 - 1}, \end{aligned} \tag{1}$$

the scalar velocity potential $\phi(r, \varphi, \zeta)$ of small irrotational disturbances of a uniform flow in the ζ-direction obeys the well-known wave equation [1]

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} - \frac{\partial^2\phi}{\partial\zeta^2} = 0. \tag{2}$$

The vector of the velocity perturbation is given by the gradient of the scalar velocity potential ϕ, and, consequently, the velocity components (u, v, w) in the (r, φ, ζ)-system are given by [1]:

$$\frac{u}{w_0} = \frac{\partial\phi}{\partial r}, \qquad \frac{v}{w_0} = \frac{1}{r}\frac{\partial\phi}{\partial\varphi}, \qquad \frac{w}{w_0} = 1 + \frac{1}{\sqrt{M_0^2 - 1}}\frac{\partial\phi}{\partial\zeta}. \tag{3}$$

If, furthermore, the fluid under consideration is assumed to be an ideal gas with constant ratio κ of its specific heats c_p and c_v, then the pressure p and density ρ are connected to the scalar velocity potential ϕ by its first derivative with respect to ζ, i.e., the ζ-component of the velocity perturbation [1]:

$$\frac{p}{p_0} = 1 - \kappa\frac{M_0^2}{\sqrt{M_0^2 - 1}}\frac{\partial\phi}{\partial\zeta}, \tag{4a}$$

$$\frac{\rho}{\rho_0} = 1 - \frac{M_0^2}{\sqrt{M_0^2 - 1}}\frac{\partial\phi}{\partial\zeta}, \tag{4b}$$

where, analogously to p_0, ρ_0 designates the density in the undisturbed uniform parallel flow, which, however, is not necessarily equal to the density ρ_∞ of the surrounding medium. By means of the ideal gas equation of state, a similar relation can also be given for the temperature T in the flow field.

In order to establish a well-posed Cauchy problem for the wave equation (2), we have to specify appropriate boundary conditions for the surface of the semi-infinite cylinder defined by $0 \le r \le 1$, $\zeta \ge 0$. For the lateral surface $r = 1$, $\zeta > 0$, which is identical with the mean physical jet boundary, we demand that the pressure is constant and equal to that of the surrounding medium. From Eq. (4a), it is then obvious that

$$\left.\frac{\partial\phi}{\partial\zeta}\right|_{r=1} \stackrel{!}{=} 0 \tag{5}$$

must be fulfilled for any φ and $\zeta > 0$. For the base surface $0 \le r \le 1$, $\zeta = 0$, which is identical with the nozzle exit cross section, we demand that the velocity potential ϕ and its first derivative with respect to ζ are both *known* functions $\phi_0(r, \varphi)$ and $\phi_0'(r, \varphi)$, which are determined mainly by the geometrical shape of the particular nozzle used for the generation of the free jet:

$$\phi|_{\zeta=0} \stackrel{!}{=} \phi_0(r, \varphi), \tag{6a}$$

$$\left.\frac{\partial\phi}{\partial\zeta}\right|_{\zeta=0} \stackrel{!}{=} \phi_0'(r, \varphi). \tag{6b}$$

From Eqs. (3) and (4), the physical meaning of ϕ_0 and ϕ_0' is obvious: while ϕ_0' describes the perturbation of *axial* velocity and, therefore, of pressure and density at the nozzle exit cross section, ϕ_0 corresponds to the perturbations of the velocity components *perpendicular* to the jet axis and, therefore, to the disturbance of parallel efflux from the nozzle.

By means of a separation ansatz under inclusion of the boundary condition (5), it is easy to show that the general solution $\phi(r, \varphi, \zeta)$ of the wave equation (2) arises as an infinite Fourier-Bessel double series [13]:

$$\begin{aligned}\phi(r, \varphi, \zeta) = &\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\beta_{mn} r)\cos\beta_{mn}\zeta[A_{mn}\cos m\varphi + B_{mn}\sin m\varphi] \\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\beta_{mn} r)\sin\beta_{mn}\zeta[C_{mn}\cos m\varphi + D_{mn}\sin m\varphi],\end{aligned} \tag{7}$$

where $J_m(x)$ are the well-known Bessel functions of the first kind of order m and the β_{mn} designate the zeros of $J_m(x)$ arranged in ascending order of magnitude. By use of the well-known orthogonality relations of the Bessel functions [5] and the harmonic functions, the Fourier-Bessel coefficients A_{mn}, B_{mn}, C_{mn}, and D_{mn} can be obtained from the boundary conditions (6) by performing the following integrations over the unit circle:

$$A_{mn} = \frac{2}{\pi J_{m+1}^2(\beta_{mn})}\int_0^{2\pi}\int_0^1 \phi_0(r, \varphi) J_m(\beta_{mn} r)\cos m\varphi\, r\, dr\, d\varphi, \tag{8a}$$

$$B_{mn} = \frac{2}{\pi J_{m+1}^2(\beta_{mn})}\int_0^{2\pi}\int_0^1 \phi_0(r, \varphi) J_m(\beta_{mn} r)\sin m\varphi\, r\, dr\, d\varphi, \tag{8b}$$

$$C_{mn} = \frac{2}{\pi \beta_{mn} J_{m+1}^2(\beta_{mn})}\int_0^{2\pi}\int_0^1 \phi_0'(r, \varphi) J_m(\beta_{mn} r)\cos m\varphi\, r\, dr\, d\varphi, \tag{8c}$$

$$D_{mn} = \frac{2}{\pi \beta_{mn} J_{m+1}^2(\beta_{mn})}\int_0^{2\pi}\int_0^1 \phi_0'(r, \varphi) J_m(\beta_{mn} r)\sin m\varphi\, r\, dr\, d\varphi. \tag{8d}$$

In the case $m = 0$, (8a) and (8c) have to be divided by 2, whereas (8b) and (8d) are vanishing [13]. Hence, the special solution for any particular free jet problem is obtained in a straightforward manner by evaluating the coefficient integrals (8) for the given functions $\phi_0(r, \varphi)$ and $\phi_0'(r, \varphi)$ and introducing the resulting coefficients into the general solution (7).

3. Special solutions containing singularities. In this section, we will restrict ourselves to the investigation of the case that at the nozzle exit cross section $\zeta = 0$, the boundary conditions $\phi_0(r, \varphi)$ and $\phi_0'(r, \varphi)$ are given by the product of the simple harmonic function $\cos m\varphi$ $(m = 0, 1, 2, \dots)$ and some functions $f(r)$, $g(r)$ that are bounded in $0 \le r \le 1$ and whose order of magnitude is small compared to unity:

$$\begin{aligned}\phi_0(r, \varphi) &= f(r)\cos m\varphi \\ \phi_0'(r, \varphi) &= g(r)\cos m\varphi\end{aligned} \qquad |f(r)|, |g(r)| \ll 1. \tag{9}$$

It is easy to show that in this case, the double sum (7) reduces to the single sum

$$\phi(r, \varphi, \zeta) = \cos m\varphi \sum_{n=1}^{\infty} J_m(\beta_{mn} r)[A_n \cos\beta_{mn}\zeta + C_n \sin\beta_{mn}\zeta], \tag{10}$$

where the coefficients A_n and C_n are given via (8a) and (8c) for all m by

$$A_n = \frac{2}{J^2_{m+1}(\beta_{mn})} \int_0^1 rf(r)J_m(\beta_{mn}r)\,dr\,, \tag{11a}$$

$$C_n = \frac{2}{\beta_{mn}J^2_{m+1}(\beta_{mn})} \int_0^1 rg(r)J_m(\beta_{mn}r)\,dr\,. \tag{11b}$$

There are very few functions $f(r)$, $g(r)$ for which the integration (11) can be performed analytically. We will therefore restrict ourselves to cases where $f(r)$ and $g(r)$ can be expressed as simple (distinct) polynomials $\chi_m(r)$, which start with m as the lowest power of r and continue with the next P odd or even powers if m is odd or even, respectively:

$$\chi_m(r) = \sum_{p=0}^{P} a_{m+2p} r^{m+2p}\,. \tag{12}$$

Here, a_{m+2p} are the polynomial coefficients, whose absolute values are assumed to be small compared to unity. (The case of $\chi_m(r)$ being a general polynomial including all integer powers of r can also be treated, but leads to more complicated results.) Hence, the coefficients (11) of the solution series (10) can be written in the general form

$$X_n = \frac{2}{\beta^{\alpha}_{mn}J^2_{m+1}(\beta_{mn})} \sum_{p=0}^{P} a_{m+2p} \int_0^1 r^{m+2p+1} J_m(\beta_{mn}r)\,dr\,, \qquad \alpha = 0\,,\,1\,, \tag{13}$$

where $\alpha = 0$ corresponds to (11a) and $\alpha = 1$ corresponds to (11b).

The integration in (13) can be performed analytically [5] and by defining coefficients b_k, which are connected with the polynomial coefficients a_{m+2p} of (12) via

$$b_k = (-1)^k 2^{2k} (k!)^2 \sum_{p=k}^{P} \binom{p}{k} \binom{m+p}{k} a_{m+2p}\,, \tag{14a}$$

the general form (13) of the series coefficients can finally be expressed as

$$X_n = \frac{2}{\beta^{\alpha+1}_{mn} J_{m+1}(\beta_{mn})} \sum_{k=0}^{P} \frac{b_k}{\beta^{2k}_{mn}}\,, \tag{14b}$$

where, as above, $\alpha = 0$ and $\alpha = 1$ correspond to (11a) and (11b), respectively. Thus, via (14), the coefficients (11) are determined for the case that the functions $f(r)$, $g(r)$ in the boundary conditions (9) are given by polynomials of the form (12).

We now have to answer the basic question, for which polynomials (12), the velocity potential (10) with its coefficients (11) determined via (14) gives rise to flow fields containing mathematical singularities. From (3) and (4), it is obvious that this is to be expected if the velocity potential (10) or its first derivatives with respect to r and ζ do not converge uniformly to continuous functions. The above question can thus be answered by use of the following theorem:

THEOREM. Let $f(r, \zeta)$ be a bounded continuous function in the domain $D(f)$ possessing bounded continuous first derivatives with respect to r and ζ. Let furthermore

$$|c_n| \leq \frac{c}{\beta_{mn}^{2+\varepsilon}}$$

with $\varepsilon > 0$ and $c = \text{const.}$ be valid for sufficiently large n. Then the series

$$\sum_{n=1}^{\infty} c_n f(\beta_{mn} r, \beta_{mn} \zeta)$$

as well as its first derivatives with respect to r and ζ converge uniformly in $D(f)$ to continuous functions.

The proof of this theorem is an elementary generalization of a corresponding proof in the theory of Fourier-Bessel series of a single variable (cf. [6]) and is therefore omitted. By means of the well-known inequality [6]

$$|J_{m+1}(\beta_{mn})| \geq \frac{c}{\sqrt{\beta_{mn}}}, \qquad c = \text{const.}, \tag{15}$$

we obtain, for a single term of (14b),

$$\left| \frac{2b_k}{\beta_{mn}^{2k+\alpha+1} J_{m+1}(\beta_{mn})} \right| \leq \frac{c}{\beta_{mn}^{2k+\alpha+1/2}}, \qquad c = \text{const.}, \tag{16}$$

and, thus, it is easy to see from (14) and the above theorem, that for $\alpha = 0$ and $\alpha = 1$, uniform convergence of (10) and its first derivatives is only certain if the coefficient b_0 in (14) is vanishing:

$$b_0 = \sum_{p=0}^{P} a_{m+2p} \stackrel{!}{=} 0. \tag{17}$$

It is obvious that this condition requires the polynomial (12) to vanish at $r = 1$.

Since both the velocity potential (10) and its first derivative with respect to ζ are vanishing for all $\zeta > 0$ at $r = 1$, we obtain the important result that for the case considered here, discontinuous flow fields are to be expected if the boundary conditions are discontinuous at the nozzle edge $\zeta = 0$, $r = 1$. Because discontinuities of the flow variables (e.g., pressure) at the nozzle edge are a basic feature of supersonic free jet flow, this situation is of great physical significance and deserves a detailed investigation, which however can only be performed if the corresponding solutions can be evaluated and discussed.

4. Application of Kummer's series transformation. From the results of the previous section, it has become obvious that in the linear theory of supersonic flow, solutions of physical relevance can occur which are described by nonuniformly convergent Fourier-Bessel series and whose direct numerical evaluation is therefore hardly feasible. Fortunately, however, the situation is not so hopeless as it appears to be at a first glance, since there exist a couple of analytical methods to improve the convergence of a given series. One of these methods and probably the best candidate for handling series with singularities is an algebraic transformation which was found by E. E. Kummer in 1837 [2]. The basic idea behind Kummer's series transformation is to subtract from a given series a suitably chosen one, the so-called comparison series, which has about the same poor rate of convergence but whose sum can be determined in closed analytical form. To fix notation, suppose that the given series is

$$S = \sum_{n=1}^{\infty} s_n , \tag{18}$$

and suppose that we know the sum

$$\tilde{S} = \sum_{n=1}^{\infty} \tilde{s}_n , \tag{19}$$

where s_n and $\tilde{s}_n$ are asymptotically equal, i.e.,

$$\lim_{n\to\infty} \frac{\tilde{s}_n}{s_n} = 1 . \tag{20}$$

Then, by simply subtracting (19) from (18), the unknown sum S can be written as

$$S = \tilde{S} + \sum_{n=1}^{\infty} (s_n - \tilde{s}_n) = \tilde{S} + R . \tag{21}$$

Thus, we are reduced to computing the residual series R, which by definition converges more rapidly than the given series (18). If the comparison series (19) is well-chosen, then the residual series R converges uniformly and therefore, approaches a continuous function rather rapidly, while the singularities of the given series (18) are now buried in the closed analytical expression $\tilde{S}$ for the comparison sum (19).

As has been demonstrated by Grabitz [7, 8] and Grabitz and Burmann [9] for the particular series describing the density field and the slope of the jet boundary of the axisymmetric free jet, Kummer's method can be used successfully if for the Bessel function $J_0(x)$ and its zeros β_{0n}, the corresponding asymptotic expansions are introduced into the given series to obtain a suitable comparison series which also allows for an analytical summation. However, their method can be applied only to particular series and is valid only for axisymmetric solutions, i.e., $m = 0$. Therefore, it appears well appropriate to generalize this procedure for *all* nonnegative integer orders $m = 0, 1, 2, \ldots$ of Bessel functions $J_m(x)$ and for *all* series that can arise from the special solutions of the supersonic jet free problem discussed in the previous section.

First, we have to find suitable comparison terms for all components of the series of interest. By use of the well-known asymptotic expansions of the Bessel functions $J_m(x)$, their first derivatives $J'_m(x)$ and their zeros β_{mn} [3, 5], we obtain the following asymptotic expressions:

$$\frac{1}{\beta_{mn}} \sim \frac{1}{\mu_{mn}} + O(\mu_{mn}^{-3}), \tag{22a}$$

$$J_m(\beta_{mn} r) \sim (-1)^n \sqrt{\frac{2}{\pi \mu_{mn} r}} \times \left[\sin \mu_{mn}(r-1) - \frac{\lambda - 1}{8\mu_{mn}} \left(r - \frac{1}{r} \right) \cos \mu_{mn}(r-1) \right] + O(\mu_{mn}^{-5/2}), \tag{22b}$$

$$J'_m(\beta_{mn} r) \sim (-1)^n \sqrt{\frac{2}{\pi \mu_{mn} r}} \times \left[\cos \mu_{mn}(r-1) + \frac{\lambda - 1}{8\mu_{mn}} \left(r - \frac{\lambda+3}{\lambda-1} \frac{1}{r} \right) \sin \mu_{mn}(r-1) \right] + O(\mu_{mn}^{-5/2}), \tag{22c}$$

$$\frac{1}{J_{m+1}(\beta_{mn})} \sim (-1)^{n+1} \sqrt{\frac{\pi}{2}\mu_{mn}} + O(\mu_{mn}^{-7/2}), \tag{22d}$$

$$\cos \beta_{mn}\zeta \sim \cos \mu_{mn}\zeta + \frac{\lambda-1}{8\mu_{mn}} \zeta \sin \mu_{mn}\zeta + O(\mu_{mn}^{-2}), \tag{22e}$$

$$\sin \beta_{mn}\zeta \sim \sin \mu_{mn}\zeta - \frac{\lambda-1}{8\mu_{mn}} \zeta \cos \mu_{mn}\zeta + O(\mu_{mn}^{-2}). \tag{22f}$$

In the above equations, λ and μ_{mn} are given by

$$\lambda = 4m^2, \quad \mu_{mn} = \left(n + \frac{m}{2} - \frac{1}{4} \right) \pi. \tag{22g}$$

By means of the asymptotic expressions (22), it is now possible to construct appropriate comparison series for any physical flow variable of interest that can be derived from the scalar potential function (10) with its coefficients (11) given by the general form (14), if the condition of continuity (17) is not fulfilled and, thus, series with nonuniform convergence are to be expected. In any such case, after performing Kummer's series transformation according to (21), we are left with a uniformly (and thus, rather rapidly) converging residual series and the problem of summing the badly converging comparison series analytically. A closer inspection of all possibly occurring cases shows that this task can always be reduced to the analytical summation of the two complex trigonometric series

$$\Psi_m(s, x) = \sum_{n=1}^{\infty} \psi_{m,n}(s, x) = \sum_{n=1}^{\infty} \frac{e^{i\mu_{mn}x}}{\mu_{mn}^s}, \qquad s = 0, 1, \tag{23a}$$

$$\Omega_m(s, x) = \sum_{n=1}^{\infty} \omega_{m,n}(s, x) = \sum_{n=1}^{\infty} (-1)^n \frac{e^{i\mu_{mn}x}}{\mu_{mn}^s} \qquad s = \pm\tfrac{1}{2}. \tag{23b}$$

The summation of the series (23), which can be done by standard analytical methods, is a somewhat lengthy procedure and is therefore given in the appendix. Hence, any nonuniformly converging series having been derived from (10), (11), and (14) can be transformed to a uniformly converging series by means of the following procedure:

- Find a suitable comparison series by introducing the asymptotic expressions (22) into the given series and truncating the resulting comparison term at $O(\mu_{mn}^{1+\varepsilon})$ where $\varepsilon > 0$.
- Sum the comparison series by using (23) and the results given in the appendix.
- Perform Kummer's series transformation (21).

In the following section, this procedure will be applied to evaluate the complete flow field of an axisymmetric supersonic free jet with homogeneous pressure perturbation at the nozzle exit cross section.

5. The axisymmetric supersonic free jet. In the following, we consider an axisymmetric supersonic free jet emerging from the nozzle in parallel flow with a superimposed small pressure perturbation Δp, which is assumed to be constant throughout the nozzle exit cross section. The general solution for the axisymmetric case is given by (10) for $m = 0$, and the special solution is via (3) and (4) obtained by evaluating the coefficient integrals (11) for the following boundary conditions:

$$\phi_0(r) = 0, \tag{24a}$$

$$\phi_0'(r) = -\frac{1}{\kappa}\frac{\sqrt{M_0^2 - 1}}{M_0^2}\frac{\Delta p}{p_0} = \text{const.} \tag{24b}$$

Hence, $A_n = 0$ and C_n is easily determined from (14) for $\alpha = 1$ and $P = 0$. Thus, we obtain the scalar velocity potential $\phi(r, \zeta)$ in the simple form

$$\phi(r, \zeta) = -\frac{1}{\kappa}\frac{\sqrt{M_0^2 - 1}}{M_0^2}\frac{\Delta p}{p_0}\sum_{n=1}^{\infty}\frac{2J_0(\beta_{0n}r)\sin\beta_{0n}\zeta}{\beta_{0n}^2 J_1(\beta_{0n})}. \tag{25}$$

By defining two series $S_r(r, \zeta)$, $S_\zeta(r, \zeta)$ as the first derivatives of (25)

$$S_r(r, \zeta) = \frac{\partial}{\partial r}\sum_{n=1}^{\infty}\frac{2J_0(\beta_{0n}r)\sin\beta_{0n}\zeta}{\beta_{0n}^2 J_1(\beta_{0n})} = \sum_{n=1}^{\infty}\frac{2J_0'(\beta_{0n}r)\sin\beta_{0n}\zeta}{\beta_{0n} J_1(\beta_{0n})}, \tag{26a}$$

$$S_\zeta(r, \zeta) = \frac{\partial}{\partial \zeta}\sum_{n=1}^{\infty}\frac{2J_0(\beta_{0n}r)\sin\beta_{0n}\zeta}{\beta_{0n}^2 J_1(\beta_{0n})} = \sum_{n=1}^{\infty}\frac{2J_0(\beta_{0n}r)\cos\beta_{0n}\zeta}{\beta_{0n} J_1(\beta_{0n})}, \tag{26b}$$

we can easily write all physical flow variables (3) and (4) in terms of (26):

$$\frac{u}{w_0} = -\frac{1}{\kappa}\frac{\sqrt{M_0^2 - 1}}{M_0^2}\frac{\Delta p}{p_0} S_r(r, \zeta), \tag{27a}$$

$$\frac{w}{w_0} = 1 - \frac{1}{\kappa M_0^2}\frac{\Delta p}{p_0} S_\zeta(r, \zeta), \tag{27b}$$

$$\frac{p}{p_0} = 1 + \frac{\Delta p}{p_0} S_\zeta(r, \zeta), \tag{27c}$$

$$\frac{\rho}{\rho_0} = 1 + \frac{1}{\kappa}\frac{\Delta p}{p_0} S_\zeta(r, \zeta). \tag{27d}$$

Thus, the problem of the axisymmetric supersonic free jet with constant initial pressure perturbation is formally solved. The scalar velocity potential (25) was first derived by D. C. Pack in 1950 [4], but Pack did not evaluate the relations (27) because of their nonuniform convergence which, according to the theorem in Sec. 3, is due to the mathematical singularities contained in the two series (26). We will now show that these series can nevertheless be evaluated by use of Kummer's series transformation as outlined in the previous section.

In order to apply Kummer's series transformation (21) to the series (26), we first have to determine the terms of the corresponding comparison series. By introducing the asymptotic expansions (22) into the terms $s_{r,n}(r, \zeta)$ and $s_{\zeta,n}(r, \zeta)$ of the series (26a) and (26b), we obtain

$$\begin{aligned} s_{r,n}(r, \zeta) &= \frac{2J_0'(\beta_{0n} r)\sin\beta_{0n}\zeta}{\beta_{0n} J_1(\beta_{0n})} \\ &\sim -\frac{1}{\sqrt{r}}\left[\frac{\sin\mu_{0n}(r+\zeta-1)}{\mu_{0n}} - \frac{\sin\mu_{0n}(r-\zeta-1)}{\mu_{0n}}\right] + O(\mu_{0n}^{-2}) \end{aligned} \tag{28a}$$

and

$$\begin{aligned} s_{\zeta,n}(r, \zeta) &= \frac{2J_0(\beta_{0n} r)\cos\beta_{0n}\zeta}{\beta_{0n} J_1(\beta_{0n})} \\ &\sim -\frac{1}{\sqrt{r}}\left[\frac{\sin\mu_{0n}(r+\zeta-1)}{\mu_{0n}} + \frac{\sin\mu_{0n}(r-\zeta-1)}{\mu_{0n}}\right] + O(\mu_{0n}^{-2}) \end{aligned} \tag{28b}$$

for the case $r > 0$; in the case $r = 0$, we introduce $J_0(0) = 1$, $J_0'(0) = 0$ into the terms of (26a) and (26b):

$$s_{r,n}(0, \zeta) = 0, \tag{28c}$$

$$\begin{aligned} s_{\zeta,n}(0, \zeta) &= \frac{2\cos\beta_{0n}\zeta}{\beta_{0n} J_1(\beta_{0n})} \\ &\sim -\sqrt{2\pi}(-1)^n \frac{\cos\mu_{0n}\zeta}{\sqrt{\mu_{0n}}} + O(\mu_{0n}^{-3/2}). \end{aligned} \tag{28d}$$

Since $s_r(r, \zeta)$ vanishes at $r = 0$, no comparison term is needed for this case.

Obviously, Kummer's series transformation (21) can be applied successfully if, in terms of the notation (23), the following comparison terms are used (in the remainder of this work, the superscripts R and I denote the real and imaginary part of a complex number, respectively):

$$\tilde{s}_{r,n}(r, \zeta) = -r^{-1/2}[\psi^I_{0,n}(1, r+\zeta-1) - \psi^I_{0,n}(1, r-\zeta-1)], \qquad r > 0, \tag{29a}$$

$$\tilde{s}_{\zeta,n}(r, \zeta) = \begin{cases} -r^{-1/2}[\psi^I_{0,n}(1, r+\zeta-1) + \psi^I_{0,n}(1, r-\zeta-1)], & r > 0, \\ -\sqrt{2\pi}\omega^R_{0,n}(\tfrac{1}{2}, \zeta), & r = 0. \end{cases} \tag{29b}$$

Consequently, the analytical sums of the comparison series can be written as

$$\widetilde{S}_r(r, \zeta) = \sum_{n=1}^{\infty} \tilde{s}_{r,n} = -r^{-1/2}[\Psi^I_0(1, r+\zeta-1) - \Psi^I_0(1, r-\zeta-1)], \; r > 0, \tag{30a}$$

$$\widetilde{S}_\zeta(r, \zeta) = \sum_{n=1}^{\infty} \tilde{s}_{\zeta,n} = \begin{cases} -r^{-1/2}[\Psi^I_0(1, r+\zeta-1) + \Psi^I_0(1, r-\zeta-1)], \; r > 0. \\ -\sqrt{2\pi}\Omega^R_0(\tfrac{1}{2}, \zeta), \; r = 0, \end{cases} \tag{30b}$$

where the functions $\Psi^I_0(1, x)$ and $\Omega^R_0(\frac{1}{2}, x)$ are given by Eqs. (A11b) and (A15a) for $m = 0$ in the appendix. Hence, by (29) and (30), Kummer's transformation can finally be applied to the series (26a) and (26b). The singularities of the original series (26) are now given by the singularities of the closed expressions (30), whereas the remaining residual series, whose terms are formed by the difference between the terms of the original and the comparison series, converge according to (28) uniformly and absolutely to continuous functions, thus allowing for the complete evaluation of the flow field described by Eqs. (27).

In Fig. 1 (see p. 346), the efficiency of the presented method is demonstrated for the series $S_\zeta(0, \zeta)$, which, according to Eqs. (27), describes the perturbations of pressure, density, and axial velocity on the jet axis. Fig. 1(a) shows the result of a direct evaluation of (26b) for $r = 0$ without performing Kummer's transformation. The bad convergence produced by the singularities of $S_\zeta(0, \zeta)$ at $\zeta = 2k+1$ ($k = 0, \pm 1, \pm 2, \ldots$) is obvious and manifests itself in strong oscillations of the curve, which do not vanish no matter how many terms are included in the summation (30 terms in Fig. 1(a)). This behaviour of nonuniformly convergent series is well known from the theory of Fourier series as Gibbs' phenomenon [6]. If now Kummer's transformation (21) is applied to $S_\zeta(0, \zeta)$, we obtain

$$\begin{aligned} S_\zeta(0, \zeta) &= \widetilde{S}_\zeta(0, \zeta) + \sum_{n=1}^{\infty}[s_{\zeta,n}(0, \zeta) - \tilde{s}_{\zeta,n}(0, \zeta)] \\ &= \widetilde{S}_\zeta(0, \zeta) + R_\zeta(0, \zeta), \end{aligned} \tag{31}$$

and, thus, the singularities of $S_\zeta(0, \zeta)$ are shifted to the closed analytical expression (30b) for $\widetilde{S}_\zeta(0, \zeta)$, whereas according to (28d), the residual series $R_\zeta(0, \zeta)$ converges uniformly and thus, approaches a continuous function after inclusion of 20–30 terms. Both $\widetilde{S}_\zeta(0, \zeta)$ and $R_\zeta(0, \zeta)$ are presented in Fig. 1(b). Finally, $S_\zeta(0, \zeta)$ as obtained from Kummer's transformation (31) (i.e., by simply adding $\widetilde{S}_\zeta(0, \zeta)$ and $R_\zeta(0, \zeta)$) is presented in Fig. 1(c). Hence, $S_\zeta(0, \zeta)$ is identified as a function that is piecewise continuous except at the locations $\zeta = 2k+1$, where it exhibits the same

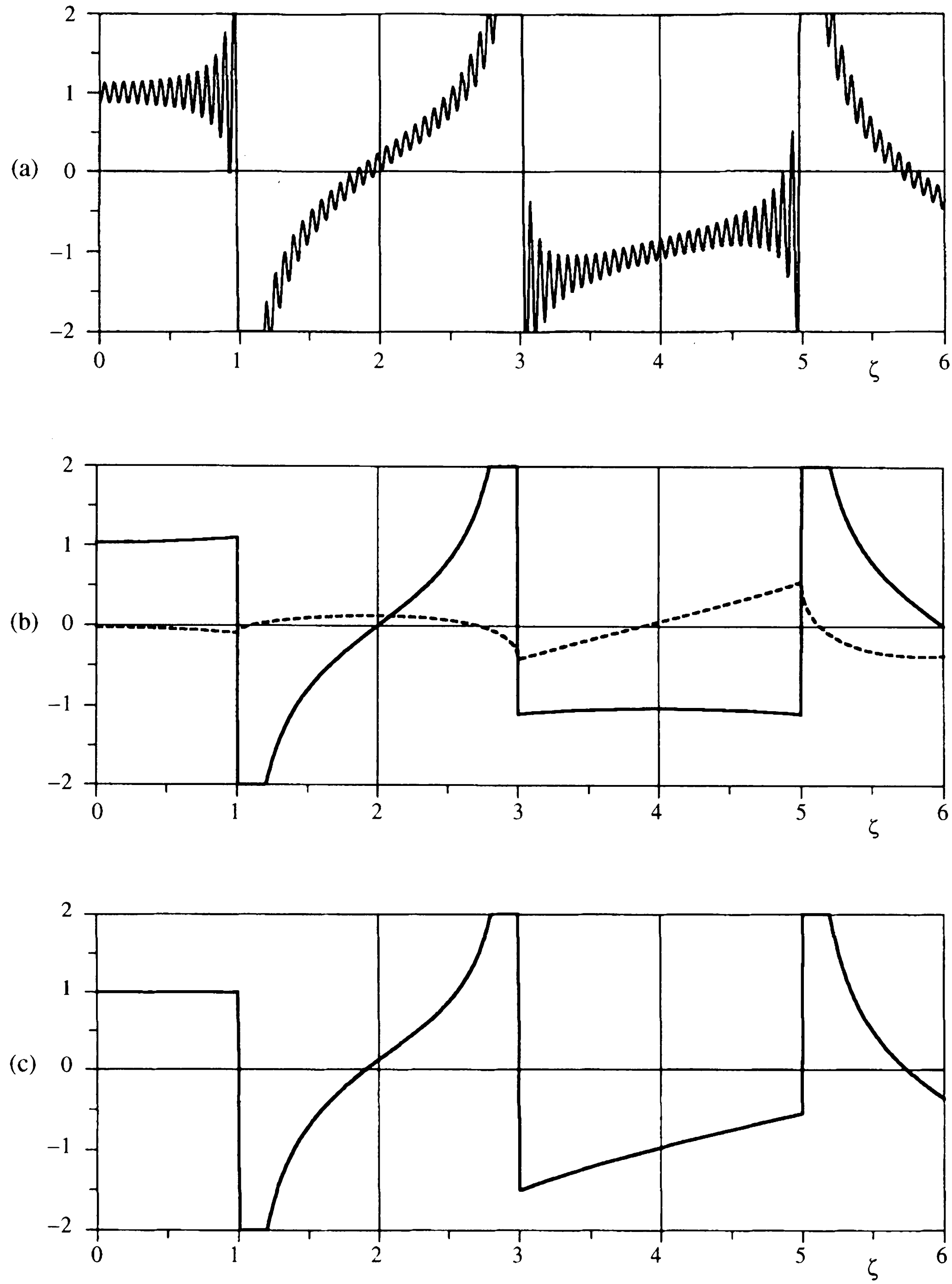

FIG. 1. The efficiency of Kummer's series transformation demonstrated for the series $S_\zeta(0, \zeta)$: (a) Result of a direct evaluation (30 terms) of (26b). (b) Comparison sum $\widetilde{S}_\zeta(0, \zeta)$ (solid line) and continuous residual series $R_\zeta(0, \zeta)$ (dashed line) (30 terms). (c) $S_\zeta(0, \zeta) = \widetilde{S}_\zeta(0, \zeta) + R_\zeta(0, \zeta)$.

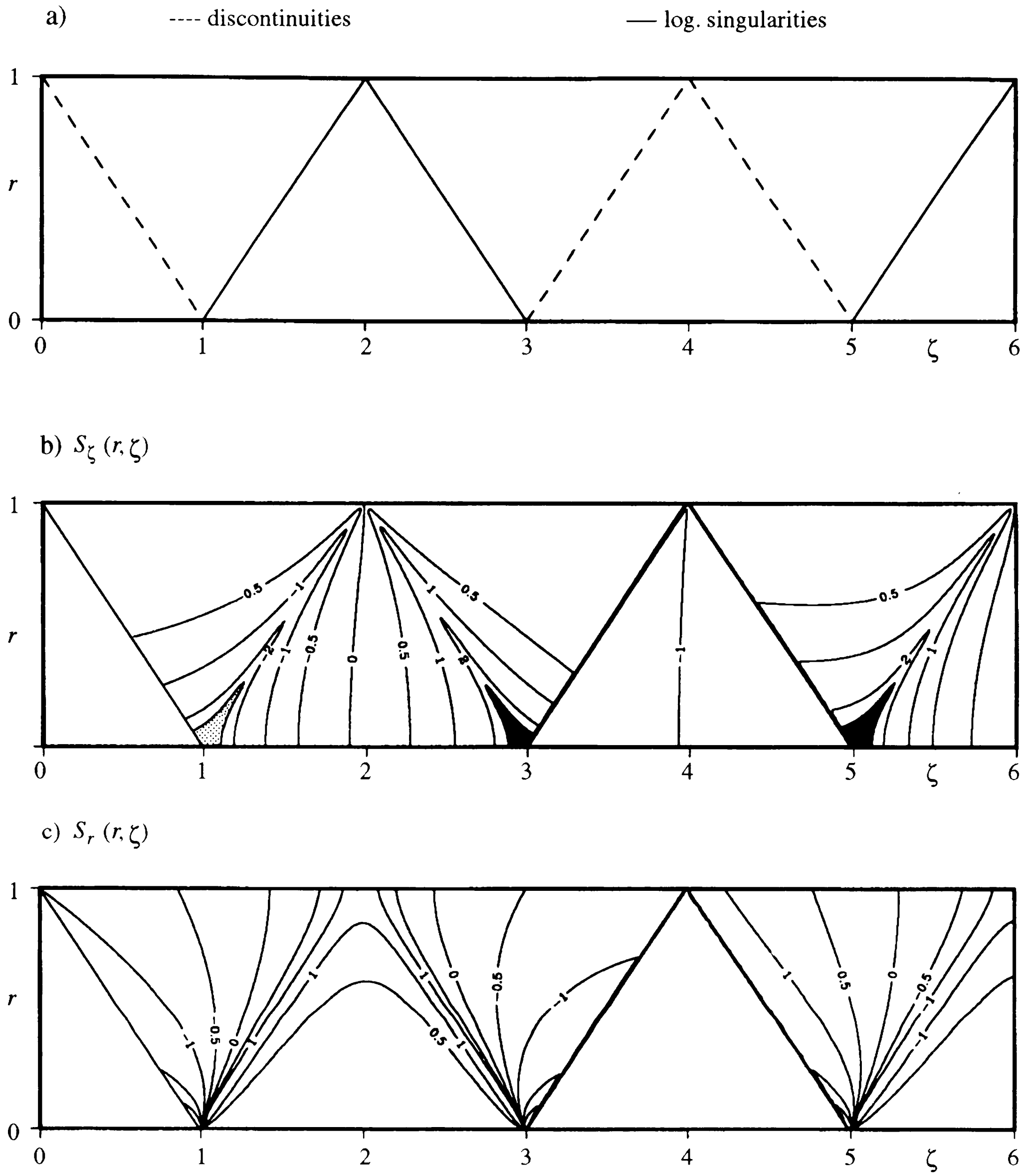

FIG. 2. Solution for the axisymmetric free jet: (a) Location of the singularities of $S_\zeta(r\,,\,\zeta)$ and $S_r(r\,,\,\zeta)$. (b) Contour plot of $S_\zeta(r\,,\,\zeta)$. Areas of extremely low and high values are indicated by bright and dark shading, respectively. (c) Contour plot of $S_r(r\,,\,\zeta)$.

inverse-square-root singularities as $\Omega_0(\frac{1}{2}\,,\,x)$ for $x = 2k + 1$ (cf. appendix). This result was first obtained by Grabitz [8].

In Fig. 2, both series $S_r(r\,,\,\zeta)$ and $S_\zeta(r\,,\,\zeta)$ are evaluated by means of Kummer's transformation in the range $0 \le r \le 1$, $0 \le \zeta \le 6$. Since for $r > 0$, both series exhibit the same singularities as their corresponding comparison series (30), it is easy to show that they represent continuous functions everywhere in the (r, ζ)-plane except at the locations $\zeta + r = 2k+1$ and $\zeta - r = 2k+1$, where they possess either finite discontinuities or logarithmic singularities (cf. appendix). It has been shown by Grabitz and Burmann [9], that $S_\zeta(r, \zeta)$ is continuous for $r \to 0$ except at the focus points $\zeta = 2k+1$, where the nature of the singularities changes from the logarithmic to the inverse-square-root type as mentioned above.

In Fig. 2(a), the locations of the discontinuities are indicated by broken lines, whereas the full lines correspond to the logarithmic singularities. Fig. 2(b) and Fig. 2(c) show contour plots of $S_\zeta(r, \zeta)$ and $S_r(r, \zeta)$ which, via Eqs. (27), correspond to radial velocity and the perturbations of axial velocity, pressure, and density, respectively. Thus, the complete free jet flow field has been evaluated.

While the singularities of the flow field are distributed periodically, the flow pattern itself shows no strictly periodic behaviour. In the following, we discuss briefly the example of an underexpanded free jet (i.e., $\Delta p > 0$). From Fig. 2(c) it is obvious that by a radial focussing effect, the simple expansion waves propagating from the nozzle edge into the inner portions of the jet form a region of very low pressure close to the jet axis near $\zeta = 1$, which is the starting point for a combined expansion-compression wave. By reflection at the jet boundary, this wave is transformed into a combined compression-expansion wave that ends as an ordinary compression wave close to the jet axis at $\zeta = 3$. This process is continued for increasing values of ζ, but because of the cylindrical geometry of the jet, the wave pattern is not strictly periodic and thus, much more complicated compared to the plane supersonic free jet, where only simple expansion and compression waves alternate to produce a strictly periodic flow field.

This characteristic behaviour of cylindrical jets has been confirmed experimentally by Grabitz, Hiller, and Meier [14] by means of Mach-Zehnder interferometry. They also demonstrated the general applicability of linear potential flow theory by computing theoretical Mach-Zehnder interferograms from the density relation (27d), which agree qualitatively well with the experimental patterns.

6. Conclusions. In this paper, the general solution for the problem of linearized three-dimensional supersonic free jet flow with cylindrical geometry has been presented. It has been shown that the special flow patterns are determined by boundary condition functions $\phi'(r, \varphi)$ and $\phi(r, \varphi)$, which describe the perturbations of axial velocity, pressure, and density and the perturbation of parallel efflux from the nozzle, respectively. For a broad class of such functions, it has been shown that the resulting flow field contains singularities if the boundary conditions at the nozzle edge are discontinuous, i.e., if the functions $\phi'(r, \varphi)$ and $\phi(r, \varphi)$ do not vanish at $r = 1$. In these cases, the Fourier-Bessel series representing the physical flow variables exhibit very poor convergence properties and thus, their direct numerical evaluation is hardly feasible. It has been shown that this problem can completely be resolved by means of a simple method, which is based on Kummer's series transformation and allows for a complete evaluation and discussion of the flow field and its singularities in the two- and three-dimensional case. All comparison terms and analytical sums necessary for the general practical applicability of the presented method have been derived. The capability of the presented method has been demonstrated for D. C. Pack's classical solution [4] for the axisymmetric supersonic free jet with initial constant pressure perturbation, which, although having been derived in 1950, has not been evaluated completely yet.

By means of the presented method, a discussion of the three-dimensional free jet flow patterns described by (10), (11), and (14) for $m > 0$ is also possible. During the last decades, considerable progress has been made in the development of numerical methods and experimental techniques for the investigation of three-dimensional supersonic flow. Unfortunately, however, there is a certain lack of exact solutions of the fundamental equations of fluid mechanics for this case. The three-dimensional solutions presented in Sec. 3, which are discussed elsewhere [10], may therefore be a contribution to improve our understanding of the basic physical mechanisms of three-dimensional supersonic flow; besides, they could also serve as reference solutions to verify numerical and experimental methods.

7. Appendix. As mentioned in Sec. 4, the task of summing the comparison series can always be reduced to the analytical summation of the real or imaginary part of the complex trigonometric series (23), which represent periodic functions with period 8:

$$\Psi_m(s, x) = \Psi_m^R(s, x) + i\Psi_m^I(s, x) = \sum_{n=1}^{\infty} \frac{e^{i\mu_{mn}x}}{\mu_{mn}^s}, \qquad s = 0, 1, \qquad \text{(A1a)}$$

$$\Omega_m(s, x) = \Omega_m^R(s, x) + i\Omega_m^I(s, x) = \sum_{n=1}^{\infty} (-1)^n \frac{e^{i\mu_{mn}x}}{\mu_{mn}^s}, \qquad s = \pm\frac{1}{2}, \qquad \text{(A1b)}$$

where

$$\mu_{mn} = \left(n + \frac{m}{2} - \frac{1}{4}\right)\pi .$$

For both series (A1), the summation to a closed form can be done by use of Lerch's transcendent $\Phi(z, s, a)$, which is defined by [11, 12]

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \qquad |z| < 1, \ a \neq 0, -1, -2, \dots . \qquad \text{(A2)}$$

Replacing the summation index n in (A1) by $n + 1$, we can easily write $\Psi_m(s, x)$ and $\Omega_m(s, x)$ in terms of $\Phi(z, s, a)$:

$$\Psi_m(s, x) = +\pi^{-s} e^{i(m/2-3/4)\pi x} \Phi\left(e^{i\pi x}, s, \frac{m}{2} + \frac{3}{4}\right), \qquad \text{(A3a)}$$

$$\Omega_m(s, x) = -\pi^{-s} e^{i(m/2+3/4)\pi x} \Phi\left(e^{i\pi(x+1)}, s, \frac{m}{2} + \frac{3}{4}\right). \qquad \text{(A3b)}$$

For $s = 0$, Lerch's transcendent $\Phi(z, s, a)$ is independent of its third argument and known in closed form [12]:

$$\Phi(z, 0, a) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad \text{(A4)}$$

and hence, by substituting (A4) into (A3a) and decomposing the result into its real and imaginary part, we obtain

$$\Psi_m^R(0, x) = -\frac{\sin(\frac{m}{2} + \frac{1}{4})\pi x}{2\sin\frac{\pi}{2}x}, \tag{A5a}$$

$$\Psi_m^I(0, x) = +\frac{\cos(\frac{m}{2} + \frac{1}{4})\pi x}{2\sin\frac{\pi}{2}x}. \tag{A5b}$$

Thus, the analytical summation of $\Psi_m(0, x)$ is already done. It is easy to show that $\Psi_m^R(0, x)$ has hyperbolic singularities at $x = 4k + 2$ and removable singularities at $x = 4k$, whereas $\Psi_m^I(0, x)$ exhibits hyperbolic singularities at $x = 4k$ and removable singularities at $x = 4k + 2$ $(k = 0, \pm 1, \pm 2, \ldots)$.

For $s \neq 0$, $\Phi(z, s, a)$ can be expressed in terms of other functions in the special case $a = 1$. By use of the transformation rule [12]

$$\Phi(z, s, a + \nu) = z^{-a}\left[\Phi(z, s, \nu) - \sum_{n=0}^{a-1} \frac{z^n}{(n+\nu)^s}\right] \tag{A6}$$

and by defining the finite trigonometric series

$$\sigma_m(s, x) = -\begin{cases} \sum_{n=1}^{m/2} \frac{\exp(i(n-1/4)\pi x)}{[(n-1/4)\pi]^s}, & m \text{ even}, \\ \sum_{n=1}^{\frac{m+1}{2}} \frac{\exp(i(n-3/4)\pi x)}{[(n-3/4)\pi]^s}, & m \text{ odd} \end{cases} \tag{A7a}$$

and

$$\tau_m(s, x) = -\begin{cases} (-1)^{m/2} \sum_{n=1}^{m/2} (-1)^n \frac{\exp(i(n-1/4)\pi x)}{[(n-1/4)\pi]^s}, & m \text{ even}, \\ (-1)^{(m+1)/2} \sum_{n=1}^{(m+1)/2} (-1)^n \frac{\exp(i(n-3/4)\pi x)}{[(n-3/4)\pi]^s}, & m \text{ odd} \end{cases} \tag{A7b}$$

we obtain from (A3a)

$$\Psi_m(s, x) = \sigma_m(s, x) + \begin{cases} \pi^{-s} e^{i(3\pi/4)x} \Phi(e^{i\pi x}, s, \frac{3}{4}), & m \text{ even}, \\ \pi^{-s} e^{i(\pi/4)x} \Phi(e^{i\pi x}, s, \frac{1}{4}), & m \text{ odd} \end{cases} \tag{A8a}$$

and from (A3b)

$$\Omega_m(s, x) = \tau_m(s, x) - \begin{cases} (-1)^{m/2} \pi^{-s} e^{i(3\pi/4)x} \Phi(e^{i\pi(x+1)}, s, \frac{3}{4}), & m \text{ even}, \\ (-1)^{(m+1)/2} \pi^{-s} e^{i(\pi/4)x} \Phi(e^{i\pi(x+1)}, s, \frac{1}{4}), & m \text{ odd}. \end{cases} \tag{A8b}$$

Furthermore, it is easy to show by means of (A2), that $\Phi(z, s, \frac{3}{4})$ and $\Phi(z, s, \frac{1}{4})$ can be written as

$$z^{3/4}\Phi(z, s, \tfrac{3}{4}) = 4^{s-1} z^{1/4}[\Phi(z^{1/4}, s, 1) + \Phi(-z^{1/4}, s, 1) - \Phi(iz^{1/4}, s, 1) - \Phi(-iz^{1/4}, s, 1)] \tag{A9a}$$

$$z^{1/4}\Phi(z, s, \tfrac{1}{4}) = 4^{s-1} z^{1/4}[\Phi(z^{1/4}, s, 1) + \Phi(-z^{1/4}, s, 1) + \Phi(iz^{1/4}, s, 1) + \Phi(-iz^{1/4}, s, 1)] \tag{A9b}$$

and thus, via (A7), (A8), and (A9), $\Psi_m(s, x)$ and $\Omega_m(s, x)$ are finally expressed in terms of $\Phi(z, s, 1)$.

For $s = 1$, $\Phi(z, s, 1)$ is known in closed form [12]:

$$\Phi(z, 1, 1) = \sum_{n=0}^{\infty} \frac{z^n}{n+1} = -\frac{1}{z} \ln(1-z) \tag{A10}$$

and hence, by introducing (A10) into (A9) and (A8a), and decomposing the result into its real and imaginary part, we obtain

$$\Psi_m^R(1, x) = \sigma_m^R(1, x) \mp \frac{1}{2} \operatorname{sgn} \cos \frac{\pi x}{4} - \frac{1}{2\pi} \ln \frac{1 - \cos \frac{\pi x}{4}}{1 + \cos \frac{\pi x}{4}}, \tag{A11a}$$

$$\Psi_m^I(1, x) = \sigma_m^I(1, x) + \frac{1}{2} \operatorname{sgn} \sin \frac{\pi x}{4} \pm \frac{1}{2\pi} \ln \frac{1 - \sin \frac{\pi x}{4}}{1 + \sin \frac{\pi x}{4}}, \tag{A11b}$$

where the upper and lower signs correspond to even and odd values of m, respectively, and $\operatorname{sgn} x$ is the signum function, which gives -1, 0, or $+1$ depending on whether x is negative, zero, or positive. It is easy to show that $\Psi_m^R(1, x)$ has logarithmic singularities at $x = 4k$ and discontinuities of magnitude unity at $x = 4k+2$, whereas $\Psi_m^I(1, x)$ exhibits logarithmic singularities at $x = 4k+2$ and discontinuities of magnitude unity at $x = 4k$ $(k = 0, \pm 1, \pm 2, \ldots)$.

For s being an arbitrary noninteger real number, $\Phi(z, s, a)$ cannot be expressed in terms of elementary functions. However, via Lerch's functional equation for $\Phi(z, s, a)$ in the special case $a = 1$ [11, 12]:

$$z\Phi(z, s, 1) = i(2\pi)^{s-1}\Gamma(1-s)\left[e^{-i\pi s/2}\zeta\left(1-s, \frac{\ln z}{2\pi i}\right) - e^{+i\pi s/2}\zeta\left(1-s, 1-\frac{\ln z}{2\pi i}\right)\right], \tag{A12}$$

there exists a connection to the Hurwitz zeta function $\zeta(p, q)$ [11, 12]:

$$\zeta(p, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^p}, \qquad q \neq 0, -1, -2, \ldots, \tag{A13}$$

whose numerical computation is easily feasible by means of appropriate asymptotic expansions and recurrence relations [11]. Hence, by introducing (A12) into (A9) and (A8b) and by defining the periodic function

$$Z(p, q) = \begin{cases} \zeta(p, q), & 0 \leq q \leq 1, \\ Z(p, q-1), & q > 1, \\ Z(p, q+1), & q < 0, \end{cases} \tag{A14}$$

in order to account for the periodicity of the complex argument z in (A12), we can easily express the real and imaginary part of $\Omega(s, x)$ for *any* noninteger value of s in terms of the Hurwitz zeta function. For the special cases $s = \pm\frac{1}{2}$, we obtain

$$\begin{aligned}\Omega_m^R(\tfrac{1}{2}, x) &= \tau_m^R(\tfrac{1}{2}, x) \\ &+ \frac{1}{2\sqrt{2\pi}} \cdot \begin{cases} (-1)^{m/2}[Z(\frac{1}{2}, \frac{7+x}{8}) + Z(\frac{1}{2}, \frac{7-x}{8}) - Z(\frac{1}{2}, \frac{3+x}{8}) - Z(\frac{1}{2}, \frac{3-x}{8})] \\ (-1)^{(m+1)/2}[Z(\frac{1}{2}, \frac{5+x}{8}) + Z(\frac{1}{2}, \frac{5-x}{8}) - Z(\frac{1}{2}, \frac{1+x}{8}) - Z(\frac{1}{2}, \frac{1-x}{8})] \end{cases}\end{aligned} \tag{A15a}$$

$$\Omega^I_m(\tfrac{1}{2}, x) = \tau^I_m(\tfrac{1}{2}, x) + \frac{1}{2\sqrt{2\pi}} \cdot \begin{cases} (-1)^{m/2}[Z(\frac{1}{2}, \frac{1+x}{8}) - Z(\frac{1}{2}, \frac{1-x}{8}) - Z(\frac{1}{2}, \frac{5+x}{8}) + Z(\frac{1}{2}, \frac{5-x}{8})] \\ (-1)^{(m+1)/2}[Z(\frac{1}{2}, \frac{3+x}{8}) - Z(\frac{1}{2}, \frac{3-x}{8}) - Z(\frac{1}{2}, \frac{7+x}{8}) + Z(\frac{1}{2}, \frac{7-x}{8})] \end{cases} \tag{A15b}$$

and

$$\Omega^R_m(-\tfrac{1}{2}, x) = \tau^R_m(-\tfrac{1}{2}, x) + \frac{1}{32\sqrt{2\pi}} \cdot \begin{cases} (-1)^{m/2}[Z(\frac{3}{2}, \frac{5+x}{8}) + Z(\frac{3}{2}, \frac{5-x}{8}) - Z(\frac{3}{2}, \frac{1+x}{8}) - Z(\frac{3}{2}, \frac{1-x}{8})] \\ (-1)^{(m+1)/2}[Z(\frac{3}{2}, \frac{7+x}{8}) + Z(\frac{3}{2}, \frac{7-x}{8}) - Z(\frac{3}{2}, \frac{3+x}{8}) - Z(\frac{3}{2}, \frac{3-x}{8})] \end{cases} \tag{A16a}$$

$$\Omega^I_m(-\tfrac{1}{2}, x) = \tau^I_m(-\tfrac{1}{2}, x) + \frac{1}{32\sqrt{2\pi}} \cdot \begin{cases} (-1)^{m/2}[Z(\frac{3}{2}, \frac{7+x}{8}) - Z(\frac{3}{2}, \frac{7-x}{8}) - Z(\frac{3}{2}, \frac{3+x}{8}) + Z(\frac{3}{2}, \frac{3-x}{8})] \\ (-1)^{(m+1)/2}[Z(\frac{3}{2}, \frac{5+x}{8}) - Z(\frac{3}{2}, \frac{5-x}{8}) - Z(\frac{3}{2}, \frac{1+x}{8}) + Z(\frac{3}{2}, \frac{1-x}{8})]. \end{cases} \tag{A16b}$$

In the formulas (A15) and (A16), the upper and lower parts correspond to even and odd values of m, respectively. Hence, the analytical summation of $\Omega(\pm\frac{1}{2}, x)$ is done. It is easy to show that both the real and imaginary parts of (A15) and (A16) exhibit singularities at $x = 2k+1$ $(k = 0, \pm 1, \pm 2, \dots)$, whose nature is determined by the singularity of $\zeta(1-s, x)$ at $x = 0$. By means of the functional equations given for the Hurwitz zeta function [11], it can be shown that $\zeta(1-s, x)$ approaches infinity as x^{-s} for $x \to 0$.

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