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A VARIATION OF THE INTRINSIC MULTIPLE-SCALE HARMONIC BALANCE METHOD

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Abstract. A variation of the intrinsic multiple-scale harmonic balance method is introduced by combining the intrinsic multiple-scale harmonic balance method with the ideas introduced to modify the method of multiple-scales. The combined method has the advantage of having the desirable characteristics of both techniques. This is demonstrated by solving the Duffing equation.

Introduction. The classical harmonic balance technique is a relatively simple method for investigating nonlinear oscillations. However, it has serious shortcomings and therefore is justifiably discarded by some authors [1]. The shortcoming of the classical harmonic balance method was later eliminated by Atadan and Huseyin to give consistent results [2, 3]. More recently, a combination of the intrinsic harmonic balancing technique and the multiple-time-scales (derivative expansion) method was introduced by Huseyin and Lin [4].

In a separate development, a modification of the derivative expansion method was presented by Veronis [5]. This was done by replacing the natural frequency of the given equation by a series representation of frequency.

In this paper, a combination of the intrinsic multiple-scale harmonic balance method introduced by Huseyin and Lin [4], and the modification of the derivative expansion method given by Veronis [5] are presented. The two methods [4, 5] are combined in such a way that the proposed technique has the desirable characteristics of both methods.

Firstly, it is conceptually simple and does not involve secular terms as in the method presented by Veronis [5] (see, e.g., Eq. (14) in [5]). The absence of secular terms follows from the intrinsic method of harmonic balancing [2, 3] which is the foundation of the intrinsic multiple-scale harmonic balancing technique [4] used here. Secondly, the frequency-parameter relationship obtained is in the form introduced by Veronis [5]. However, this is achieved by introducing a time-warping transformation rather than expanding the natural frequency as in Veronis [5]. For comparison reasons the method is introduced by solving the Duffing equation as in Veronis [5].

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Derivative expansion method and its modification. First, consider the following linear harmonic oscillator with a small damping:

$$\frac{d^2}{dt^2}u + 2\varepsilon\frac{d}{dt}u + \omega_0^2 u = 0. \tag{1}$$

The (closed-form) solution of the linear problem (1) can be expressed as

$$u = \tfrac{1}{2}ae^{-\varepsilon t}e^{i\sqrt{\omega_0^2-\varepsilon^2 t}+\beta} + \text{c.c.} \tag{2}$$

where c.c. indicates complex conjugate. Since the developments in this paper involve the derivative expansion method, we obtain the solution of the same problem via the derivative expansion method:

$$u = \tfrac{1}{2}ae^{-\varepsilon t}e^{i\omega_0(1-\varepsilon^2/2\omega_0^2)t} + \text{c.c.} \tag{3}$$

where the number of scales is limited to T_0, T_1, and T_2 as in references [5, 6].

In the case of the modified derivative expansion method introduced by Veronis [5], however, the solution given by (2) is obtained when the time-scales are limited to T_0, T_1, and T_2 as was done to derive (3). Veronis achieves this by replacing ω_0 in Eq. (1) with a power series of the form

$$\omega_0^2 = \omega^2 + \varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \cdots . \tag{4}$$

Consider now the Duffing equation

$$\frac{d^2}{dt^2}u + \omega_0^2 u = \varepsilon\alpha u^3 = 0 \tag{5}$$

where the odd nonlinearities represent a soft or hard spring depending on the sign of the coefficient of u^3. Solution of the Duffing equation using the derivative expansion method can be obtained as [6]:

$$\begin{aligned} u =& \frac{1}{2}ae^{i(\omega t+\chi)} + \varepsilon\alpha a^3\frac{1}{64\omega_0^2}\left[1 - \varepsilon\alpha a^2\frac{21}{32\omega_0^2} + \cdots\right]e^{i3(\omega t+\chi)} \\ &+ \varepsilon^2\alpha^2 a^5\frac{1}{2048\omega_0^4}e^{i5(\omega t+\chi)} + \text{c.c.} + \cdots \end{aligned} \tag{6}$$

where the frequency is

$$\omega = \omega_0 + \varepsilon\alpha a^2\frac{3}{8\omega_0} - \varepsilon^2\alpha^2 a^4\frac{15}{256\omega_0^3} + \cdots . \tag{7}$$

Following Veronis [5], however, the frequency is obtained as

$$\omega^2 = \omega_0^2 + \varepsilon\alpha a^2\frac{3}{4}\left(1 + \varepsilon\alpha a^2\frac{1}{32\omega_0^2}\right) + \cdots \tag{8}$$

where the expression (4) has been substituted into the Duffing equation given by (5).

A variation of the intrinsic multiple-scale harmonic balance method. In this section we will introduce a variation of the intrinsic multiple-scale harmonic balance method presented by Huseyin and Lin [4]. To start the development we seek an approximate solution to the Duffing equation given by (5) as

$$u = u(t; \varepsilon), \tag{9}$$

which, warping the time via

$$\tau = \omega(\varepsilon) t, \tag{10}$$

becomes

$$u = u(\tau; \varepsilon). \tag{11}$$

Following (10) and (11), the Duffing equation can be rewritten as

$$\omega^2(\varepsilon)\Delta^2 u(\tau; \varepsilon) + \omega_0^2 u(\tau; \varepsilon) + \varepsilon\alpha u^3 = 0 \tag{12}$$

where Δ is the derivative operator indicating differentiation with respect to τ. Note that in (12) ω_0 is not replaced by $\omega(\varepsilon)$ as was done by Veronis [5], but the frequency of solution is brought into the equation explicitly via the transformation (10).

Following the derivative expansion method, the scaled times are written as

$$\tau_n = \frac{1}{n!}\varepsilon^n t, \quad n = 0, 1, 2, \dots \tag{13}$$

where the coefficient of ε^n is chosen as $(1/n!)$ for reasons that will become clear shortly.

Introducing the time scaling into Eq. (12) and letting $\Omega(\varepsilon) = \omega^2(\varepsilon)$ results in

$$\Omega(\varepsilon)\Delta^2(\varepsilon) u(\tau_0, \tau_1, \tau_2, \dots; \varepsilon) + \omega_0^2 u(\tau_0, \tau_1, \tau_2, \dots; \varepsilon) + \varepsilon\alpha u^3(\tau_0, \tau_1, \tau_2, \dots; \varepsilon) = 0. \tag{14}$$

The Taylor expansions of the frequency $\Omega(\varepsilon)$ and the derivative operator $\Delta(\varepsilon)$ are expressed as

$$\Omega(\varepsilon) = \omega_0^2 + \sum_{n=1}^{N} \varepsilon^n \frac{1}{n!}\Omega_n \tag{15}$$

and

$$\Delta(\varepsilon) = \sum_{n=0}^{N} \varepsilon^n \frac{1}{n!}\Delta_n \tag{16}$$

where

$$\Delta_n \equiv \frac{\partial}{\partial \tau_n}. \tag{17}$$

Following (16) one has

$$\Delta^2(\varepsilon) = \Delta_0^2 + \varepsilon 2\Delta_0\Delta_1 + \varepsilon^2[\Delta_0\Delta_2 + \Delta_1^2] + \cdots. \tag{18}$$

It should be pointed out here that, rather than substituting the Taylor series expansion of the derivative operator into (14) as was done in [4], the perturbation equations are generated by differentiating (14) with respect to ε. Note that differentiation is the reason why $(1/n!)$ is introduced in the time scales given by (13).

Following Huseyin [7–9], here the perturbation equations are generated by differentiation. The significance of generating perturbation equations by differentiation is that the number of perturbation equations need not be decided in advance. The standard perturbation techniques, however, do not have this advantage because they involve the substitution of a Taylor series expansion into the given differential equation.

Following this, the solution of Eq. (14) is assumed as

$$\begin{aligned} u &= u(\tau_0, \tau_1, \tau_2; \varepsilon) \\ &= \sum_{m=-M}^{+M} a_{(m)}(\tau_1, \tau_2, ; \varepsilon) e^{im\tau_0} \end{aligned} \tag{19}$$

where, in order to compare the results obtained in this paper to those of Veronis [5] and Nayfeh [6], the number of time scales is limited to τ_0, τ_1, and τ_2 (i.e., $n = 0, 1, 2$). The coefficient $a_{(m)}$ in series (19) is complex and given in the exponential form

$$a_{(m)}(\tau_1, \tau_2; \varepsilon) = d_{(m)}(\tau_1, \tau_2; \varepsilon) e^{im\phi(\tau_1, \tau_2; \varepsilon)} \tag{20}$$

with $d_{(m)} \in \mathfrak{R}$ and $\phi_{(m)} \in \mathfrak{R}$.

Evaluating (14) at $\varepsilon = 0$ results in the zero-order perturbation equation

$$(\Delta_0^2 + 1) u_0 = 0. \tag{21}$$

Differentiating (14) with respect to ε as many times as required and evaluating at $\varepsilon = 0$ yields the hierarchy of remaining perturbation equations, namely,

$$(\Delta_0^2 + 1) u_1 = -[\omega_0^{-2} \Omega_1 \Delta_0 + 2\Delta_1] \Delta_0 u_0 - \omega_0^{-2} \alpha u_0^3, \tag{22}$$

$$\begin{aligned} (\Delta_0^2 + 1) u_2 = &- [\omega_0^{-2} \Omega_2 \Delta_0 + 2\Delta_2] \Delta_0 u_0 - 2[2\omega_0^{-2} \Omega_1 \Delta_0 + \Delta_1] \Delta_1 u_0 \\ &- 2[\omega_0^{-2} \Omega_1 \Delta_0 + 2\Delta_1] \Delta_0 u_1 - 3\omega_0^{-2} \alpha u_0^2 u_1, \end{aligned} \tag{23}$$

$$\vdots$$

From the perturbation Eq. (21) one obtains

$$(1 - m^2) a_{0,(m)}(\tau_1, \tau_2) = 0, \tag{24}$$

which yields

$$a_{0,(m)}(\tau_1, \tau_2) = 0 \quad \text{for } m \neq \pm 1. \tag{25}$$

Following (25) and $a_{0,(-1)}(\tau_1, \tau_2) = a^*_{0,(+1)}(\tau_1, \tau_2)$, the solution of (21) can be written as

$$u_0(\tau_0, \tau_1, \tau_2) = a_{0,(+1)}(\tau_1, \tau_2) e^{i\tau_0} + \text{c.c.}, \tag{26}$$

where the superscript "$*$" indicates the complex conjugate.

Substituting (26) into the r.h.s. of the first-order perturbation equation given by (22) results in

$$\sum_{m=-M}^{+M}(1-m^2)a_{1,(m)}(\tau_1,\tau_2)e^{im\tau_0} = Me^{i\tau_0} + Ne^{i3\tau_0} + \text{c.c.} \tag{27}$$

where

$$M = \omega_0^{-2}\Omega_1 a_{0,(+1)} - i2a_{0,(+1)\tau_1} - 3\alpha\omega_0^{-2}a^*_{0,(+1)}a^2_{0,(+1)}, \tag{28}$$

$$N = -\alpha\omega_0^{-2}a^3_{0,(+1)}, \tag{29}$$

and

$$a_{0,(+1)} = a_{0,(+1)}(\tau_1,\tau_2). \tag{30}$$

Coefficients of $e^{im\tau}$ $(m \neq \pm 1, \pm 3)$ and $e^{i\tau}, e^{i3\tau}$ can be obtained from (27) as

$$a_{1,(m)} = 0 \quad \text{for } m \neq \pm 1, \pm 3, \tag{31}$$

$$\omega_0^{-2}\Omega_1 a_{0,(+1)}(\tau_1,\tau_2) - i2a_{0,(+1)\tau_1}(\tau_1,\tau_2) - 3\alpha\omega_0^{-2}a^*_{0,(+1)}(\tau_1,\tau_2)a^2_{0,(+1)}(\tau_1,\tau_2) = 0, \tag{32}$$

and

$$8a_{1,(3)}(\tau_1,\tau_2) = \alpha\omega_0^{-2}a^3_{0,(+1)}(\tau_1,\tau_2), \tag{33}$$

respectively.

Substituting

$$a_{0,(+1)} = a_{0,(+1)}(\tau_1,\tau_2) = \tfrac{1}{2}a_0(\tau_1,\tau_2)e^{i\phi_0(\tau_1,\tau_2)} \tag{34}$$

into (32), and separating real and imaginary parts leads to

$$\frac{\partial}{\partial\tau_1}a_0(\tau_1,\tau_2) = 0. \tag{35}$$

Assuming that $a_0(\tau_1,\tau_2) \neq 0$,

$$\frac{\partial}{\partial\tau_1}\phi_0(\tau_1,\tau_2) = -\tfrac{1}{2}\omega_0^{-2}\Omega_1 + \tfrac{3}{8}\alpha\omega_0^{-2}a_0^2(\tau_1,\tau_2). \tag{36}$$

The steady-state solutions of (35) and (36) result in

$$a_0 = a_0(\tau_2), \tag{37}$$

$$\phi_0 = \phi_0(\tau_2), \tag{38}$$

$$\Omega_1 = \tfrac{3}{4}\alpha a_0^2(\tau_2), \tag{39}$$

following which (33) and (34) take the form

$$a_{0,(+1)} = a_{0,(+1)}(\tau_2) = \tfrac{1}{2}a_0(\tau_2)e^{i\phi_0(\tau_2)}, \tag{40}$$

$$a_{1,(+3)} = a_{1,(+3)}(\tau_2) = \alpha\tfrac{1}{64}\omega_0^{-2}a_0^3(\tau_2)e^{i3\phi_0(\tau_2)}, \tag{41}$$

where $a_{0,(-m)}(\tau_2) = a^*_{0,(+m)}(\tau_2) \quad (m = 1, 3)$.

From the results of the foregoing analysis the second-order perturbation equation (23) becomes

$$\sum_{m=-M}^{+M} (1 - m^2) a_{2,(m)}(\tau_1, \tau_2) e^{im\tau_0} = P e^{i\tau_0} + Q e^{i3\tau_0} + R e^{i3\tau_0} + \text{c.c.} \tag{42}$$

where

$$\begin{aligned} P = &- 6\alpha[a_{0,(-1)}^2 a_{1,(+3)} + 2a_{0,(-1)} a_{0,(+1)} a_{1,(+1)} + a_{0,(+1)}^2 a_{1,(-3)}] \\ &+ 2\Omega_1 a_{1,(+1)} + \Omega_2 a_{0,(+1)} - i2\omega_0^2 a_{0,(+1)\tau_2}. \end{aligned} \tag{43}$$

The remaining coefficients Q and R are not important for the order of approximation considered.

For ($m = 1$) Eq. (42) yields

$$P = 0. \tag{44}$$

Substituting (40), (41), (44), and $a_{1,(+1)} = 0$ into (43), and separating the real and imaginary parts result in

$$\frac{\partial}{\partial \tau_2} a_0 = 0, \tag{45}$$

$$\frac{\partial}{\partial \tau_0} \phi_0 = \omega_0^{-3} \left[\frac{3}{128} \omega_0^{-2} \alpha^2 a_0^4 - \frac{1}{2} \Omega_2 \right]. \tag{46}$$

The steady-state solutions of (45) and (46) lead to

$$a_0 = \text{constant}, \qquad \Omega_2 = \frac{3}{64} \omega_0^{-2} \alpha^2 a_0^4. \tag{47}$$

Substituting the derivatives (39) and (47) into (15) and noting that $\Omega(\varepsilon) = \omega^2(\varepsilon)$ lead to

$$\omega(\varepsilon) = \sqrt{\omega_0^2 + \varepsilon \frac{3}{4} \alpha a_0^2 + \varepsilon^2 \frac{3}{128} \omega_0^{-2} \alpha^2 a_0^4 + \cdots}, \tag{48}$$

which is the same as the frequency amplitude relationship obtained by Veronis (see Eq. (8)) [5].

For comparison reasons, one can expand (48) into a binomial series and show that it is in full agreement with the relationship given by Nayfeh [6] (see Eq. (7)).

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