QUARTERLY OF APPLIED MATHEMATICS

VOLUME LIV MARCH · 1996 NUMBER 1

MARCH 1996, PAGES 1-19

TEMPORARY RANGE OF VALIDITY FOR SOME MATHEMATICAL MODELS WHICH INVOLVE THE HEAT-DIFFUSION EQUATION

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Abstract. We study different initial and boundary value problems for the one-dimensional heat-diffusion equation, with the purpose of establishing conditions on initial and boundary data that ensure the solution is, during a certain interval of time, bounded by two predetermined constants. When the constants represent phase-change temperatures of a material medium, the desired conditions can be physically interpreted by saying that the medium does not undergo a phase-change during a certain lapse of time; i.e., the heat-conduction model will preserve its validity in the meantime. The tools employed to tackle this kind of problem consist in reducing the initial and boundary value problems to a Volterra integral equation. For these equations there exist simple methods to estimate their solutions. We improve some results that appeared in [8].

1. Introduction and preliminaries. An essential part of the construction of mathematical models of a physical phenomenon is to specify their scope. Particularly, once a model of some phenomenon of evolution has been formulated, it is natural to attempt to determine its temporary range of validity. In the case of models of heat-conduction in material media, an important limitation of this range is imposed by the change of phase phenomena. Hence the necessity of modifying the model to include this characteristic occurs: free-boundaries and mushy regions could now appear. It is then desirable to establish conditions on initial and boundary data of the problem so that phase-changes of the medium during a given time cannot occur.

These theoretical considerations have an immediate practical counterpart: when we operate a thermal engine (think of a blast furnace or a nuclear reactor), it is necessary to fix conditions of operation that ensure that the temperature in certain parts of the engine never exceeds the fusion point of the constitutive material. More often, we wish to operate in a range of temperatures for which the process is, in some sense, optimal. On the other hand, in diffusion processes, we may wish that an

Received April 19, 1993.

1991 *Mathematics Subject Classification.* Primary 35K05, 35B50.

Key words and phrases. Linear models of heat conduction or mass diffusion, Volterra integral equations, Laplace transformation, maximum principles for the heat-diffusion equation.

intervening substance maintain its chemical concentration among a preestablished range of values.

In paper [8] the authors, paraphrasing the terminology employed for the porous-media equation, call "waiting-time" t^* the time that must elapse before the material undergoes a phase-change. Using methods that mainly consist of comparisons—carried out by means of the maximum principles for the heat equation—between the solution of the problem in question with the exact solution of a similar problem, the authors determine necessary or sufficient conditions on initial and boundary data so as to have one of the three possibilities: $t^* = 0$, $0 < t^* < +\infty$, or $t^* = +\infty$. Initial papers related to this subject are [6] and [7], where an explicit solution for a problem in a semi-infinite slab is considered for which conditions on data to obtain an instantaneous change of phase (i.e., $t^* = 0$) are established.

In the present paper we develop a general method to treat some problems like the above mentioned. Our method consists in reducing the initial-boundary value problem to Volterra integral equations: for these equations we are able to estimate their solutions. We restrict ourselves to linear problems, but the same techniques are applicable to nonlinear ones (cf. [1]); for example, to models with nonlinear boundary conditions or problems that involve a quasi-linear parabolic equation.

In Secs. 2 and 3 we study various problems of heat conduction with boundary conditions of a mixed type. More precisely, in Sec. 2 we establish sufficient conditions on initial and boundary data so that $t^* > 0$ holds for the problem

$$\begin{aligned} &u_t - u_{xx} = 0, && 0 < x < 1, 0 < t, \\ &u(x, 0) = \theta_0(x), && 0 < x < 1, \\ &u(0, t) = b(t), && 0 < t, \\ &u_x(1, t) = -q(t), && 0 < t, \end{aligned} \tag{1.1}$$

where θ_0 is a continuous function for $0 \leq x \leq 1$, and b, q are piecewise-continuous functions for $t \geq 0$. As is well known, the problem (1.1) models the phenomenon of heat conduction in a homogeneous slab (after normalization of the thermal coefficients of the constitutive material, likewise the length of the slab) with an initial temperature distribution given by $\theta_0(x)$, and prescribed temperature $b(t)$ and flux $q(t)$ in its end faces.

In Sec. 3 we generalize the results obtained in the preceding section to include the case in which on one face of the slab a convective condition is imposed. Specifically, we study the problem

$$\begin{aligned} &u_t - u_{xx} = 0, && 0 < x < 1, 0 < t, \\ &u(x, 0) = \theta_0(x), && 0 < x < 1, \\ &u_x(0, t) + \alpha(t)u(0, t) = q(t), && 0 < t, \\ &u(1, t) = b(t), && 0 < t, \end{aligned} \tag{1.2}$$

where θ_0 is a continuous function on $[0, 1]$, $\alpha(t)$ is continuous and $q(t)$, $b(t)$ are piecewise-continuous for $t > 0$.

Finally, in Sec. 4 we enter upon the following diffusion problem:

$$\begin{array}{ll} u_t - u_{xx} = 0, & 0 < x < +\infty, 0 < t, \\ u(x, 0) = \theta_0(x), & 0 < x < +\infty, \\ u_t(0, t) + \alpha(t)u_x(0, t) + \beta(t)u(0, t) = g(t), & 0 < t, \\ |u(x, t)| \le c_1 e^{c_2 x^2}, & 0 < x < +\infty, 0 < t \ (c_1, c_2 > 0) \end{array} \tag{1.3}$$

with α, $\beta \in \mathscr{C}^0(0, +\infty)$, g piecewise-continuous for $t > 0$, and $\theta_0 \in \mathscr{C}^2[0, +\infty)$, $|\theta_0(x)| \le c_1 e^{c_2 x^2}$, $x \ge 0$. The condition $u_t(0, t) + \alpha(t)u_x(0, t) + \beta(t)u(0, t) = g(t)$ here models some kind of chemical reaction in the boundary, with the term $\alpha(t)u_x(0, t)$ representing diffusive transport of material through this boundary.

For problem (1.3) we determine sufficient conditions on data θ_0, α, β, and g so that, for $0 \le t \le t_0$, the chemical concentration $u(x, t)$ varies among the bounds u_1 and u_2, $u_1 < u_2$, in a certain place $x_1 \le x \le x_2$ of the reactor.

Throughout this paper, we suppose T_ξ, T_ν $(T_\xi < T_\nu)$ respectively are the melting and boiling temperatures of the constitutive material which, initially, stands in the phase determined by $T_\xi < \theta_0 < T_\nu$ (liquid phase). With $K(x, t)$ we denote the one-dimensional heat kernel; i.e.,

$$K(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)},$$

while $G(x, \xi, t) = K(x-\xi, t) - K(x+\xi, t)$ and $N(x, \xi, t) = K(x-\xi, t) + K(x+\xi, t)$ will respectively indicate the Green and Neumann functions for $x > 0$.

Volterra integral equations like

$$u(t) = f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} u(s)\, ds, \quad 0 \le t < +\infty, \tag{1.4}$$

where $0 \le \gamma < 1$, f is continuous on $[0, +\infty)$, and k is continuous for $0 \le s \le t \le +\infty$, will be common in the development of the following sections. For this reason, it is convenient to state here the following theorem relative to the existence and uniqueness of its solution.

THEOREM 1.1. Under the hypothesis above noted, the integral equation (1.4) admits a unique solution $u(t)$, continuous in $[0, +\infty)$, which can be expressed in the form

$$u(t) = f(t) + \int_0^t \Gamma(t, s) f(s)\, ds, \quad 0 \le t < +\infty, \tag{1.5}$$

where the (resolvent) kernel $\Gamma(t, s)$ is given by

$$\Gamma(t, s) = \sum_{i=1}^{\infty} \kappa^{(i)}(t, s),$$

with

$$\kappa^{(1)}(t, s) = \frac{k(t, s)}{(t-s)^\gamma}; \qquad \kappa^{(i)}(t, s) = \int_s^t \kappa^{(1)}(t, r)\kappa^{(i-1)}(r, s)\, dr, \quad 0 \le s \le t, \ i > 1.$$

□

A proof of the classical Theorem 1.1 can be found in [3] or [10]. As regards Eq. (1.4), we are particularly interested in establishing conditions on data f and k so that the solution $u(x, t)$ keeps its value between the constants T_ξ and T_ν during a certain lapse of time. To this purpose, we state a useful consequence of Theorem 1.1.

COROLLARY 1.2. Let M be a nonnegative constant. Then, the solution of the equation

$$u(t) = 1 + \int_0^t \frac{M}{(t-s)^\gamma} u(s)\, ds, \quad 0 \le t < +\infty \tag{1.6}$$

verifies $u(t) \ge 1$ for all $t \ge 0$.

Proof. From Theorem 1.1 we obtain $u(t) = 1 + \int_0^t \Gamma(t, s)\, ds$, where the resolvent kernel $\Gamma(t, s)$ verifies $\Gamma(t, s) \ge 0$ for $0 \le s \le t$. □

When the kernel k of Eq. (1.4) does not change sign, we have the following important result which provides bounds for the solution of Eq. (1.4).

THEOREM 1.3. Let f be a continuous function on $[0, t_0]$, and let k be a constant-sign continuous function for $0 \le s \le t \le t_0$. If v and w are two continuous functions on $[0, t_0]$ that satisfy, when $k \ge 0$, the inequalities

$$\begin{aligned} v(t) &< f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} v(s)\, ds, \quad 0 \le t \le t_0, \\ w(t) &> f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} w(s)\, ds, \quad 0 \le t \le t_0; \end{aligned} \tag{1.7}$$

or, when $k \le 0$, the following:

$$\begin{aligned} v(t) &< f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} w(s)\, ds, \quad 0 \le t \le t_0, \\ w(t) &> f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} v(s)\, ds, \quad 0 \le t \le t_0; \end{aligned} \tag{1.8}$$

then, the solution u of Eq. (1.4) satisfies

$$v(t) < u(t) < w(t), \quad 0 \le t \le t_0. \tag{1.9}$$

Furthermore, if the inequalities (1.7) (or (1.8)) hold with a non-strict sign, the same occurs with the inequalities in (1.9).

Proof. We only prove the theorem for $k \ge 0$, because the remaining case is identical. The ideas involved can be found in the treatise [9] and also in [5].

Assuming that the inequalities (1.7) hold, by setting $t = 0$ we obtain

$$v(0) < u(0) = f(0) < w(0).$$

Now, if we suppose that (1.9) does not hold in $(0, t_0]$, there will be a first point $t_1 \in (0, t_0]$ such that

$$v(t) < u(t) < w(t), \quad 0 \le t < t_1,$$

but, for example, $v(t_1) = u(t_1)$. Since $k \ge 0$, we would then have

$$v(t_1) = u(t_1) = f(t_1) + \int_0^{t_1} \frac{k(t_1, s)}{(t_1-s)^\gamma} u(s)\, ds \ge f(t_1) + \int_0^{t_1} \frac{k(t_1, s)}{(t_1-s)^\gamma} v(s)\, ds > v(t_1),$$

which would be an absurdity. A similar absurdity is obtained by assuming that $w(t_1) = u(t_1)$, and so (1.9) must be true.

Next, assume that the inequalities (1.7) hold in a non-strict way. In this case, we will show that (1.9) also holds non-strictly. To this end, let $p(t)$ be the solution of Eq. (1.6) with $M = \sup_{0\le s\le t\le t_0} k(t, s)$. From Corollary 1.2, $p(t) > 0$ for $0 \le t \le t_0$ results, and therefore, for each $\delta > 0$, we can write

$$\begin{aligned} v(t) - \delta p(t) &\le f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} v(s)\, ds - \delta p(t) \\ &< f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} (v(s) - \delta p(t))\, ds, \quad 0 \le t \le t_0, \\ w(t) + \delta p(t) &\ge f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} w(s)\, ds + \delta p(t) \\ &> f(t) + \int_0^t \frac{k(t, s)}{(t-s)^\gamma} (w(s) + \delta p(t))\, ds, \quad 0 \le t \le t_0. \end{aligned}$$

Because of what has just been proved, for $0 \le t \le t_0$ we have

$$v(t) - \delta p(t) < u(t) < w(t) + \delta p(t).$$

By performing $\delta \downarrow 0$, these last inequalities become

$$v(t) \le u(t) \le w(t), \quad 0 \le t \le t_0,$$

which completes the proof. □

Assume for a moment that the kernel $k(t, s)$ of (1.4) is nonpositive, and let C_1 and C_2 be two real constants with $C_1 < C_2$. Under these circumstances, the conditions (1.8) in Theorem 1.3 become

$$\begin{aligned} C_1 &< f(t) + C_2 \int_0^t \frac{k(t, s)}{(t-s)^\gamma}\, ds, \quad 0 \le t \le t_0, \\ C_2 &> f(t) + C_1 \int_0^t \frac{k(t, s)}{(t-s)^\gamma}\, ds, \quad 0 \le t \le t_0; \end{aligned}$$

or, by getting together these inequalities,

$$C_1 - C_2 \int_0^t \frac{k(t, s)}{(t-s)^\gamma}\, ds < f(t) < C_2 - C_1 \int_0^t \frac{k(t, s)}{(t-s)^\gamma}\, ds, \quad 0 \le t \le t_0. \tag{1.10}$$

Thus, when the kernel $k(t, s)$ is nonpositive and the inequalities (1.10) hold we are in a position to conclude (from Theorem 1.3) that the solution $u(t)$ of the Volterra integral equation (1.4) verifies

$$C_1 < u(t) < C_2, \quad 0 \le t \le t_0.$$

This restricted form of Theorem 1.3 is the one we will repeatedly use in the following sections.

2. A case of conduction with flux-temperature boundary data. The solution of problem (1.1) can be represented (cf. [2]) in the form

$$u(x, t) = v(x, t) - 2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau)\phi_1(\tau)\, d\tau + 2\int_0^t K(x-1, t-\tau)\phi_2(\tau)\, d\tau, \tag{2.1}$$

where

$$v(x, t) = \int_{-\infty}^{+\infty} K(x - \xi, t)\theta(\xi)\, d\xi, \tag{2.2}$$

where, in turn, θ is a bounded continuous extension to $\mathbb{R}$ of θ_0. In (2.1), ϕ_1 and ϕ_2 are piecewise-continuous solutions of the Volterra integral equations

$$\begin{aligned} b(t) &= v(0, t) + \phi_1(t) + 2\int_0^t K(-1, t-\tau)\phi_2(\tau)\, d\tau, \\ q(t) &= -v_s(1, t) + 2\int_0^t \frac{\partial^2 K}{\partial x^2}(1, t-\tau)\phi_1(\tau)\, d\tau - \phi_2(t), \quad t > 0. \end{aligned} \tag{2.3}$$

If, for the purpose of maintaining the constitutive material of the slab in a determined phase, we impose the restrictions

$$T_\xi < \theta_0(x) < T_\nu, \quad 0 \le x \le 1, \tag{2.4}$$

$$T_\xi < b(t) < T_\nu, \quad 0 \le t \le t_0, \tag{2.5}$$

and we control the temperature at the right face of the slab so that

$$T_\xi < u(1, t) < T_\nu, \quad 0 \le t \le t_0, \tag{2.6}$$

then, the weak maximum principle for the heat equation (cf. [2, 4]) will enable us to conclude that

$$T_\xi < u(x, t) < T_\nu, \quad 0 \le x \le 1, 0 \le t \le t_0.$$

Now, from (2.1) it follows that

$$u(1, t) = v(1, t) - 2\int_0^t \frac{\partial K}{\partial x}(1, t-\tau)\phi_1(\tau)\, d\tau - 2\int_0^t K(0, t-\tau)\phi_2(\tau)\, d\tau,$$

where ϕ_1, ϕ_2 satisfy (2.3). Our immediate objective will be to derive a Volterra integral equation for $u(1, t)$ from the previous equations. For this, it will be useful to use a more compact notation: as is customary, we indicate the convolution of the functions f and g by $f * g$. If, for the sake of brevity, we define $u(t) = u(1, t)$, Eqs. (2.1) and (2.3) can be expressed in the form

$$u(t) = v(1, t) - 2\left(\frac{\partial K}{\partial x}(1, \cdot) * \phi_1\right)(t) + 2(K(0, \cdot) * \phi_2)(t), \tag{2.7}$$

$$b(t) = v(0, t) + \phi_1(t) + 2(K(-1, \cdot) * \phi_2)(t), \tag{2.8}$$

$$q(t) = -v_x(1, t) + 2\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * \phi_1\right)(t) - \phi_2(t), \quad t > 0. \tag{2.9}$$

By substituting the value of ϕ_1 in (2.7) which derives from (2.8), we obtain

$$\begin{aligned} u(t) &= v(1, t) - 2\left(\frac{\partial K}{\partial x}(1, \cdot) * (b - v(0, \cdot))\right)(t) \\ &\quad + 2\left(\left(K(0, \cdot) + 2\left(\frac{\partial K}{\partial x}(1, \cdot) * K(-1, \cdot)\right)\right) * \phi_2\right)(t). \end{aligned} \tag{2.10}$$

On the other hand, from (2.8) and (2.9) we deduce the following equation for ϕ_2:

$$\begin{aligned}\phi_2(t) = -(q(t) + v_x(1, t)) &+ 2\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * (b - v(0, \cdot))\right)(t) \\ &- 4\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot) * \phi_2\right)(t).\end{aligned} \tag{2.11}$$

By taking convolutions of both sides of (2.10) with the kernel $4(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot))$ and remembering that $*$ is an associative and commutative operation, we obtain

$$\begin{aligned}4&\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot) * u\right)(t) \\ &= 4\left(\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot)\right) * \left(v(1, t) - 2\left(\frac{\partial K}{\partial x}(1, \cdot) * (b - v(0, \cdot))\right)\right)\right)(t) \\ &\quad + 2\left(\left(K(0, \cdot) + 2\left(\frac{\partial K}{\partial x}(1, \cdot) * K(-1, \cdot)\right)\right)\right. \\ &\qquad\qquad \left. * \left(4\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot)\right) * \phi_2\right)\right)(t),\end{aligned}$$

or, by using (2.11),

$$\begin{aligned}4&\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot) * u\right)(t) \\ &= 4\left(\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot)\right) * \left(v(1, \cdot) - 2\left(\frac{\partial K}{\partial x}(1, \cdot) * (b - v(0, \cdot))\right)\right)\right)(t) \\ &\quad + 2\left(\left(K(0, \cdot) + 2\left(\frac{\partial K}{\partial x}(1, \cdot) * K(-1, \cdot)\right)\right)\right. \\ &\qquad\qquad \left. * \left(-\phi_2 - (q + v_x(1, \cdot)) + 2\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * (b - v(0, \cdot))\right)\right)\right)(t) \\ &= 4\left(\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * K(-1, \cdot)\right) * \left(v(1, \cdot) - 2\left(\frac{\partial K}{\partial x}(1, \cdot) * (b - v(0, \cdot))\right)\right)\right)(t) \\ &\quad + 2\left(\left(K(0, \cdot) + 2\left(\frac{\partial K}{\partial x}(1, \cdot) * K(-1, \cdot)\right)\right)\right. \\ &\qquad\qquad \left. * \left(-(q + v_x(1, \cdot)) + 2\left(\frac{\partial^2 K}{\partial x^2}(1, \cdot) * (b - v(0, \cdot))\right)\right)\right)(t) \\ &\quad - 2\left(\left(K(0, \cdot) + 2\left(\frac{\partial K}{\partial x}(1, \cdot) * K(-1, \cdot)\right)\right) * \phi_2\right)(t).\end{aligned} \tag{2.12}$$

Finally, by replacing the last term of the right-hand side of (2.12) by its value deduced from (2.10), we arrive at the desired equation for u:

$$u(t) = F(t) + (k * u)(t), \quad t > 0, \tag{2.13}$$

where, for $t > 0$, we have written

$$k(t) = -4\left(\frac{\partial^2 K}{\partial x^2}(1,\cdot) * K(-1,\cdot)\right)(t); \tag{2.14}$$

$$\begin{aligned} F(t) = v(1,t) &+ 4\left(\frac{\partial^2 K}{\partial x^2}(1,\cdot) * K(-1,\cdot) * v(1,\cdot)\right)(t) \\ &+ 2\left(\left(2\frac{\partial^2 K}{\partial x^2}(1,\cdot) * K(0,\cdot) - \frac{\partial K}{\partial x}(1,\cdot)\right) * (b - v(0,\cdot))\right)(t) \\ &- 2\left(\left(2\frac{\partial K}{\partial x}(1,\cdot) * K(-1,\cdot) + K(0,\cdot)\right) * (q + v_x(1,\cdot))\right)(t). \end{aligned} \tag{2.15}$$

With the help of the Laplace transformation, we obtain

$$\begin{aligned} -k(t) &= 4\left(\frac{\partial^2 K}{\partial x^2}(1,\cdot) * K(-1,\cdot)\right)(t) = \frac{1}{\sqrt{\pi}} t^{-3/2} e^{-1/t}; \\ k_1(t) &= 2\left(2\frac{\partial^2 K}{\partial x^2}(1,\cdot) * K(0,\cdot) - \frac{\partial K}{\partial x}(1,\cdot)\right)(t) = \frac{1}{\sqrt{\pi}} t^{-3/2} e^{-1/(4t)}; \\ k_2(t) &= 2\left(2\frac{\partial K}{\partial x}(1,\cdot) * K(-1,\cdot) + K(0,\cdot)\right)(t) = \frac{1}{\sqrt{\pi}} t^{-1/2}(1 - e^{-1/t}). \end{aligned}$$

Since $k(t) \le 0$ for each $t \ge 0$ (we define $k(0) = 0$), an application of Theorem 1.3 leads us to the following theorem.

Theorem 2.1. If, as well as conditions (2.4) and (2.5), the data of problem (1.1) satisfies

$$T_\xi - T_\nu \int_0^t k(s)\,ds < F(t) < T_\nu - T_\xi \int_0^t k(s)\,ds, \quad 0 \le t \le t_0; \tag{2.16}$$

then the solution u verifies

$$T_\xi < u(x,t) < T_\nu, \quad 0 \le x \le 1,\ 0 \le t \le t_0, \tag{2.17}$$

i.e., the model of heat conduction preserves its validity for $0 \le t \le t_0$. When the inequalities in (2.4), (2.5), and (2.16) are non-strict, the inequalities in (2.17) also become non-strict.

From Eqs. (2.2), (2.14), and (2.15) we see that the inequalities (2.16) express that if $F(t)$, the "forcing function" of the Volterra integral equation (2.13)—in practice, a kind of balance of the boundary data $\theta_0(x)$, $b(t)$, and $q(t)$ of problem (1.1)—is conveniently bounded, then the temperature $u(x,t)$ is correspondingly bounded by (2.17).

Proof. As a consequence of the weak maximum principle and the conditions (2.4) and (2.5), it is sufficient to prove that $u(1,t) = u(t)$ verifies the inequalities (2.6). In the foregoing discussion we conclude that $u(t)$ satisfies the Volterra integral equation (2.13–2.15). The kernel $k(t)$ of this equation is negative and then, by using the condition (2.16), we can apply Theorem 1.3 to obtain the desired result. □

In the sequel we assume that $T_\xi = 0$. The results that follow show how to use Theorem 2.1 to obtain estimates of the waiting-time in some simple instances of the problem thus far considered.

COROLLARY 2.2. If the data of problem (1.1) is

$$\begin{aligned} &\text{i)} \quad q(t) = \frac{q_0}{\sqrt{\pi t}}, \quad t > 0, \\ &\text{ii)} \quad \theta_0(x) = \beta_0, \quad 0 \le x \le 1, \\ &\text{iii)} \quad b(t) = \beta_0, \quad t > 0, \end{aligned}$$

with $\beta_0 \ge q_0 > 0$, then the solution u verifies

$$0 \le u(x, t) \le 2\beta_0, \quad 0 < x < 1, t > 0.$$

Proof. By assuming $T_\xi = 0$ and $T_s = 2\beta_0$ in the condition (2.16) of Theorem 2.1, we see that it will be enough to prove that

$$-2\beta_0 \int_0^t k(s)\, ds \le F(t) \le 2\beta_0, \quad t \ge 0, \tag{2.18}$$

where $F(t)$ is expressed by (2.15).

By means of linear changes of variable in the integrals, we obtain

$$v(0, t) = v(1, t) = \beta_0 \int_{-\infty}^{+\infty} K(\xi, t)\, d\xi = \beta_0; \quad v_x(1, t) = \beta_0 \int_{-\infty}^{+\infty} \frac{\partial K}{\partial x}(\xi, t)\, d\xi = 0.$$

Furthermore, by using the Laplace transformation we arrive at

$$(k_2 * q)(t) = q_0 \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right),$$

and

$$\int_0^t k(s)\, ds = -\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) = \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) - 1;$$

then (2.18) can be written in the form

$$2\beta_0 \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)\right) \le 2\beta_0 - (q_0 + \beta_0) \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) \le 2\beta_0. \tag{2.19}$$

By taking into account that $\beta_0 > q_0$, and remembering that $0 = \operatorname{erf}(0) \le \operatorname{erf}(\tau) \le \operatorname{erf}(+\infty) = 1$, it is not difficult to realize that (2.19) holds for all $t \ge 0$. □

COROLLARY 2.3. Assume that, for $\sigma > 0$, the data of problem (1.1) satisfies the following conditions:

$$\begin{aligned} &\text{i)} \quad 0 < q(t) \le q_0, \quad 0 < t < \sigma; \\ &\text{ii)} \quad \theta_0(x) \ge \beta_0 > 0, \quad 0 \le x \le 1; \\ &\text{iii)} \quad b(t) \ge \beta_0, \quad 0 < t < \sigma. \end{aligned}$$

Then, if $\beta_0 \ge 2q_0$ the solution u satisfies

$$u(x, t) \ge 0, \quad 0 \le x \le 1, 0 < t \le \sigma, \tag{2.20}$$

while if $\beta_0 < 2q_0$,

$$u(x,t) \geq 0, \quad 0 \leq x \leq 1, 0 < t \leq \min\{\sigma, t_0\} \tag{2.21}$$

holds, where

$$t_0 = \phi^{-1}\left(\frac{\beta_0}{2q_0} - 1\right),$$

$$\phi(x) = \sqrt{\frac{x}{\pi}}(1 - e^{-1/x}) - \operatorname{erf}\left(\frac{1}{\sqrt{x}}\right), \quad x > 0.$$

Proof. The weak maximum principle and the theorem on the sign of the conormal derivative at a point where the maximum is attained (cf. Theorem 3, Chap. 3 of [4]) show that problem (1.1) with data verifying the restrictions i), ii), and iii) admits a solution $u(x,t)$ bounded below, for $0 < t < \sigma$, by the solution of the same problem (1.1) with data $q(t) = q_0$, $\theta_0(x) = \beta_0$, and $b(t) = \beta_0$. After some computations that involve the use of the Laplace transformation, the function $F(t)$ of Theorem 3.2 corresponding to the latter problem can be written as follows:

$$F(t) = 2(\beta_0 - q_0) + (2q_0 - \beta_0)\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) - 2q_0\sqrt{\frac{t}{\pi}}(1 - e^{-1/t}).$$

Moreover, as we said in the proof of the previous corollary, we have

$$\int_0^t k(s)\,ds = -\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right),$$

and so, if $\beta_0 \geq 2q_0$, it is enough to prove that

$$2(\beta_0 - q_0)\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) \leq F(t) \leq 2(\beta_0 - q_0), \quad 0 \leq t, \tag{2.22}$$

while, if $\beta_0 < 2q_0$, that

$$\beta_0 \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) \leq F(t) \leq \beta_0, \quad 0 \leq t \leq t_0. \tag{2.23}$$

To this end, we observe that the function

$$\phi(t) = \sqrt{\frac{t}{\pi}}(1 - e^{-1/t}) - \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)$$

verifies

$$\phi'(t) = \frac{1}{2\sqrt{\pi t}}(1 - e^{-1/t}) > 0, \quad t > 0;$$

thus, for each $t \geq 0$, we must have

$$-1 = \phi(0^+) \leq \phi(t) < \phi(+\infty) = 0.$$

Firstly, we will study (2.22). Since

$$F(t) = 2(\beta_0 - q_0) - \beta_0 \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) - 2q_0\phi(t),$$

and as we have supposed $\beta_0 \geq 2q_0$, for $t \geq 0$ we obtain

$$\begin{aligned} 2(\beta_0 - q_0) - F(t) &= \beta_0 \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) + 2q_0\phi(t) \\ &\geq 2q_0\left(\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) + \phi(t)\right) \\ &= 2q_0\sqrt{\frac{t}{\pi}}(1 - e^{-1/t}) \geq 0. \end{aligned}$$

This proves the last inequality in (2.22); for the first one we have, for each $t \geq 0$,

$$F(t) - 2(\beta_0 - q_0)\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) = (\beta_0 - 2q_0)\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) - 2q_0\phi(t) \geq -2q_0\phi(t) \geq 0.$$

As regards (2.23), we note above all that we can write

$$F(t) = \beta_0 + \beta_0 \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) - 2q_0(1 + \phi(t)).$$

Since $\beta_0 < 2q_0$, for all $t \geq 0$ we have

$$\begin{aligned} \beta_0 - F(t) &= 2q_0(1 + \phi(t)) - \beta_0 \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) \\ &\geq 2q_0\left(1 + \phi(t) - \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)\right) \\ &= 2q_0\sqrt{\frac{t}{\pi}}(1 - e^{-1/t}) \geq 0, \end{aligned}$$

and for $0 \leq t \leq t_0$,

$$F(t) - \beta_0 \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) = \beta_0 - 2q_0(1 + \phi(t)) \geq 0.$$

This completes the proof. □

REMARK 2.4. Corollary 2.3 can be favorably compared with Theorem 1 of [8] wherein, under the supplementary hypothesis $\theta_0' \leq 0$, $b(t) \geq 0$, the authors obtain

$$u(x, t) \geq 0, \quad 0 \leq x \leq 1, 0 \leq t \leq \min\left\{\sigma, \frac{\pi}{4}\left(\frac{\beta_0}{q^0}\right)^2\right\}.$$

In fact, the regularity assumptions on data done in Corollary 2.3 are weaker than ones in the above theorem. Furthermore, the estimates of waiting time furnished in Corollary 2.3 are more accurate. In fact, if for $t > 0$ we set

$$\psi(t) = \sqrt{\frac{t}{\pi}} - 1,$$

it follows that

$$(\psi - \phi)'(t) = \frac{e^{-1/t}}{2\sqrt{\pi t}} > 0, \quad t > 0,$$

and therefore

$$\psi(t) - \phi(t) > \psi(0^+) - \phi(0^+) = -1 - (-1) = 0, \quad t > 0.$$

In this way

$$\psi^{-1}(t) - \phi^{-1}(t) < 0, \quad t > -1,$$

and, as a particular case,

$$\frac{\pi}{4}\left(\frac{\beta_0}{q_0}\right)^2 = \psi^{-1}\left(\frac{\beta_0}{2q_0} - 1\right) < \phi^{-1}\left(\frac{\beta_0}{2q_0} - 1\right).$$

REMARK 2.5. The question of the limitations of the method employed to estimate waiting times are perhaps worth examining. With the purpose of briefly discussing these limitations, we consider the problem (1.1) with data $q(t) = q_0$, $\theta_0(x) = \beta_0$, $b(t) = \beta_0$ which has been treated in the proof of Corollary 2.3. It is physically reasonable, and by starting from the maximum principles it can be shown, that the solution $u(x, t)$ of this problem will verify

$$u(x, t) \leq \beta_0, \quad 0 \leq x \leq 1, 0 < t. \tag{2.24}$$

Apparently, the method allows us to prove (2.24) only when $\beta_0 < 2q_0$, while if $\beta_0 \geq 2q_0$ we obtain

$$u(x, t) \leq 2(\beta_0 - q_0), \quad 0 \leq x \leq 1, 0 < t,$$

a rougher bound since $2(\beta_0 - q_0) \geq \beta_0$ in this case. The same is true for the problem (1.1) with the same data considered in Corollary 2.2.

On the other hand, if we realize that the steady solution of the problem is $u_\infty(x) = -q_0x + \beta_0$, $0 \leq x \leq 1$, it is easy to see that, when $\beta_0 \geq q_0$,

$$u(x, t) \geq 0, \quad 0 \leq x \leq 1, 0 < t, \tag{2.25}$$

holds. From a glance at the proof of Corollary 2.3, we conclude that our method provides (2.25) only when the condition $\beta_0 \geq 2q_0$ is fulfilled.

3. A case of conduction with convective boundary data. In this section we extend the method and results of the previous section to comprise a boundary condition of convective type like that in problem (1.2). We start writing the solution for problem (1.2) as (cf. [2])

$$u(x, t) = v(x, t) - 2\int_0^t K(x, t-\tau)\phi_1(\tau)\,d\tau + 2\int_0^t \frac{\partial K}{\partial x}(x-1, t-\tau)\phi_2(\tau)\,d\tau, \tag{3.1}$$

where, as in the previous section, we have written

$$v(x, t) = \int_{-\infty}^{+\infty} K(x-\xi, t)\theta(\xi)\,d\xi, \tag{3.2}$$

where, in turn, θ is a bounded continuous extension to $\mathbb{R}$ of θ_0. In (3.1), ϕ_1 and ϕ_2 are piecewise-continuous solutions of the Volterra integral equations

$$\begin{aligned} b(t) &= v(1, t) + \phi_2(t) - 2\int_0^t K(1, t-\tau)\phi_1(\tau)\,d\tau, \quad t > 0, \\ q(t) &= v_x(0, t) + 2\int_0^t \left(\frac{\partial^2 K}{\partial x^2}(-1, t-\tau) + \alpha(t)\frac{\partial K}{\partial x}(-1, t-\tau)\right)\phi_2(\tau)\,d\tau + \phi_1(t) \\ &\quad + \alpha(t)v(0, t) - 2\alpha(t)\int_0^t K(0, t-\tau)\phi_1(\tau)\,d\tau, \quad t > 0. \end{aligned} \tag{3.3}$$

Using the method employed in Sec. 2 for problem (1.1), we impose the restrictions

$$T_\xi < \theta_0(x) < T_\nu, \quad 0 \le x \le 1; \tag{3.4}$$

$$T_\xi < b(t) < T_\nu, \quad 0 \le t \le t_0; \tag{3.5}$$

and we maintain the temperature in the left face so that

$$T_\xi < u(1, t) < T_\nu, \quad 0 \le t \le t_0. \tag{3.6}$$

In this way, the weak maximum principle will allow us to conclude

$$T_\xi < u(x, t) < T_\nu, \quad 0 \le x \le 1, 0 \le t \le t_0.$$

The underlying idea is the same as that of Sec. 2: by defining for $t > 0$, $u(t) = u(0, t)$, from (3.1) we obtain

$$u(t) = v(0, t) - 2\int_0^t K(0, t-\tau)\phi_1(\tau)\,d\tau + 2\int_0^t \frac{\partial K}{\partial x}(-1, t-\tau)\phi_2(\tau)\,d\tau. \tag{3.7}$$

By using the operation $*$, we now express Eqs. (3.3) and (3.7):

$$u(t) = v(0, t) - 2(K(0, \cdot) * \phi_1)(t) + 2\left(\frac{\partial K}{\partial x}(-1, t-\tau) * \phi_2\right)(t), \tag{3.8}$$

$$b(t) = v(1, t) + \phi_2(t) - 2(K(1, \cdot) * \phi_1)(t), \tag{3.9}$$

$$\begin{aligned} q(t) = v_x(0, t) &+ 2\left(\left(\frac{\partial^2 K}{\partial x^2}(-1, \cdot) + \alpha(t)\frac{\partial K}{\partial x}(-1, \cdot)\right) * \phi_2\right)(t) + \phi_1(t) \\ &+ \alpha(t)v(0, t) - 2\alpha(t)(K(0, \cdot) * \phi_1)(t). \end{aligned} \tag{3.10}$$

By carrying out operations on (3.8), (3.9), and (3.10) much the same as those used in the preceding section for Eqs. (2.7), (2.8), and (2.9), we arrive at the following Volterra integral equation for u:

$$u(t) = F(t) + (k(t, \cdot) * u)(t), \quad t > 0, \tag{3.11}$$

where we have written

$$k(t, \tau) = 2\left(\alpha(t)K(0, \tau) - 2\left(\left(\frac{\partial^2 K}{\partial x^2}(-1, \cdot) + \alpha(t)\frac{\partial K}{\partial x}(-1, \cdot)\right) * K(1, \cdot)\right)(t)\right), \tag{3.12}$$

$$\begin{aligned} F(t) = v(0, t) &- 2\left(\left(\alpha(t)K(0, \cdot) - 2\left(\left(\frac{\partial^2 K}{\partial x^2}(-1, \cdot) + \alpha(t)\frac{\partial K}{\partial x}(-1, \cdot)\right) * K(1, \cdot)\right)\right) * v(0, \cdot)\right)(t) \\ &+ 2\left(\left(2\frac{\partial^2 K}{\partial x^2}(-1, \cdot) * K(0, \cdot) + \frac{\partial K}{\partial x}(-1, \cdot)\right) * (b - v(1, \cdot))\right)(t) \\ &- 2\left(\left(K(0, \cdot) - 2\frac{\partial K}{\partial x}(-1, \cdot) * K(1, \cdot)\right) * (q - (v_x(0, \cdot) + \alpha(t)v(0, \cdot)))\right)(t). \end{aligned} \tag{3.13}$$

By means of the properties of the heat kernel and some results obtained in Sec. 2, we easily find

$$k(t,\tau)=\frac{1}{\sqrt{\pi}}\tau^{-1/2}(\alpha(t)(1-e^{-1/\tau})-\tau^{-1}e^{-1/\tau}),$$

$$k_1(t)=2\left(2\frac{\partial^2 K}{\partial x^2}(-1,\cdot)*K(0,\cdot)+\frac{\partial K}{\partial x}(-1,\cdot)\right)(t)=\frac{1}{\sqrt{\pi}}t^{-3/2}e^{-1/(4t)},$$

and

$$k_2(t)=2\left(K(0,\cdot)-2\frac{\partial K}{\partial x}(-1,\cdot)*K(1,\cdot)\right)(t)=\frac{1}{\sqrt{\pi}}t^{-1/2}(1-e^{-1/t}).$$

Before establishing the major result of this section, we wish to point out that the kernel $k(t,\tau)$ of the integral equation (3.11) has an integrable singularity for $t=\tau$ unless $\alpha(t)\equiv 0$. In the latter case, we obtain Eq. (2.13) except for the symmetry $x\mapsto 1-x$. We now prove the

THEOREM 3.1. If, apart from the conditions (3.4) and (3.5), the data of problem (1.2) satisfies either

i) $\alpha(t)\le 0$, $0\le t\le t_0$, and

$$T_\xi - T_\nu\int_0^t k(t,t-s)\,ds < F(t) < T_\nu - T_\xi\int_0^t k(t,t-s)\,ds,\quad 0\le t\le t_0, \tag{3.14}$$

or

ii) $\alpha(t)\ge \dfrac{1}{t(e^{1/t}-1)}$, $0\le t\le t_0$, and

$$T_\xi\left(1-\int_0^t k(t,t-s)\,ds\right) < F(t) < T_\nu\left(1-\int_0^t k(t,t-s)\,ds\right),\ 0\le t\le t_0, \tag{3.15}$$

then the model of heat conduction preserves its validity during the lapse of time $0\le t\le t_0$. In other words, a waiting time $t^*>t_0$ exists.

Note that case ii) in the statement of the theorem above is physically meaningless within the classical theory of heat conduction. We include it because it can be of interest in other areas.

Proof. It is enough to observe that the condition $\alpha(t)\ge 1/(t(e^{1/t}-1))$ $[\alpha(t)\le 0]$ is equivalent to the positivity [negativity] of the kernel $k(t,t-\tau)$ for $0\le\tau<t\le t_0$. The rest proceeds along the lines of the proof of Theorem 2.1. □

To derive some useful consequences of Theorem 3.1 we observe that

$$\frac{1}{\sqrt{\pi}}\int_0^t s^{3/2}e^{1/s}\,ds=\frac{2}{\sqrt{\pi}}\int_{1/\sqrt{t}}^{+\infty}e^{-s^2}\,ds=\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)$$

and

$$\begin{aligned}
\int_0^t k_2(s)\,ds &= \frac{1}{\sqrt{\pi}}\int_0^t s^{-1/2}(1-e^{-1/s})\,ds = \frac{1}{\sqrt{\pi}}\left(\int_0^t s^{-1/2}\,ds - \int_0^t s^{-1/2}e^{-1/s}\,ds\right) \\
&= \frac{1}{\sqrt{\pi}}\left(2t^{1/2} - \int_{1/t}^{+\infty} s^{-3/2}e^{-s}\,ds\right) \\
&= \frac{1}{\sqrt{\pi}}\left(2t^{1/2} + 2\left(-t^{1/2}e^{-1/t} + \int_{1/t}^{+\infty} s^{-1/2}e^{-s}\,ds\right)\right) \\
&= \frac{1}{\sqrt{\pi}}\left(2t^{1/2}(1-e^{-1/t}) + 2\int_{1/t}^{+\infty} s^{-1/2}e^{-s}\,ds\right) \\
&= \frac{2}{\sqrt{\pi}}t^{1/2}(1-e^{-1/t}) + \frac{4}{\sqrt{\pi}}\int_{1/\sqrt{t}}^{+\infty} e^{-s^2}\,ds \\
&= \frac{2}{\sqrt{\pi}}t^{1/2}(1-e^{-1/t}) + 2\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right),
\end{aligned}$$

From the above equations it follows that

$$\begin{aligned}
\int_0^t k(t, t-s)\,ds = \int_0^t k(t, s)\,ds &= \alpha(t)\int_0^t k_2(s)\,ds - \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right) \\
&= \alpha(t)\left(\frac{2}{\sqrt{\pi}}t^{1/2}(1-e^{-1/t}) + 2\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)\right) - \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right).
\end{aligned}$$

Next we assume that $T_\xi = 0$. Thus, we obtain the following

COROLLARY 3.2. Assume the data of problem (1.2) satisfies

$$\begin{aligned}
&\text{i)} \quad q(t) = q_0 > 0, \quad t \geq 0; \\
&\text{ii)} \quad \theta_0(x) = \beta_0 > 0, \quad 0 \leq x \leq 1; \\
&\text{iii)} \quad b(t) = \beta_0, \quad t \geq 0; \\
&\text{iv)} \quad \alpha(t) = \alpha_0 < 0, \quad t \geq 0.
\end{aligned}$$

Then, if $\beta_0 \geq 2q_0$ the solution u verifies

$$0 \leq u(x, t) \leq 2(\beta_0 - q_0), \quad 0 < x < 1, 0 \leq t \leq t_0 = \phi^{-1}\left(\frac{-2\alpha_0(\beta_0 - q_0)}{(1+2\alpha_0)(\beta_0 - q_0) - \beta_0}\right); \tag{3.16}$$

while if $\beta_0 < 2q_0$,

$$0 \leq u(x, t) \leq \beta_0, \quad 0 < x < 1, \quad 0 \leq t \leq t_0 = \phi^{-1}\left(\frac{2q_0 - (1+2\alpha_0)\beta_0}{2(\alpha_0\beta_0 - q_0)}\right). \tag{3.17}$$

The function ϕ that appears in (3.16) and (3.17) is the same as that defined in Corollary 2.3, i.e.,

$$\phi(x) = \sqrt{\frac{x}{\pi}}(1 - e^{-1/x}) - \operatorname{erf}\left(\frac{1}{\sqrt{x}}\right), \quad x > 0.$$

Proof. The calculations here involved are very similar to those carried out in the proof of Corollary 2.3. For this reason, we shall restrict ourselves to proving (3.17).

With this objective, we suppose $\beta_0 \geq 2q_0$ and we set $T_\nu = 0$ and $T_\xi = 2(\beta_0 - q_0)$ in condition (3.14) of Theorem 3.1, which then becomes

$$-2(\beta_0 - q_0)\int_0^t k(t, s-t)\,ds \leq F(t) \leq 2(\beta_0 - q_0), \tag{3.18}$$

where $F(t)$ is defined by (3.13). By taking into account the restrictions imposed on the data, the preceding calculations allow us to write

$$\begin{aligned}\int_0^t k(t, t-s)\,ds &= \alpha_0 \int_0^t k_2(s)\,ds - \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)\\ &= \alpha_0\left(2\sqrt{\frac{t}{\pi}}(1-e^{-1/t}) + 2\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)\right) - \operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)\end{aligned}$$

and

$$\begin{aligned}F(t) &= 2\beta_0 - \beta_0 \operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) - q_0\int_0^t k_2(s)\,ds\\ &= 2(\beta_0 - q_0) + (2q_0 - \beta_0)\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) - 2q_0\sqrt{\frac{t}{\pi}}(1-e^{-1/t}).\end{aligned}$$

Note that the function $F(t)$ is the same as in Corollary 2.2 and therefore, the right inequality of (3.16) is true for all $t \geq 0$ as in this corollary. With regard to the left inequality, after the appropriate manipulations we can write it as follows:

$$4\alpha_0(\beta_0 - q_0) + (2(1+2\alpha_0)(\beta_0 - q_0) - 2\beta_0)\phi(t) + (\beta_0 - 2q_0)\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) \geq 0.$$

Since $\beta_0 \geq 2q_0$, we have

$$\begin{aligned}4\alpha_0(\beta_0 - q_0) &+ (2(1+2\alpha_0)(\beta_0 - q_0) - 2\beta_0)\phi(t) + (\beta_0 - 2q_0)\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)\\ &\geq 4\alpha_0(\beta_0 - q_0) + (2(1+2\alpha_0)(\beta_0 - q_0) - 2\beta_0)\phi(t) \geq 0.\end{aligned} \tag{3.19}$$

The last inequality in (3.19) holds for $0 \leq t \leq t_0$, with t_0 defined in (3.16). This completes the proof. □

COROLLARY 3.3. If the data of problem (1.2) satisfy the conditions

i) $q(t) = q_0 > 0, \quad t \geq 0;$
ii) $\theta_0(x) \geq \beta_0 > 0, \quad 0 \leq x \leq 1;$
iii) $b(t) \geq \beta_0, \quad t \geq 0;$
iv) $\alpha(t) = \alpha_0 < 0, \quad t \geq 0;$

then, if $\beta_0 \geq 2q_0$ the solution u verifies

$$u(x,t) \geq 0,\ 0 < x < 1, \quad 0 \leq t \leq t_0 = \phi^{-1}\left(-\frac{2\alpha_0(\beta_0 - q_0)}{(1+2\alpha_0)(\beta_0 - q_0) - \beta_0}\right);$$

while if $\beta_0 < 2q_0$,

$$u(x,t) \geq 0,\ 0 < x < 1, \quad 0 \leq t \leq t_1 = \phi^{-1}\left(\frac{2q_0 - (1+2\alpha_0)\beta_0}{2(\alpha_0\beta_0 - q_0)}\right).$$

Proof. The result follows as a direct application of Corollary 3.3 and the weak maximum principle. □

REMARK 3.4. Under the conditions of Corollary 3.3 and the additional hypothesis $\theta_0' \geq 0$, $b \geq 0$ in [8] (Theorem 6) a waiting time $t^* \geq t_2 = (\frac{1}{\alpha_0}\chi^{-1}(1 - \frac{\alpha_0\beta_0}{q_0}))^2$, with $\chi(x) = \exp(-x^2)/\operatorname{erfc}(x)$, $x > 0$, is shown to exist. By observing that, in the limit $\alpha_0 \downarrow 0$, the times t_0 and t_1 of Corollary 3.4 coincide with the corresponding times of Corollary 2.3, and taking into account that $\lim_{\alpha_0 \downarrow 0} t_2 = \frac{\pi}{4}(\beta_0/q^0)^2$, we deduce from Remark 2.4 that, at least for small α_0,

$$t_2 < \min\{t_0, t_1\}$$

must be satisfied. So, we can affirm that Corollary 3.3 improves Theorem 6 of [8], giving a more accurate estimate of the waiting time at the expense of a weaker hypothesis.

4. A diffusion problem with reaction in the boundary. We can express the unique bounded solution of problem (1.3) in the following form (cf. [2]):

$$u(x, t) = \int_0^\infty G(x, \xi, t)\theta_0(\xi)\, d\xi - 2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau)(\theta_0(0)+\phi(\tau))\, d\tau, \quad x > 0, t > 0, \tag{4.1}$$

where ϕ is the solution of the Volterra integral equation

$$\phi(t) = f(t) + \int_0^t (2\alpha(t)K(0, t-\tau) - \beta(t))\phi(\tau)\, d\tau, \quad t > 0, \tag{4.2}$$

with

$$f(t) = g(t) - \alpha(t)\int_0^\infty N(0, \xi, t)\theta_0'(\xi)\, d\xi - \theta_0(0)\beta(t). \tag{4.3}$$

By using the operation $*$, Eqs. (4.1) and (4.2) can be written as

$$u(x, t) = \int_0^\infty G(x, \xi, t)\theta_0(\xi)\, d\xi - 2\theta_0(0)\int_0^t \frac{\partial K}{\partial x}(x, \tau)\, d\tau - 2\left(\frac{\partial K}{\partial x}(x, \cdot) * \phi\right)(t), \quad x > 0, t > 0, \tag{4.4}$$

and

$$\phi(t) = f(t) + ((2\alpha(t)K(0, \cdot) - \beta(t)) * \phi)(t), \quad t > 0. \tag{4.5}$$

By making $k(t, \tau) = 2\alpha(t)K(0, \tau) - \beta(t)$, manipulations wholly analogous to those employed in the above sections lead to the following family of Volterra integral equations for u:

$$u(x, t) = F(x, t) + (k(t, \cdot) * u(x, \cdot))(t), \quad t > 0, x > 0, \tag{4.6}$$

where

$$\begin{aligned} F(x, t) = {} & \int_0^\infty G(x, \xi, t)\theta_0(\xi)\, d\xi - 2\theta_0(0)\int_0^t \frac{\partial K}{\partial x}(x, \tau)\, d\tau \\ & - \left(k(t, \cdot) * \left(\int_0^\infty G(x, \xi, \cdot)\theta_0(\xi)\, d\xi - 2\theta_0(0)\int_0^{\cdot} \frac{\partial K}{\partial x}(x, \tau)\, d\tau\right)\right)(t) \\ & - 2\left(\frac{\partial K}{\partial x}(x, \cdot) * \left(g - \alpha\int_0^\infty N(0, \xi, \cdot)\theta_0'(\xi)\, d\xi - \theta_0(0)\beta\right)\right)(t). \end{aligned} \tag{4.7}$$

The theorem proved subsequently states sufficient conditions on the data so that, for $0 \le t \le t_0$, the chemical concentration u verifies $u_1 \le u \le u_2$, $u_1 < u_2$, in a portion $x_1 \le x \le x_2$ of the reactor.

THEOREM 4.1. Let $x_i, u_i \in \mathbb{R}$, $i = 1, 2$, be such that $0 \le x_1 < x_2$, $0 \le u_1 < u_2$. Assume that the data of problem (1.3) verify the regularity and growth conditions stated in Sec. 1 and, for $t_0 > 0$,

i) $u_1 \le \theta_0(x) \le u_2, \quad x_1 \le x \le x_2$

holds. Moreover, suppose the conditions

ii) $\beta(t) \ge 0$ and $\alpha(t) \ge \sqrt{\pi t}\beta(t)$, $0 \le t \le t_0$
(or $\beta(t) \le 0$ and $\alpha(t) \ge 0$, $0 \le t \le t_0$);

iii) $u_1\left(1 - \int_0^t k(t, s)\,ds\right) \le F(x_i, t) \le u_2\left(1 - \int_0^t k(t, s)\,ds\right)$,
$0 \le t \le t_0$, $i = 1, 2$;

or

ii′) $\beta(t) \le 0$ and $\alpha(t) \le \sqrt{\pi t}\beta(t)$, $0 \le t \le t_0$
(or $\beta(t) \ge 0$ and $\alpha(t) \le 0$, $0 \le t \le t_0$);

iii′) $u_1 - u_2 \int_0^t k(t, s)\,ds \le F(x_i, t) \le u_2 - u_1 \int_0^t k(t, s)\,ds$, $0 \le t \le t_0$, $i = 1, 2$,

where $F(t, x)$ is given by (4.7). Under these assumptions, the solution u verifies

$$u_1 \le u(x, t) \le u_2, \quad x_1 \le x \le x_2, 0 \le t \le t_0. \tag{4.8}$$

If i) and iii) or iii′) hold with strict inequalities, then the same occurs with (4.8).

Proof. Because of condition i) and the weak maximum principle, to prove that the solution u of problem (1.3) satisfies (4.8), it is enough to show that

$$u_1 \le u(x_i, t) \le u_2, \quad 0 \le t \le t_0, i = 1, 2. \tag{4.9}$$

By realizing that $u(x_i, t)$ is the solution of the integral equation (4.6) with $x = x_i$, $i = 1, 2$, and that condition ii) [ii′)] ensures the positivity [negativity] of the kernel $k(t, \tau)$ of (4.6), we can apply Theorem 1.3 to condition iii) [iii′)] to obtain (4.9). □

Next we illustrate the use of Theorem 4.1 in the case of constant data. Concretely, suppose that $\theta_0(x) \equiv c_0$, $\alpha(t) \equiv \alpha_0$, $\beta(t) \equiv \beta_0$, $g(t) \equiv g_0$; then, we have the following corollary.

COROLLARY 4.2. Let $x_i, u_i \in \mathbb{R}$, $i = 1, 2$, be such that $0 \le x_1 < x_2$, $0 \le u_1 < u_2$, and suppose the data of problem (1.3) satisfies the following conditions:

i) $u_1 \le c_0 \le u_2$;
ii) $\beta_0 \ge 0$ and $\alpha_0 \le 0$.

If, for each $0 \le t \le t_0$ and $i = 1, 2$, the inequalities

$$u_1 - c_0 + (u_2 - c_0)\left(\frac{2}{\sqrt{\pi}}\alpha_0 t^{1/2} - \beta_0 t\right) \le (g_0 - \beta_0 c_0)\operatorname{erfc}\left(\frac{x_i}{2\sqrt{t}}\right) \le u_2 - c_0 - (c_0 - u_1)\left(\frac{2}{\sqrt{\pi}}\alpha_0 t^{1/2} - \beta_0 t\right)$$

hold, then

$$u_1 \le u(x, t) \le u_2, \quad x_1 \le x \le x_2, \quad 0 \le t \le t_0.$$

Proof. We will check that condition iii$'$) of Theorem 4.1 holds. For this we note that, by means of appropriate changes of variable in the integrals, we can obtain

$$\int_0^\infty G(x, \xi, t)\, d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) ; -2\int_0^t \frac{\partial K}{\partial x}(x, \tau)\, d\tau = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{4.10}$$

Moreover, we have

$$\int_0^t k(t, s)\, ds = \frac{2}{\sqrt{\pi}}\alpha_0 t^{1/2} - \beta_0 t. \tag{4.11}$$

from (4.10) and (4.11) it follows that (4.7) becomes

$$F(x, t) = c_0 - c_0\left(\frac{2}{\sqrt{\pi}}\alpha_0 t^{1/2} - \beta_0 t\right) + (g_0 - \beta_0 c_0)\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) ; \tag{4.12}$$

and therefore, condition iii$'$) of Theorem 4.1 can be written as

$$u_1 - c_0 + (u_2 - c_0)\left(\frac{2}{\sqrt{\pi}}\alpha_0 t^{1/2} - \beta_0 t\right) \le (g_0 - \beta_0 c_0)\operatorname{erfc}\left(\frac{x_i}{2\sqrt{t}}\right) \le u_2 - c_0 - (c_0 - u_1)\left(\frac{2}{\sqrt{\pi}}\alpha_0 t^{1/2} - \beta_0 t\right),$$

with $i = 1, 2$. These inequalities are exactly the same as those we assumed in the hypothesis, which provides the proof. □

Acknowledgment. This paper has been partially supported by the Project "Problemas de Frontera Libre de la Física Matemática", CONICET-UNR, Rosario, Argentina.

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