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THE CAHN-HILLIARD EQUATION AS DEGENERATE LIMIT OF THE PHASE-FIELD EQUATIONS

By

BARBARA E. E. STOTH

Department of Applied Mathematics, University of Bonn, Germany

Abstract. We show that the Cahn-Hilliard equation occurs as a special scaling limit of the phase-field equation.

Introduction and statement of the results. Within this paper we study the relationship between the Cahn-Hilliard equation and the phase-field model studied by Caginalp (1986). We recollect that the Cahn-Hilliard equation may be written as

$$\begin{aligned} \partial_t \varphi &= \Delta u, \\ u &= -\varepsilon \Delta \varphi + \frac{\sigma^2}{\varepsilon} f'(\varphi), \end{aligned} \tag{CH}$$

where $f(\varphi) = \frac{1}{2}(1-\varphi^2)^2$ is a double well potential, σ a measure for surface tension, and ε a (mostly) small parameter measuring the width of the transition layer, where φ varies quickly from -1 to $+1$.

The phase-field equations we consider (see Caginalp 1989) have the form

$$\begin{aligned} \alpha \partial_t u + \partial_t \varphi - \Delta u &= 0, \\ \alpha \varepsilon \partial_t \varphi - \varepsilon \Delta \varphi + \frac{\sigma^2}{\varepsilon} f'(\varphi) - u &= 0. \end{aligned} \tag{PhF}$$

Formally we expect the Cahn-Hilliard equation as the limit α to zero (see Elliott 1989), and in fact it turns out that this is true rigorously under reasonable restrictions on the initial data.

Our assumptions are the following.

A) The functions u, $\varphi = u_\alpha$, $\varphi_\alpha \in C^\infty(\overline{\Omega_T})$ are solutions of (PhF) in some smooth space-time domain $\Omega_T := (0, T) \times \Omega$.

B) The boundary values $u_\alpha|_{\partial\Omega} =: u_D$ are independent of time and the parameter α and either the boundary values $\varphi_\alpha|_{\partial\Omega} =: \varphi_D$ are independent of time and the parameter α or $\nabla\varphi_\alpha \cdot \nu|_{\partial\Omega} = 0$.

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C) The initial data $u_\alpha(0, \cdot)$ and $\varphi_\alpha(0, \cdot)$ satisfy

$$\left(\frac{\varepsilon}{2}\int_\Omega |\nabla\varphi_\alpha|^2\,dx + \frac{\sigma^2}{\varepsilon}\int_\Omega f(\varphi_\alpha)\,dx + \frac{\alpha}{2}\int_\Omega u_\alpha^2\,dx\right)(0) \le C,$$

$$\left(\frac{\alpha\varepsilon^2}{\sigma^2}\int_\Omega |\nabla u_\alpha|^2\,dx + \frac{\varepsilon}{\sigma^2}\int_\Omega u_\alpha^2\,dx + \frac{\varepsilon^3}{\sigma^2}\int_\Omega |\Delta\varphi_\alpha|^2\,dx + \|\varphi_\alpha\|_{L^\infty(\Omega)}\right)(0) \le C$$

with some constant C independent of α, and the initial data of the phase-field φ_α converge in $L^1(\Omega)$ to some $\varphi^0 \in H^{2,2}(\Omega)$.

As an example we may take smooth initial data independent of α.

All subsequently introduced constants C will depend implicitly only on the boundary values, the above constant C, the domain Ω and T, but not on the parameters ε, σ, and α.

The existence of such solutions of the phase-field equations has been shown by Elliott and Zheng (1989).

Theorem 1. Under the above hypotheses a subsequence $(u_\alpha, \varphi_\alpha)$ converges to limiting functions (u, φ) with

$$u \in L^2(0, T; H^{1,2}(\Omega)) \quad \text{and} \quad \varphi \in L^\infty(0, T; H^{1,2}(\Omega)) \cap L^\infty(0, T; L^4(\Omega))$$

in the sense that, as $\alpha \to 0$,

$$\begin{aligned} u_\alpha, \nabla u_\alpha, \nabla\varphi_\alpha &\to u, \nabla u, \nabla\varphi \quad &&\text{weakly in } L^2(\Omega_T),\\ \varphi_\alpha &\to \varphi &&\text{in } L^1(\Omega_T) \text{ and pointwise a.e.} \end{aligned}$$

The limiting functions are weak solutions of the Cahn-Hilliard equation

$$\begin{aligned} \partial_t\varphi &= \Delta u,\\ u &= -\varepsilon\Delta\varphi + \frac{\sigma^2}{\varepsilon}f'(\varphi) \end{aligned} \tag{CH}$$

in Ω_T with boundary values

$$\begin{aligned} u|_{\partial\Omega} &= u_D,\\ \varphi|_{\partial\Omega} &= \varphi_D \quad (\nabla\varphi\cdot\nu|_{\partial\Omega} = 0, \text{ resp.}), \end{aligned}$$

and initial values $\varphi(0, \cdot) = \varphi^0$.

The weak formulation of (CH) does not allow one to resubstitute u by $-\varepsilon\Delta\varphi + \frac{\sigma^2}{\varepsilon}f'(\varphi)$ in the first equation of (CH). In order to obtain the usual Cahn-Hilliard equations we need higher regularity of the solution.

Proposition 2. If $n \le 3$, then the limiting phase-field φ belongs to $L^2(0, T; H^{3,2}(\Omega))$. If, in addition, the initial values satisfy the compatibility condition

$$\varphi^0 \in H^{4,2}(\Omega) \quad \text{and} \quad \begin{cases} -\varepsilon\Delta\varphi^0 + \frac{\sigma^2}{\varepsilon}f'(\varphi^0) = u_D,\\ \varphi^0 = \varphi_D \quad (\nabla\varphi^0\cdot\nu = 0, \text{ resp.}) \end{cases} \quad \text{on } \partial\Omega,$$

then

$$\varphi \in H^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^{4,2}(\Omega))$$

and φ is the unique solution of the Cahn-Hilliard equation

$$\partial_t \varphi + \Delta\left(\varepsilon\Delta\varphi - \frac{\sigma^2}{\varepsilon} f'(\varphi)\right) = 0 \quad \text{on } \Omega_T$$

with initial values φ^0 and Dirichlet boundary values φ_D (Neumann-0 boundary values, resp.) for φ and Dirichlet boundary values $(-\varepsilon\Delta\varphi + \frac{\sigma^2}{\varepsilon} f'(\varphi))|_{\partial\Omega} = u_D$.

In particular, the whole sequence φ_α converges to φ.

A priori bounds, compactness, and regularity. We start collecting some useful a priori estimates.

LEMMA 3. The sequence $(u_\alpha, \varphi_\alpha)$ satisfies

(i)

$$\alpha\varepsilon \int_0^T \int_\Omega |\partial_t \varphi_\alpha|^2 \, dx\, dt + \int_0^T \int_\Omega |\nabla u_\alpha|^2 \, dx\, dt + \sup_{t\in[0,T]} \left(\varepsilon \int_\Omega |\nabla \varphi_\alpha|^2 \, dx + \frac{\sigma^2}{\varepsilon} \int_\Omega f(\varphi_\alpha)\, dx + \alpha \int_\Omega u_\alpha^2 \, dx\right)(t) \le C,$$

(ii)

$$\sup_{t\in[0,T]} \left(\alpha\varepsilon \int_\Omega |\partial_t \varphi_\alpha|^2 \, dx + \int_\Omega |\nabla u_\alpha|^2 \, dx\right) + \varepsilon \int_0^T \int_\Omega |\nabla \partial_t \varphi_\alpha|^2 \, dx\, dt + \alpha \int_0^T \int_\Omega |\partial_t u_\alpha|^2 \, dx\, dt \le \frac{C}{\alpha\varepsilon^2}$$

with some constant C independent of α.

Proof. The first estimate (i) can be derived by multiplying the first of the phase-field equations (PhF) by $(u - u_D)$, the second by $\partial_t \varphi$, adding both results and integrating over Ω_T. Then hypothesis B on the boundary data and the first of the estimates of the initial data in C implies the result.

To obtain (ii) differentiate the second of the phase-field equations (PhF) with respect to t, multiply by $\partial_t \varphi$, multiply the first equation by $\partial_t u$, add the results and integrate. Note that $-f'' \le 2$ to estimate the nonquadratic term, apply the first part of this lemma to estimate $\iint |\partial_t \varphi|^2$, and use the second estimate of hypothesis C to control the initial values.

Proof of Theorem 1. Since we deal with a nonlinear equation we have to show strong compactness of the sequence φ_α in $L^1(\Omega_T)$. Lemma 3 only furnishes estimates of derivatives in the x-direction, but no uniform bounds on time derivatives. Compactness follows from interpolating the fact that we know $\partial_t \varphi_\alpha \in L^2(0, T; H^{-1,2}(\Omega))$ and $\nabla \varphi_\alpha \in L^\infty(0, T; L^2(\Omega))$. We define the time differences

$$\chi_\alpha(t) := \varphi_\alpha(t+h) - \varphi_\alpha(t).$$

The differential equation

$$\partial_t \varphi_\alpha = -\alpha \partial_t u_\alpha + \Delta u_\alpha$$

immediately implies

$$\int_0^{T-h} \|\chi_a(t)\|_{H^{-1,2}}\,dt \le T^{\frac12} h \left(\left(\int_0^T \int_\Omega |\nabla u_\alpha|^2\,dx\,dt \right)^{\frac12} + \alpha \left(\int_0^T \int_\Omega |\partial_t u_\alpha|^2\,dx\,dt \right)^{\frac12} \right).$$

The right-hand side can be estimated by Lemma 3 and we find

$$\int_0^{T-h} \|\chi_a(t)\|_{H^{-1,2}}\,dt \le C\cdot h.$$

We apply the following interpolation inequality, valid for all $w \in L^1(0, T; H^{1,1}(\Omega))$ and $\rho > 0$ (see Luckhaus 1990):

$$\int_0^T \int_\Omega |w|\,dx\,dt \le \rho \int_0^T \int_\Omega |\nabla w|\,dx\,dt + \frac{C}{\rho} \int_0^T \|w(t)\|_{H^{-1,2}}\,dt + \int_0^T \int_{B_{2\rho}(\partial\Omega)} |w|\,dx\,dt.$$

We set $w = \chi_\alpha$, use the above estimate to obtain bounds for the right-hand side and arrive at

$$\int_0^{T-h} \int_\Omega |\varphi_\alpha(t+h, x) - \varphi_\alpha(t, x)|\,dt\,dx \le \left(\rho + \frac{h}{\rho} + \rho^{1/2} \right).$$

Now choose $\rho = h^{1/4}$. This estimate, together with Lemma 3, then implies the compactness of φ_α in $L^1(\Omega_T)$ such that we can select a subsequence of φ_α converging in $L^1(\Omega_T)$ and pointwise almost everywhere to some limiting phase-field φ. Lemma 3 allows to assume that, for a further subsequence, $\nabla\varphi_\alpha$, ∇u_α, and u_α converge weakly in $L^2(\Omega_T)$ to $\nabla\varphi$, ∇u, and u (resp.). The uniform bounds from Lemma 3 carry over to the limits φ and u.

Now we pass to the limit in the phase-field equations (PhF). To this end multiply $(\mathrm{PhF})_1$ by a smooth test function ζ with $\zeta(T, \cdot) = 0$ and $\zeta(t, \cdot) \in C_0^\infty(\Omega)$ in the case of prescribed Dirichlet values and $(\mathrm{PhF})_2$ by a smooth test function η with $\eta(T, \cdot) = 0$:

$$-\int_0^T \int_\Omega (\alpha u_\alpha + \varphi_\alpha)\partial_t\zeta\,dx\,dt - \int_\Omega (\alpha u_\alpha + \varphi_\alpha)\zeta(0, x)\,dx + \int_0^T \int_\Omega \nabla u_\alpha \cdot \nabla\zeta\,dx\,dt = 0,$$

$$\alpha\varepsilon \int_0^T \int_\Omega \varphi_\alpha \partial_t\eta\,dx\,dt - \alpha\varepsilon \int_\Omega (\varphi_\alpha\eta)(0, x)\,dx + \varepsilon \int_0^T \int_\Omega \nabla\varphi_\alpha \cdot \nabla\eta\,dx\,dt$$
$$+ \int_0^T \int_\Omega \left(\frac{\sigma^2}{\varepsilon} f'(\varphi_\alpha) - u_\alpha \right) \eta\,dx\,dt = 0.$$

We note that αu_α and $\alpha\varphi_\alpha$ converge weakly to zero. Furthermore, $f'(\varphi_\alpha) = -2(\varphi_\alpha - \varphi_\alpha^3)$ converges in $L^1(\Omega_T)$, since it converges pointwise and $\varphi_\alpha \in L^4(\Omega_T)$ uniformly in α. All the other terms are linear and converge as a consequence of the weak convergence of the respective terms. We obtain

$$-\int_0^T \int_\Omega \varphi\partial_t\zeta\,dx\,dt - \int_\Omega \varphi^0\zeta(0)\,dx + \int_0^T \int_\Omega \nabla u \cdot \nabla\zeta\,dx\,dt = 0,$$

$$\varepsilon \int_\Omega \nabla\varphi \cdot \nabla\eta\,dx + \int_\Omega \frac{\sigma^2}{\varepsilon} f'(\varphi)\eta\,dx - \int_\Omega u\eta\,dx = 0$$

for all $\zeta \in C_0^\infty([0, T) \times \Omega)$ and for all $\eta \in C_0^\infty(\Omega)$ ($\eta \in C^\infty(\Omega)$ resp.). This is a weak formulation of the Cahn-Hilliard equation.

Proof of Proposition 2. By elliptic and parabolic regularity theory, weak solutions of the Cahn-Hilliard equation (CH) as in Theorem 1 are in fact pointwise solutions: we first observe that, since $f''(\varphi) = 6\varphi^2 - 2 \in L^\infty(0, T; L^2(\Omega))$, we have

$$\nabla f'(\varphi) = f''(\varphi)\nabla\varphi \in L^\infty(0, T; L^1(\Omega)),$$

and thus by the Sobolev imbedding theorem

$$f'(\varphi) \in L^\infty(0, T; L^{n/(n-1)}(\Omega)).$$

We apply elliptic regularity theory to the second equation of (CH) and obtain (since $u \in L^2(\Omega_T)$)

$$\varphi \in L^2(0, T; H^{2, n/(n-1)}(\Omega)),$$

such that in return

$$-\varepsilon\Delta\varphi + \frac{\sigma^2}{\varepsilon} f'(\varphi) = u \quad \text{almost everywhere in } \Omega_T.$$

If $n \le 3$ the Sobolev imbedding theorem yields for all $q < \infty$

$$\varphi \in L^2(0, T; L^q(\Omega)).$$

This implies for all $p < 2$

$$\nabla f'(\varphi) = f''(\varphi)\nabla\varphi \in L^1(0, T; L^p(\Omega)),$$

and together with $\nabla f'(\varphi) \in L^p(0, T; L^1(\Omega))$ that for all $m < 3/2$

$$\nabla f'(\varphi) \in L^m(0, T; L^m(\Omega)).$$

Then elliptic regularity theory implies (since $\nabla u \in L^2(\Omega_T)$)

$$\nabla\varphi \in L^m(0, T; H^{2, m}(\Omega))$$

and so

$$\varphi \in L^m(0, T; H^{3, m}(\Omega)) \subset L^m(0, T; H^{2, 3m/(3-m)}(\Omega)) \subset L^m(0, T; H^{1, 3m/(3-2m)}(\Omega)).$$

In addition since $\varphi \in L^\infty(0, T; H^{1,2}(\Omega))$ we have $\varphi \in L^\infty(0, T; L^6(\Omega))$.

We substitute u by $-\varepsilon\Delta\varphi + \frac{\sigma^2}{\varepsilon} f'(\varphi)$ in the first equation of (CH) and obtain a weak formulation of

$$\partial_t\varphi + \varepsilon\Delta^2\varphi = \frac{\sigma^2}{\varepsilon}(f'''(\varphi)|\nabla\varphi|^2 + f''(\varphi)\Delta\varphi) =: g$$

with a right-hand side

$$g \in L^m(0, T; L^m(\Omega)).$$

For this calculate

$$\begin{aligned}\int_0^T \|f''(\varphi)\Delta\varphi\|_m^m(t)\, dt &\le C \int_0^T (\|\varphi(t)\|_6^3 + 1)\|\Delta\varphi(t)\|_{2m}^m\, dt \\ &\le C \sup_t(\|\varphi(t)\|_6^3 + 1) \int_0^T \|\Delta\varphi(t)\|_{2m}^m\, dt\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \|f'''(\varphi)|\nabla\varphi|^2\|_m^m(t)\,dt \\
&\quad \le C\int_0^T \|\nabla\varphi(t)\|_{3m/(3-2m)}^m \left(\int_\Omega (|\varphi|^m+1)|\nabla\varphi|^m(t,x)\,dx\right)^{2m/3} dt \\
&\quad \le C\int_0^T \|\nabla\varphi(t)\|_{3m/(3-2m)}^m \|\varphi(t)\|_6^m \|\nabla\varphi\|_2^m\,dt \\
&\quad \le C\sup_t \|\varphi(t)\|_6^m \sup_t \|\nabla\varphi(t)\|_2^m \int_0^T \|\nabla\varphi(t)\|_{3m/(3-2m)}^m\,dt\,.
\end{aligned}$$

Parabolic regularity theory together with the compatibility condition of the initial data then implies that

$$\varphi \in H^{1,m}(0,T;L^m(\Omega)) \cap L^m(0,T;H^{4,m}(\Omega)),$$

such that in return

$$\partial_t\varphi + \Delta\left(\varepsilon\Delta\varphi - \frac{\sigma^2}{\varepsilon}f'(\varphi)\right) = 0 \quad \text{almost everywhere in } \Omega_T\,.$$

Elliott and Zheng proved (1986) that a solution of this class is unique.

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References

[1] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rat. Mech. Anal. **92**, 205–245 (1986)

[2] G. Caginalp, *Stefan and Hele Shaw type models as asymptotic limits of the phase-field equations*, Physical Review **A39**, No. 11, 5887–5896 (1989)

[3] C. M. Elliott, *The Cahn-Hilliard model for the kinetics of phase separation*, Mathematical models for phase change problems, Proc. European Workshop, Obidos/Port. 1989, Internat. Ser. Numer. Math. **88**, 35–73 (1989)

[4] C. M. Elliott and S. Zheng, *On the Cahn-Hilliard equation*, Arch. Rat. Mech. Anal. **96**, 339–357 (1986)

[5] C. M. Elliott and S. Zheng, *Global Existence and Stability of Solutions to the Phase Field Equations*, SFB 256-Report No. 74, Bonn, 1989

[6] St. Luckhaus, *Solutions of the two phase Stefan problem with the Gibbs-Thomson law for the melting temperature*, European J. Appl. Math. **1**, 101–111 (1990)