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A GENERALISATION OF PRÜFER'S TRANSFORMATION

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Abstract. A canonical version of the Prüfer system for Sturm-Liouville equations is presented. The partial uncoupling of the first-order equations remains, providing a basis for their solution in succession.

1. The Prüfer transformation. The standard Prüfer transformation for the Sturm-Liouville equation [1, 2]

$$\frac{d}{dx}\left[P(x)\frac{du}{dx}\right] + Q(x)u = 0 \tag{1}$$

consists of defining new functions $r(x)$ and $\theta(x)$ by the relations

$$\begin{aligned} P(x)\frac{du}{dx} &= r(x)\cos[\theta(x)], \\ u &= r(x)\sin[\theta(x)]. \end{aligned} \tag{2}$$

Then we have

$$\begin{aligned} r^2(x) &= u^2 + (Pu')^2, \\ \sin[\theta(x)] &= u/r, \\ \cos[\theta(x)] &= Pu'/r. \end{aligned} \tag{3}$$

From equations (3) we obtain

$$\cot[\theta(x)] = Pu'/u, \tag{4}$$

which on differentiation, and using equation (1), gives a first-order nonlinear differential equation for the function $\theta(x)$

$$\theta'(x) = Q(x)\sin^2[\theta(x)] + \frac{1}{P(x)}\cos^2[\theta(x)]. \tag{5}$$

Differentiating the first equation in (3), and again using equation (1), we obtain

$$r'(x) = \frac{1}{2}\left[\frac{1}{P(x)} - Q(x)\right] r(x)\sin[2\theta(x)],$$

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which integrates to give

$$r(x) = K \exp\left\{\frac{1}{2}\int^x \left[\frac{1}{P(t)} - Q(t)\right]\sin[2\theta(t)]\,dt\right\}. \tag{6}$$

In principle we solve Eq. (5) for the function $\theta(t)$ and insert this into Eq. (6) to obtain the function $r(x)$.

The main purpose of the Prüfer transformation is to enable one to obtain results about the oscillatory properties of the solutions of the original Sturm-Liouville equation (1) by only considering first-order equations.

2. A generalised Prüfer transformation. We shall now consider a generalisation of the Prüfer transformation that achieves the same result of reducing the system to a pair of first-order equations that can be solved in succession and that has the added advantage of being related to the general theory of canonical transformations that play such a large part in analytical mechanics and in the theory of first-order partial differential equations.

First we write the Sturm-Liouville equation (1) as a pair of canonical first-order equations:

$$\begin{aligned} \frac{du}{dx} &= \frac{v}{P(x)} = \frac{\partial H}{\partial v}, \\ -\frac{dv}{dx} &= Q(x)u = \frac{\partial H}{\partial u}, \end{aligned} \tag{7}$$

with the Hamiltonian function H given by

$$H(x, u, v) = \frac{1}{2}\left[\frac{v^2}{P(x)} + Q(x)u^2\right]. \tag{8}$$

Now define, by means of a canonical transformation, new variables (z, θ) and a new Hamiltonian $K(x, z, \theta)$ such that

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{\partial K}{\partial z}, \\ -\frac{dz}{dx} &= \frac{\partial K}{\partial \theta}. \end{aligned} \tag{9}$$

Let $\Phi(x, u, \theta)$ be the generating function of the canonical transformation between the old and new variables. Then we must have

$$v\,du - H(x, u, v)\,dx = z\,d\theta - K(x, z, \theta) + d\Phi(x, u, \theta), \tag{10}$$

which implies that

$$\begin{aligned} \frac{\partial \Phi}{\partial u} &= v, \\ \frac{\partial \Phi}{\partial \theta} &= -z, \\ \frac{\partial \Phi}{\partial x} &= K - H. \end{aligned} \tag{11}$$

Suppose that we try to obtain a transformation of the form

$$\begin{aligned} u &= \alpha z^{\lambda} \sin\theta, \\ v &= \beta z^{\lambda} \cos\theta, \end{aligned} \tag{12}$$

where α, β, and λ are constants to be determined. If the transformation is to be canonical then from Eqs. (12) and (11) we see that

$$\begin{aligned}\frac{\partial \Phi}{\partial u} &= v = \frac{\beta}{\alpha} u \cot\theta , \\ \frac{\partial \Phi}{\partial \theta} &= -z = -\frac{u^{1/\lambda}}{\alpha^{1/\lambda}} \operatorname{cosec}^{1/\lambda}\theta .\end{aligned} \tag{13}$$

For the existence of a generating function we must have

$$\frac{\partial}{\partial \theta}\left(\frac{\partial \Phi}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial \Phi}{\partial \theta}\right),$$

which gives

$$\lambda = 1/2, \quad \alpha\beta = 2. \tag{14}$$

For simplicity we shall take $\alpha = \beta = 2^{1/2}$, to obtain

$$\begin{aligned}u &= (2z)^{1/2} \sin\theta , \\ v &= (2z)^{1/2} \cos\theta ,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\frac{\partial \Phi}{\partial u} &= u \cot\theta , \\ \frac{\partial \Phi}{\partial \theta} &= -\frac{1}{2} u^2 \operatorname{cosec}^2\theta .\end{aligned} \tag{16}$$

Equations (16) can be satisfied by the generating function

$$\Phi(x, u, \theta) = \tfrac{1}{2} u^2 \cot\theta . \tag{17}$$

This generating function does not contain the independent variable x explicitly and so by (11) the new Hamiltonian K is given by

$$K(x, z, \theta) = H(x, u, v) = z\left[Q(x)\sin^2\theta + \frac{1}{P(x)}\cos^2\theta\right]. \tag{18}$$

Since the new Hamiltonian is linear in the variable z we see that the equations separate as before:

$$\begin{aligned}\theta'(x) &= \frac{\partial K}{\partial z} = Q(x)\sin^2[\theta(x)] + \frac{1}{P(x)}\cos^2[\theta(x)], \\ z'(x) &= -\frac{\partial K}{\partial \theta} = \left[\frac{1}{P(x)} - Q(x)\right] z(x)\sin[2\theta(x)].\end{aligned} \tag{19}$$

We could have obtained the generating function Φ of the canonical transformation from the Hamilton-Jacobi equation for this problem. If $S(x, u)$ is the action function for the problem then the Hamilton-Jacobi equation that S has to satisfy is

$$H\left(x, u, \frac{\partial S}{\partial u}\right) + \frac{\partial S}{\partial x} = 0 \tag{20}$$

or

$$\frac{1}{P(x)}\left(\frac{\partial S}{\partial u}\right)^2 + Q(x)u^2 + 2\frac{\partial S}{\partial x} = 0. \tag{21}$$

If we put

$$S(x, u) = R(x)u^2, \tag{22}$$

then we obtain the function $R(x)$ as a solution of the differential equation

$$R'(x) = -\frac{1}{2}\left[Q(x) + \frac{4R^2(x)}{P(x)}\right]. \tag{23}$$

Although Eq. (23) is extremely difficult to solve in general, on comparing Eq. (23) with Eq. (4), leading to (5), we can see that a particular solution is given by

$$R(x) = \tfrac{1}{2}\cot[\theta(x)], \tag{24}$$

which together with Eq. (22) leads to (17).

3. Alternative transformation. If we use the generating function

$$\Phi(x, u, \theta) = -\tfrac{1}{2}u^2\theta, \tag{25}$$

then we obtain a new Hamiltonian

$$K_1(x, \theta, z) = z\left[\frac{\theta^2}{P(x)} + Q(x)\right] \tag{26}$$

and first-order differential equations of a seemingly simpler nature than before:

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{\partial K_1}{\partial z} = \frac{\theta^2}{P(x)} + Q(x), \\ \frac{dz}{dx} &= -\frac{\partial K_1}{\partial \theta} = \frac{-2z\theta}{P(x)}. \end{aligned} \tag{27}$$

We have shown that Prüfer's idea of changing the variables in a Sturm-Liouville problem, which is essentially moving to a sort of polar coordinates, can be generalised so that the resulting system of first-order equations is canonical in the Hamiltonian sense, a result not achieved by Prüfer's original transformation. We have also looked at this transformation from the point of view of the Hamilton-Jacobi theory for the problem.

References

[1] G. Birkhoff and G. C. Rota, *Ordinary Differential Equations*, Wiley, New York, fourth edition, 1989

[2] H. Prüfer, *Neue Herleitung der Sturm-Liouvilleschen Reihentwicklung stetiger Funktionen*, Math. Ann. **95**, 499–518 (1926)