QUARTERLY OF APPLIED MATHEMATICS
VOLUME LIII, NUMBER 3
SEPTEMBER 1995, PAGES 551-558

ON THE WELL-POSEDNESS OF THE INITIAL VALUE PROBLEM FOR ELASTIC-PLASTIC OSCILLATORS WITH ISOTROPIC WORK-HARDENING

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Abstract. The system of differential equations of elastic-plastic oscillators with isotropic work-hardening is converted to a system of differential inclusions and well-posedness is established using maximal monotone operator theory when the external force $f \in W^{1,1}(0, T; R)$. By a more delicate analysis, well-posedness is also established for $f \in L^1(0, T; R)$.

1. Introduction. In this paper, we study the well-posedness of the system of differential equations for elastic-plastic oscillators with isotropic work-hardening. By converting the system of differential equations to a system of differential inclusions, well-posedness is established using maximal monotone operator theory.

The existence and uniqueness of the system of differential equations for elastic-plastic oscillators have been a subject of many studies [1, 2, 3, 4, 5]. [1] used Filippov solutions to show the existence of the solutions for the system of differential equations of elastic-plastic oscillators without work-hardening, and that method can also be applied to the isotropic work-hardening model. By a limiting argument of viscoplastic solutions, [3] established the existence for the material with both kinematic and isotropic work-hardening. [2] and [4] proved the uniqueness of the solutions for the isotropic work-hardening model by different arguments. Some abstract models for elastoplastic systems have been proposed and the well-posedness for those models is established [6, 7]. However, the connections between the model of the elastic-plastic oscillators with isotropic work-hardening and the assumptions of constitutive relations for their abstract models is not clear.

Following the idea of [5], we introduce a new variable that enables us to establish the equivalence of the system of differential equations for elastic-plastic oscillators with isotropic work-hardening and a system of differential inclusions. The operator associated with this system of differential inclusions turns out to be maximal monotone. Hence, the well-posedness of the system of differential inclusions can easily be obtained when the external force $f \in W^{1,1}(0, T; R)$. By a more delicate analy-

Received January 6, 1993.
1991 *Mathematics Subject Classification.* Primary 73E50, 34A34.

sis, well-posedness is established when $f \in L^1(0, T; R)$. Therefore, we prove that the system of differential equations for elastic-plastic oscillators with isotropic work-hardening is well-posed when $f \in L^1(0, T; R)$. This method can also be applied to other models of elastic-plastic materials, and some properties of the solutions can be discussed by using maximal monotone operator theory.

2. Preliminaries. Let v, σ, f, u^p, and E be the velocity, stress, external force, plastic deformation, and elastic constant, respectively. Let $H : (-\delta, \infty) \to (0, \infty)$ be a concave strictly increasing function. Here δ is a positive number.

We consider the following system of differential equations for elastic-plastic oscillators with isotropic work-hardening:

$$\text{(EPO)} \quad \begin{cases} \dot{v}(t) = f(t) - \sigma(t), \\ \dot{\sigma}(t) = \begin{cases} \dfrac{EH'(\bar{u}^p(t))}{E + H'(\bar{u}^p(t))} v(t) & \text{if } |\sigma(t)| = H(\bar{u}^p(t)) \text{ and } \sigma(t)v(t) \geq 0, \\ Ev(t) & \text{otherwise,} \end{cases} \\ \dot{u}^p(t) = \begin{cases} \dfrac{E}{E + H'(\bar{u}^p(t))} v(t) & \text{if } |\sigma(t)| = H(\bar{u}^p(t)) \text{ and } \sigma(t)v(t) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \\ (v(0), \sigma(0), u^p(0)) = (v_0, \sigma_0, u_0^p), \end{cases} \tag{1}$$

subject to the constraint

$$|\sigma(t)| \leq H(\bar{u}^p(t)), \tag{2}$$

where

$$\bar{u}^p(t) = \bar{u}^p(u_0^p) + \int_0^t |\dot{u}^p(s)|\, ds. \tag{3}$$

For simplicity, we assume that the constants $\bar{u}^p(u_0^p) = 0$ and $E = 1$.

DEFINITION 1. Let $T > 0$. We say that a set of absolutely continuous functions $\{(v(t), \sigma(t), u^p(t)) \mid |\sigma(t)| \leq H(\bar{u}^p(t))\}$ is a solution of (EPO) on $[0, T]$ if it satisfies (EPO) for almost every $t \in [0, T]$ and $(v(0), \sigma(0), u^p(0)) = (v_0, \sigma_0, u_0^p)$.

We wish to establish the well-posedness of (EPO). More precisely, we want to show (EPO) has a unique solution that depends continuously on the given data, i.e., on the external force and initial values.

Since the function $H(\bar{u}^p)$ is monotonically increasing, we can define a one-to-one transformation by

$$\hat{u}^p(\bar{u}^p) = \int_0^{\bar{u}^p} \sqrt{H'(s)}\, ds \tag{4}$$

and introduce a new function $G(\hat{u}^p)$ such that

$$G(0) = H(0) \quad \text{and} \quad G'(\hat{u}^p) = \sqrt{H'(\bar{u}^p)}. \tag{5}$$

Hence, for each $\hat{u}_a^p = \hat{u}^p(\bar{u}_a^p)$ it follows that

$$G(\hat{u}_a^p) = \int_0^{\hat{u}_a^p} G'(\hat{u}^p)\,d\hat{u}^p + G(0) = \int_0^{\bar{u}_a^p} \sqrt{H'(\bar{u}^p)}\sqrt{H'(\bar{u}^p)}\,d\bar{u}^p + H(0) = H(\bar{u}_a^p), \tag{6}$$

and we have

$$G(\hat{u}^p) = H(\bar{u}^p). \tag{7}$$

It is easy to see that the function $G : (-\delta_0, \infty) \to (0, \infty)$ is concave strictly increasing, where $\delta_0 = \int_{-\delta}^0 \sqrt{H'(s)}\,ds$.

It follows from (3), (4), and (5) that

$$\dot{\hat{u}}^p(t) = \sqrt{H'(\bar{u}^p(t))}\dot{\bar{u}}^p(t) = G'(\hat{u}^p(t))|\dot{u}^p(t)| = G'(\hat{u}^p(t))\dot{u}^p(t)\,\mathrm{Sign}(\sigma(t)), \tag{8}$$

where

$$\mathrm{Sign}(\sigma(t)) := \begin{cases} 1 & \text{if } \sigma(t) > 0, \\ 0 & \text{if } \sigma(t) = 0, \\ -1 & \text{if } \sigma(t) < 0. \end{cases} \tag{9}$$

By (5) and (EPO), (8) yields

$$\dot{\hat{u}}^p(t) = \begin{cases} \dfrac{G'(\hat{u}^p(t))}{1 + G'^2(\hat{u}^p(t))} v(t)\,\mathrm{Sign}(\sigma(t)) & \text{if } |\sigma(t)| = G(\hat{u}^p(t)) \text{ and } \sigma(t)v(t) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

where we have used $E = 1$. Therefore, we get the following lemma.

LEMMA 1. (EPO) is equivalent to the following system of differential equations:

$$(\overline{\mathrm{EPO}}) \begin{cases} \dot{v}(t) = f(t) - \sigma(t), \\ \dot{\sigma}(t) = \begin{cases} \dfrac{G'^2(\hat{u}^p(t))}{1 + G'^2(\hat{u}^p(t))} v(t) & \text{if } |\sigma(t)| = G'(\hat{u}^p(t)) \quad \text{and} \quad \sigma(t)v(t) \geq 0, \\ v(t) & \text{otherwise,} \end{cases} \\ \dot{\hat{u}}^p(t) = \begin{cases} \dfrac{G'(\hat{u}^p(t))}{1 + G'^2(\hat{u}^p(t))} v(t)\,\mathrm{Sign}(\sigma(t)) & \text{if } |\sigma(t)| = G(\hat{u}^p(t)),\ \sigma(t)v(t) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \\ (v(0), \sigma(0), \hat{u}^p(0)) = (v_0, \sigma_0, 0), \end{cases} \tag{11}$$

subject to the constraint

$$|\sigma(t)| \leq G(\hat{u}^p(t)), \tag{12}$$

where $G : (-\delta_0, \infty) \to (0, \infty)$ is a concave strictly increasing function.

3. Main results. We define

$$C := \{(\sigma, \hat{u}^p) \in R \times (-\delta_0, \infty) \mid |\sigma| \leq G(\hat{u}^p)\}. \tag{13}$$

The indicator function I_c of the subset C of R^2 is defined by

$$I_c(\sigma, \hat{u}^p) = \begin{cases} 0 & \text{if } (\sigma, \hat{u}^p) \in C, \\ +\infty & \text{if } (\sigma, \hat{u}^p) \in R^2 \backslash C. \end{cases} \tag{14}$$

I_c is proper convex and lower semicontinuous and the multivalued subgradient ∂I_c of I_c is

$$\partial I_c(\sigma, \hat{u}^p) = \begin{cases} \lambda(\mathrm{Sign}(\sigma), -G'(\hat{u}^p)), & \text{if } |\sigma| = G(\hat{u}^p), \\ (0, 0) & \text{if } |\sigma| < G(\hat{u}^p), \end{cases} \tag{15}$$

where λ is a positive number.

Let us consider the following system of differential inclusions:

$$\text{(DI)} \qquad \begin{cases} \dot{v}(t) = f(t) - \sigma(t), \\ (\dot{\sigma}(t), \dot{\hat{u}}^p(t)) \in (v(t), 0) - \partial I_c(\sigma(t), \hat{u}^p(t)) \end{cases}$$

with initial condition

$$(v(0), \sigma(0), \hat{u}^p(0)) = (v_0, \sigma_0, 0). \tag{16}$$

DEFINITION 2. Let $T > 0$. We say that a set of absolutely continuous functions $(v(t), \sigma(t), \hat{u}^p(t))$ is a solution of (DI) on $[0, T]$ if it satisfies (DI) for almost every $t \in [0, T]$ and $(v(0), \sigma(0), \hat{u}^p(0)) = (v_0, \sigma_0, 0)$.

THEOREM 1. (EPO) is equivalent to (DI).

Proof. It suffices to show that (DI) and $(\overline{\text{EPO}})$ are equivalent by Lemma 1.

First, we want to show that the solution of $(\overline{\text{EPO}})$ is a solution of (DI). Let $(u(t), \sigma(t), \hat{u}^p(t))$ be the solution of the $(\overline{\text{EPO}})$. Let

$$E_t := \{t \in [0, T] \mid |\sigma(t)| < G(\hat{u}^p(t))\}$$

and

$$P_t := \{t \in [0, T] \mid |\sigma(t)| = G(\hat{u}^p(t))\}.$$

When $|\sigma(t)| < G(\hat{u}^p(t))$, it follows from $(\overline{\text{EPO}})$ that for a.e. $t \in E_t$

$$\dot{\sigma}(t) = v(t) \quad \text{and} \quad \dot{\hat{u}}^p(t) = 0.$$

Hence,

$$(\dot{\sigma}(t) - v(t), \dot{\hat{u}}^p(t)) = (0, 0) = \partial I_c(\sigma(t), \hat{u}^p(t)) \tag{17}$$

for a.e. $t \in E_t$ and (DI) is satisfied.

When $|\sigma(t)| = G(\hat{u}^p(t))$, it follows from (5) and $(\overline{\text{EPO}})$ that for a.e. $t \in P_t$

$$\dot{\hat{u}}^p(t) = -G'(\hat{u}^p(t))\dot{\hat{u}}^p(t)$$

and

$$v(t) - \dot{\sigma}(t) = \mathrm{Sign}(\sigma(t))\dot{\hat{u}}^p(t)$$

for both cases $\sigma(t)v(t) \geq 0$ and $\sigma(t)v(t) < 0$. Therefore, for a.e. $t \in P_t$,

$$(v(t) - \dot{\sigma}(t), -\dot{\hat{u}}^p(t)) \in \lambda(\mathrm{Sign}(\sigma(t)), -G'(\hat{u}^p)(t)) = \partial I_c(\sigma(t), \hat{u}^p(t)). \tag{18}$$

Hence, (DI) is satisfied.

Next, we want to show that the solution of (DI) is a solution of $(\overline{\text{EPO}})$. Let $(u(t), \sigma(t), \hat{u}^p(t))$ be the solution of (DI).

When $t \in E_t$, we have $|\sigma(t)| < G(\hat{u}^p(t))$ and for a.e. $t \in E_t$

$$\dot{\sigma}(t) = v(t) \quad \text{and} \quad \dot{\hat{u}}^p(t) = 0 \tag{19}$$

by (DI).

When $t \in P_t$, we have $|\sigma(t)| = G(\hat{u}^p(t))$ and for a.e. $t \in P_t$

$$\operatorname{Sign}(\sigma(t))\dot{\sigma}(t) = G'(\hat{u}^p(t))\dot{\hat{u}}^p(t)$$

by differentiating both sides with respect to t. It follows from (DI) that

$$(v(t) - \dot{\sigma}(t), -\dot{\hat{u}}^p(t)) = (\operatorname{Sign}(\sigma(t)), -G'(\hat{u}^p(t)))\lambda \tag{20}$$

for a.e. $t \in P_t$; here $\lambda \geq 0$. Therefore, we have

$$\operatorname{Sign}(\sigma(t))\dot{\sigma}(t) = \lambda(G'(\hat{u}^p(t)))^2 \tag{21}$$

and

$$0 \leq \lambda = \frac{1}{G'^2}\operatorname{Sign}(\sigma(t))\dot{\sigma}(t) \tag{22}$$

for a.e. $t \in P_t$. It follows from (20) and (22) that

$$(v(t) - \dot{\sigma}(t), -\dot{\hat{u}}^p(t)) = \left(\frac{\dot{\sigma}(t)}{G'^2}, -\frac{1}{G'}\operatorname{Sign}(\sigma(t))\dot{\sigma}(t)\right), \tag{23}$$

and $\operatorname{Sign}(\sigma(t))\dot{\sigma}(t) \geq 0$ for a.e. $t \in P_t$. By solving $\dot{\sigma}(t)$ from (23), we get

$$\begin{aligned} \dot{\sigma}(t) &= \frac{G'^2(\hat{u}^p(t))}{1 + G'^2(\hat{u}^p(t))}v(t), \\ \dot{\hat{u}}^p(t) &= \frac{G'(\hat{u}^p(t))}{1 + G'^2(\hat{u}^p(t))}v(t)\operatorname{Sign}(\sigma(t)), \end{aligned} \tag{24}$$

and $\sigma(t)v(t) \geq 0$ for a.e. $t \in P_t$.

By (19) and (24), we obtain for a.e. $t \in [0, T]$

$$\dot{\sigma}(t) = \begin{cases} \dfrac{G'^2(\hat{u}^p(t))}{1 + G'^2(\hat{u}^p(t))}v(t) & \text{if } |\sigma(t)| = G'(\hat{u}^p(t)) \text{ and } \sigma(t)v(t) \geq 0, \\ v(t) & \text{otherwise,} \end{cases}$$

$$\dot{\hat{u}}^p(t) = \begin{cases} \dfrac{G'(\hat{u}^p(t))}{1 + G'^2(\hat{u}^p(t))}v(t)\operatorname{Sign}(\sigma(t)) & \text{if } |\sigma(t)| = G(\hat{u}^p(t)) \text{ and } \sigma(t)v(t) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since the domain of ∂I_c is the set C and the solution of (DI) lies in the domain of ∂I_c, we have $|\sigma(t)| \leq G(\hat{u}^p(t))$ for all $t \in [0, T]$. Hence, $(u(t), \sigma(t), \hat{u}^p(t))$ is a solution of $(\overline{\text{EPO}})$.

Now, we want to apply the maximal monotone operator theory to show the well-posedness of (DI).

Theorem 2. Let $T > 0$ be given and let $f \in W^{1,1}(0, T; R)$. (DI) has a unique solution that depends continuously on given data if $(\sigma_0, \hat{u}_0^p) \in C$.

Proof. Let A be an operator that maps $(v, \sigma, \hat{u}^p)$ to $(\sigma, -v, 0)$. Let

$$\overline{C} := \{(v, \sigma, \hat{u}^p) \in R \times R \times (-\delta_0, \infty) \mid |\sigma| \leq G(\hat{u}^p)\} \tag{25}$$

and let $I_{\bar{c}}$ be the indicator function of $\overline{C}$.

Denote $x(t) = (v(t), \sigma(t), \hat{u}^p(t))$. (DI) can be written as

$$\dot{x}(t) + (A + \partial I_{\bar{c}})x(t) = \bar{f}(t),$$

where $\bar{f}(t) = (f(t), 0, 0)$.

Since $\overline{C}$ is a convex closed subset of R^3, $\partial I_{\bar{c}}$ is a maximal monotone operator. Because A is a continuous and everywhere defined monotone operator, $A + \partial I_{\bar{c}}$ is also a maximal monotone operator by Theorem 3.2 of [8] (page 158). Since $(\sigma_0, \hat{u}_0^p)$ is in the domain of ∂I_n, we have $(v_0, \sigma_0, \hat{u}_0^p)$ is in the domain of $A + \partial I_{\bar{c}}$. Hence, (DI) is well-posed [9].

Next, we want to weaken the regularity of f. More precisely, we wish to show the well-posedness of (DI) when $f \in L^1(0, T; R)$.

Lemma 2 (see [10]). Let $m \in L^1(0, T; R)$ be such that $m(t) \geq 0$ for a.e. $t \in (0, T)$ and $b \geq 0$ a constant. Let $\phi : [0, T] \to R$ be a continuous function that satisfies

$$\phi^2(t) \leq b^2 + 2\int_0^t m(s)\phi(s)\, ds \quad \forall t \in [0, T]. \tag{26}$$

Then

$$|\phi(t)| \leq b + \int_0^t m(s)\, ds \quad \forall t \in [0, T]. \tag{27}$$

Theorem 3. Let $f_i \in L^1(0, T; R)$, $i = 1, 2$ and let $(v_i(t), \sigma_i(t), \hat{u}_i^p(t))$ be a solution of (DI) with initial condition $(v_i(0), \sigma_i(0), \hat{u}_i^p(0))$ and external force f_i. Then

$$\begin{aligned}(|v_1 - v_2| + |\sigma_1 - \sigma_2| + |\hat{u}_1^p - \hat{u}_2^p|)(t)\\ \leq (|v_1 - v_2| + |\sigma_1 - \sigma_2| + |\hat{u}_1^p - \hat{u}_2^p|)(0) + \int_0^t |f_1(s) - f_2(s)|\, ds.\end{aligned} \tag{28}$$

Proof. Denote $x_i(t) = (v_i(t), \sigma_i(t), \hat{u}_i^p(t))$. By the monotonicity of $A + \partial I_{\bar{c}}$ we have

$$\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle \leq \langle f_1(t) - f_2(t), x_1(t) - x_2(t)\rangle,$$

where $\langle\ ,\ \rangle$ stands for inner product. Hence,

$$(x_1 - x_2)^2(t) \leq (x_1 - x_2)^2(0) + 2\int_0^t |(f_1(s) - f_2(s))||(x_1(s) - x_2(s))|\, ds.$$

Therefore,

$$(x_1 - x_2)(t) \leq (x_1 - x_2)(0) + \int_0^t |(f_1(s) - f_2(s))|\, ds$$

by Lemma 2.

THEOREM 4. Let $f \in L^1(0, T; R)$. (DI) has a solution if $(\sigma_0, \hat{u}_0^p) \in C$.

Proof. Let $n \in N$ and $(\partial I_{\bar{c}})_n$ be a sequence of Yosida approximation of $\partial I_{\bar{c}}$. Since $(v_0, \sigma_0, \hat{u}_0^p)$ is in the domain of $\partial I_{\bar{c}}$ and the domain of $\partial I_{\bar{c}}$ is a subset of the domain of $(\partial I_{\bar{c}})_n$ for each $n \in N$, $(v_0, \sigma_0, \hat{u}_0^p)$ is in the domain of $(\partial I_{\bar{c}})_n$ for each $n \in N$. Because $(\partial I_{\bar{c}})_n$ is Lipschitz continuous and $f \in L^1(0, T; R)$, the following equation has a unique solution for each $n \in N$:

$$\dot{x}_n(t) + (A + (\partial I_{\bar{c}})_n)x_n(t) = \bar{f}(t) \tag{29}$$

with initial condition

$$x_n(0) = (v_0, \sigma_0, \hat{u}_0^p),$$

where $\bar{f}(t) = (f(t), 0, 0)$.

Since

$$(\partial I_{\bar{c}})_n x = \tfrac{1}{n}(x - \mathrm{Proj}_{\bar{c}}\, x), \tag{30}$$

we have $(\partial I_{\bar{c}})_n 0 = 0$. Hence, for a.e. $t \in [0, T]$

$$\langle (A + (\partial I_{\bar{c}})_n)x_n(t), x_n(t)\rangle = \langle (A + (\partial I_{\bar{c}})_n)x_n(t) - (A + (\partial I_{\bar{c}})_n)0, x_n(t) - 0\rangle \leq 0. \tag{31}$$

It follows from (29) and (31) that for a.e. $t \in [0, T]$

$$\langle \dot{x}_n(t), x_n(t)\rangle \leq \langle \bar{f}(t), x_n(t)\rangle.$$

Therefore,

$$x_n^2(t) \leq x_n^2(0) + \int_0^t |\bar{f}(s)||x_n(s)|\, ds \leq x^2(0) + \int_0^T |\bar{f}(s)||x_n(s)|\, ds \tag{32}$$

for each $n \in N$ and $t \in [0, T]$. Thus, $x_n(t)$ is uniformly bounded on $[0, T]$ by Lemma 2.

Since for a.e. $t \in [0, T]$

$$(\dot{\sigma}_n(t), \dot{\hat{u}}_n^p(t)) = (v_n(t), 0) - \partial_n I_c(\sigma_n(t), \hat{u}_n^p(t)),$$

we have

$$\begin{aligned}\langle (\dot{\sigma}_n(t), \dot{\hat{u}}_n^p(t)), (\dot{\sigma}_n(t), \dot{\hat{u}}_n^p(t))\rangle + \langle \partial_n I_c(\sigma_n(t), \hat{u}_n^p(t)), (\dot{\sigma}_n(t), \dot{\hat{u}}_n^p(t))\rangle \\ = \langle (v_n(t), 0), (\dot{\sigma}_n(t), \dot{\hat{u}}_n^p(t))\rangle\end{aligned} \tag{33}$$

for a.e. $t \in [0, T]$. It follows from (33) that for each $n \in N$

$$\int_0^T (\dot{\sigma}_n^2(s) + (\dot{\hat{u}}_n^p)^2(s))\, ds + (I_c)_n(\sigma_n(T), \hat{u}_n^p(T)) - (I_c)_n(\sigma_n(0), \hat{u}_n^p(0)) = \int_0^T v_n(s)\dot{\sigma}_n(s)\, ds, \tag{34}$$

where

$$(I_c)_n(x) = \tfrac{1}{2n}|x - \mathrm{Proj}_c\, x|^2.$$

Since $(\sigma_n(0), \hat{u}_n^p(0)) = (\sigma_0, \hat{u}_0^p) \in C$, we have

$$(I_c)_n(\sigma_n(0), \hat{u}_n^p(0)) = 0 \quad \text{and} \quad (I_c)_n(\sigma_n(T), \hat{u}_n^p(T)) \geq 0.$$

Hence, (34) yields

$$\int_0^T (\dot{\sigma}_n^2(s) + (\dot{\hat{u}}_n^p)^2(s))\, ds \leq \left(\int_0^T |v_n(s)|^2\, ds\right)^{1/2} \left(\int_0^T |\dot{\sigma}_n(s)|^2\, ds\right)^{1/2} \tag{35}$$

for each $n \in N$. We conclude that the L^2 norms of $\dot{\sigma}_n$, $\dot{u}_n^p$ are uniformly bounded by the uniform boundedness of v_n. Therefore, the L^1 norms of $\dot{\sigma}_n$, $\dot{u}_n^p$ are also uniformly bounded.

Since $f(t) \in L^1(0, T; R)$ and

$$\dot{v}_n(t) = \sigma_n(t) + f(t),$$

the L^1 norm of $\dot{v}_n$ is uniformly bounded. Hence, $(v_n, \sigma_n, \hat{u}_n^p)$ converges uniformly on $[0, T]$ to a set $(v, \sigma, \hat{u}^p)$ of absolutely continuous functions. Moreover, $(\dot{v}_n, \dot{\sigma}_n, \dot{u}_n^p)$ converges weakly to $(\dot{v}, \dot{\sigma}, \dot{u}^p)$ in $L^1(0, T; R)$. Therefore, $(\dot{v}, \dot{\sigma}, \dot{u}^p)$ is the solution of (DI) by the demiclosedness of the maximal monotone operator $A + \partial I_{\bar{C}}$.

It follows from Theorems 1, 3, and 4 that the following theorem holds.

Theorem 5. Let $f \in L^1(0, T; R)$. (EPO) has a unique solution that depends continuously on initial data and the external force if $|\sigma_0| \leq H(\bar{u}^p(0))$.

Acknowledgment. I would like to thank Professor D. R. Owen for helpful discussions.

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