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ON THE DUAL BOUNDARY CONDITIONS OF THE FOURTH DERIVATIVE OPERATORS

By

PENG-FEI YAO

Institute of Systems Science, Academia Sinica, Beijing, China

Abstract. A dual relationship between the boundary condition of the fourth derivative operator

$$Au = \frac{\partial^4 u}{\partial x^4}$$

is established and examined in this paper. Some dual properties determined by the dual boundary conditions are also considered.

1. Introduction. Let $l > 0$ and A be the fourth derivative operator on $L^2(0, l)$ with the domain

$$D(A) \subset H^4(0, l), \qquad Au = u^{(4)}(\cdot). \tag{1.1}$$

The inner product in $L^2(0, l)$ is

$$(u, v) = \int_0^l u(x)\overline{v(x)}\, dx, \quad \forall u, v \in L^2(0, l).$$

Assume $D(A)$ is such that A is selfadjoint. Integration by parts yields

$$\begin{aligned}(Au, v) &= \int_0^l u^{(4)}(x)\overline{v(x)}\, dx \\ &= (u'''(x)\overline{v(x)} - u''(x)\overline{v'(x)})|_0^l + \int_0^l u''(x)\overline{v''(x)}\, dx, \quad \forall u, v \in D(A).\end{aligned}$$

Specializing to (Au, u), A is seen to be nonnegative if

$$(u'''(x)\overline{u(x)} - u''(x)\overline{u'(x)})|_0^l \geq 0, \quad \forall u \in D(A). \tag{1.2}$$

In particular, A is said to have SDE (strictly distributed energy [1]) boundary conditions if

$$(u'''(x)\overline{u(x)} - u''(x)\overline{u'(x)})|_0^l = 0, \quad \forall u \in D(A). \tag{1.3}$$

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When (1.3) holds, the system energy becomes

$$\frac{1}{2}\int_0^l \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2\right) dx.$$

Now we introduce one of the significant contributions made by D. L. Russell [1]. Consider the real Hilbert space $L^2(0,l)$ and a nonnegative selfadjoint operator A, as described above, with $D(A) \subset L^2(0,l)$. If A is nonnegative it is well known that the spectrum of A consists of eigenvalues:

$$0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

with $\lim_{n\to\infty} \lambda_n = +\infty$, where the multiple eigenvalues are listed according to their algebraic multiplicities. There exists an orthonormal basis $\{\varphi_n\}$ of $L^2(0,l)$ such that

$$A\varphi_n = \lambda_n \varphi_n, \quad n = 1, 2, 3, \ldots .$$

Let k be the nonnegative integer such that

$$\lambda_j = 0 \quad \text{if } j \le k; \qquad \lambda_j > 0 \quad \text{if } j \ge k+1.$$

If A is positive, then $k = 0$. Set

$$\psi_n(\cdot) = -\frac{1}{\sqrt{\lambda_n}}\varphi''(\cdot), \quad n \ge k+1; \tag{1.4}$$

$$B(u,v) = u'''(\cdot)v(\cdot) - u''(\cdot)v'(\cdot) - u(\cdot)v'''(\cdot) + u'(\cdot)v''(\cdot), \quad \forall u, v \in D(A).$$

D. L. Russell [1] drew the following conclusion (see [1, pp. 761–765]): *if*

$$B(u,v)|_{x=0} = B(u,v)|_{x=l} = 0, \quad \forall u, v \in D(A), \tag{1.5}$$

then $\{\psi_n | n \ge k+1\}$ *is an orthogonal set in* $L^2(0,l)$. In addition, since $\varphi_n^{(4)}(\cdot) = \lambda_n \varphi_n(\cdot)$, we have that $(\varphi''(\cdot))^{(4)} = \lambda_n \varphi''(\cdot)$, for any $n \ge k+1$. Therefore, D. L. Russell [1] pointed out that ψ_n is the eigenfunction of some symmetric operator $\hat{A}$ corresponding to the eigenvalue λ_n and called $\hat{A}$ the operator with the dual boundary conditions. Unfortunately, in general, $\{\psi_n | n \ge k+1\}$ is not an orthogonal set in $L^2(0,l)$ under the assumptions (1.5) only. A counterexample is given in [2, Proposition 3.2].

This paper is a correction and development of the above idea. In Sec. 2, the boundary characterizations are given, which assure that A should be selfadjoint, or nonnegative selfadjoint, or should have SDE boundary conditions. In Sec. 3, the dual boundary condition operator $\hat{A}$ is defined, by which one can easily obtain the boundary conditions for $\hat{A}$ from those for A. Furthermore, some dual properties between A and $\hat{A}$ are studied, such as nonnegativity, SDE boundary conditions, eigenvalues, eigenfunctions, nonnegative square roots, etc.

2. Boundary characterizations. Set

$$N = \{\varphi \mid \varphi \in H^8(0,l), \varphi^{(8)} + \varphi = 0\}.$$

In this paper N will be called the "boundary" space of the fourth derivative operator defined by (1.1) partially because of the following Proposition 2.1. Define the operator A_0 on $L^2(0,l)$ by

$$D(A_0) = H_0^4(0,l), \qquad A_0 u = u^{(4)}(\cdot), \tag{2.1}$$

where

$$H_0^4(0,l) = \{u \mid u, u', u'', u^{(3)}, u^{(4)} \in L^2(0,l), u^{(j)}(0) = u^{(j)}(l) = 0,\ j = 0,1,2,3\}.$$

It is easily checked that A_0 is a closed symmetric operator on $L^2(0,l)$ and the adjoint operator A_0^* of A_0 is

$$D(A_0^*) = H^4(0,l), \qquad A_0^* u = u^{(4)}(\cdot). \tag{2.2}$$

Set

$$N_+ = \{\varphi \mid \varphi \in H^4(0,l), \varphi^{(4)} = i\varphi\},$$
$$N_- = \{\varphi \mid \varphi \in H^4(0,l), \varphi^{(4)} = -i\varphi\},$$

where i is the imaginary unit. The following proposition shows that N is in fact the boundary space of $H^4(0,l)$.

PROPOSITION 2.1. (i) $H^4(0,l) = H_0^4(0,l) \oplus N_+ \oplus N_-$,
(ii) $N = N_+ \oplus N_-$,
where $\oplus$ is the direct sum of sets.

Proof. From (2.1) and (2.2) and by the first formula of von Neumann [3], we obtain (i). It is easy to verify that $N_+ \subset N$, $N_- \subset N$, $\dim N_+ = \dim N_- = 4$, and $\dim N = 8$. Thus (i) implies (ii). $\square$

For any $\varphi \in H^8(0,l)$, denote

$$\tilde{\varphi}(x) = (\varphi^{(7)}(x), p^{(6)}(x), \varphi^{(5)}(x), \varphi^{(4)}(x), \varphi'''(x), \varphi''(x), \varphi'(x), \varphi(x))^{\tau}.$$

A direct calculation yields

PROPOSITION 2.2. $\varphi \in N$ if and only if $\tilde{\varphi}(x)$ is a solution of the initial value problem

$$\begin{cases} \tilde{\varphi}'(x) = E\tilde{\varphi}(x), & 0 < x \le l, \\ \tilde{\varphi}(0) = \alpha, \end{cases} \tag{2.3}$$

where

$$E = \begin{pmatrix} 0 & -1 \\ I_7 & 0 \end{pmatrix},$$

$\alpha \in \mathbb{C}^8$, I_7 is the 7×7 unit matrix, and $\mathbb{C}^8$ is the 8-dimensional complex Euclidean space.

Let $\varphi \in N$. Then by Proposition 2.2, we have

$$\tilde{\varphi}(x) = e^{Ex}\tilde{\varphi}(0), \quad 0 \le x \le l.$$

Set

$$e^{El} = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}, \tag{2.4}$$

where E_j are 4×4 matrices, for $j = 1,2,3,4$.

PROPOSITION 2.3. $\det E_3 \neq 0$.

Proof. Suppose $\det E_3 = 0$. Then there is $\beta \in \mathbb{C}^4$, $\beta \neq 0$, such that

$$E_3\beta = 0.$$

By Proposition 2.2, there is $\varphi \in N$ such that

$$\tilde{\varphi}(x) = e^{Ex}\begin{pmatrix}\beta\\0\end{pmatrix}, \quad 0 \leq x \leq l.$$

Since

$$\tilde{\varphi}(l) = \begin{pmatrix}E_1 & E_2\\E_3 & E_4\end{pmatrix}\begin{pmatrix}\beta\\0\end{pmatrix} = \begin{pmatrix}E_1\beta\\E_3\beta\end{pmatrix} = \begin{pmatrix}E_1\beta\\0\end{pmatrix},$$

it follows that

$$\varphi^{(j)}(0) = \varphi^{(j)}(l) = 0, \quad j = 0,1,2,3,$$

i.e., $\varphi \in H_0^4(0,l)$. From Proposition 2.1, we obtain $\varphi = 0$. Thus $\tilde{\varphi}(0) = (\beta^\tau, 0)^\tau = 0$, contradicting $\beta \neq 0$. □

PROPOSITION 2.4. Define the mapping $\partial : H^4(0,l) \to \mathbb{C}^8$ by

$$\partial\varphi = (\varphi'''(l), \varphi''(l), \varphi'(l), \varphi(l), \varphi'''(0), \varphi''(0), \varphi'(0), \varphi(0))^\tau,$$

$$\forall\varphi \in H^4(0,l).$$

Then $\partial : N \to \mathbb{C}^8$ is a linear isomorphism.

Proof. Let $\varphi \in N$ be such that $\partial\varphi = 0$. Then $\varphi \in H_0^4(0,l)$. From Proposition 2.1, it follows that $\varphi = 0$. Thus $\partial : N \to \mathbb{C}^8$ is injective. For any $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)^\tau \in \mathbb{C}^8$, by Proposition 2.2, there is $\varphi \in N$ such that

$$\tilde{\varphi}(x) = e^{Ex}\begin{pmatrix}E_3^{-1} & -E_3^{-1}E_4\\0 & I_4\end{pmatrix}\alpha, \quad \forall 0 \leq x \leq l. \tag{2.5}$$

Letting $x = 0$ in (2.5) yields

$$\varphi^{(j)}(0) = \alpha_{8-j}, \quad j = 0,1,2,3.$$

Letting $x = l$ yields

$$\varphi^{(j)}(l) = \alpha_{4-j}, \quad j = 0,1,2,3.$$

Thus $\partial\varphi = \alpha$. Then the mapping $\partial : N \to \mathbb{C}^8$ is surjective. The proof is complete. □

Let A be defined by (1.1) and A_0 by (2.1). Since $A \subset A_0^*$, we have that $A_0 = (A_0^*) \subset A^*$. If A is selfadjoint, i.e., $A = A^*$, then A is a selfadjoint extension of the closed symmetric operator A_0. By the second formula of von Neumann [3], there is an isometric, surjective, linear operator $\mathcal{V} : N_+ \to N_-$ such that

$$D(A) = \{u \mid u = u_0 + \varphi - \mathcal{V}\varphi, u_0 \in H_0^4(0,l), \varphi \in N_+\}, \tag{2.6}$$

$$Au = u_0^{(4)}(\cdot) + i\varphi(\cdot) + i(\mathcal{V}\varphi)(\cdot). \tag{2.7}$$

However, the calculation of $\mathcal{V} : N_+ \to N_-$ is in general difficult. As concerns the fourth derivative operator A, we now give a counterpart of the von Neumann theorem. Let $\phi_j \in H^4(0,l)$, for $j = 1,2,3,4$. Denote

$$\mathbb{C}_{\phi_{1,4}}(x) = \begin{pmatrix} \phi_1'''(x) & \phi_2'''(x) & \phi_3'''(x) & \phi_4'''(x) \\ \phi_1''(x) & \phi_2''(x) & \phi_3''(x) & \phi_4''(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \phi_4'(x) \\ \phi_1(x) & \phi_2(x) & \phi_3(x) & \phi_4(x) \end{pmatrix}, \quad 0 \le x \le l.$$

Set

$$\mathcal{B}_{\phi_{1,4}} = \overline{\mathbb{C}^\tau_{\phi_{1,4}}(l)} \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(l) - \overline{\mathbb{C}^\tau_{\phi_{1,4}}(0)} \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(0),$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

THEOREM 2.1. Let A be a fourth derivative operator defined by (1.1). Then A is selfadjoint if and only if there are linearly independent elements $\phi_1, \phi_2, \phi_3, \phi_4$ in N such that

(i)

$$\overline{\mathcal{B}}^\tau_{\phi_{1,4}} = \mathcal{B}_{\phi_{1,4}}; \tag{2.8}$$

(ii)

$$D(A) = \left\{ u \;\middle|\; u = u_0 + \sum_{j=1}^{4} \alpha_j \phi_j, u_0 \in H_0^4(0,l), \alpha_j \in \mathbb{C}, 1 \le j \le 4 \right\}. \tag{2.9}$$

Proof of necessity. Let A be selfadjoint. By the von Neumann theorem, there is a unique isometric, surjective, linear operator $\mathcal{V} : N_+ \to N_-$ such that (2.6) and (2.7) hold. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ be a basis of N_+. Set

$$\phi_j = \varphi_j - \mathcal{V}\varphi_j, \quad j = 1,2,3,4.$$

It follows from (2.6) that (ii) holds. To prove that $\phi_1, \phi_2, \phi_3, \phi_4$ are linearly independent, let $\sum_{j=1}^4 \alpha_j \phi_j = 0$. Letting $\varphi = \sum_{j=1}^4 \alpha_j \varphi_j$, we have

$$\varphi = \mathcal{V}\varphi.$$

Since $\varphi \in N_+$ and $\mathcal{V}\varphi \in N_-$, we have $\varphi = 0$ from Proposition 2.1 (ii). Thus $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

For any $\alpha, \beta \in \mathbb{C}^4$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\tau$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^\tau$. Set

$$u = \sum_{j=1}^4 \alpha_j \phi_j, \qquad v = \sum_{j=1}^4 \beta_j \phi_j.$$

Since $u, v \in D(A)$, the selfadjointness of A yields

$$
\begin{aligned}
0 &= (Au, v) - (u, Av) \\
&= (u'''(x)\overline{v(x)} - u''(x)\overline{v'(x)} - u(x)\overline{v'''(x)} + u'\overline{v''(x)})|_0^l \\
&= (\overline{v'''(x)}, \overline{v''(x)}, \overline{v'(x)}, \overline{v(x)}) \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u'''(x) \\ u''(x) \\ u'(x) \\ u(x) \end{pmatrix} \Bigg|_0^l \\
&= \overline{\beta^\tau} \left[\overline{\mathbb{C}^\tau_{\phi_{1,4}}(l)} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(l) - \overline{\mathbb{C}^\tau_{\phi_{1,4}}(0)} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(0) \right] \alpha = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \alpha, \beta \in \mathbb{C}^4.
\end{aligned}
$$

Thus,

$$
\overline{\mathbb{C}^\tau_{\phi_{1,4}}(l)} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(l) - \overline{\mathbb{C}^\tau_{\phi_{1,4}}}(0) \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(0) = 0. \tag{2.10}
$$

It follows that $\overline{\mathcal{B}^\tau_{\phi_{1,4}}} = \mathcal{B}_{\phi_{1,4}}$ since $\overline{J^\tau} = -J$.

Sufficiency. Let $\phi_1, \phi_2, \phi_3, \phi_4 \in N$ be linearly independent elements such that (2.8) and (2.9) hold. Then (2.10) holds. By Proposition 2.1(ii), ϕ_j have the unique direct sums

$$
\phi_j = \varphi_j - \psi_j, \quad \varphi_j \in N_+, \quad \psi_j \in N_-, \quad j = 1, 2, 3, 4.
$$

We will prove that $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are linearly independent. Suppose that $\sum_{j=1}^4 \alpha_j \varphi_j = 0$. Set $\phi = \sum_{j=1}^4 \alpha_j \phi_j$. Then $\phi = -\sum_{j=1}^4 \alpha_j \psi_j \in N_-$, i.e., $\phi^{(4)} = -i\phi$. From (2.10) and by integration by parts, it follows that

$$
\begin{aligned}
-2i(\phi, \phi) &= (\phi^{(4)}, \phi) - (\phi, \phi^{(4)}) \\
&= \overline{\alpha^\tau \mathbb{C}^\tau_{\phi_{1,4}}}(x) \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(x) \alpha|_0^l = 0,
\end{aligned}
$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\tau$, i.e., $\phi = \sum_{j=1}^4 \alpha_j \phi_j = 0$. Therefore $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

Define the operator $\mathcal{V} : N_+ \to N_-$ by

$$
\mathcal{V}\varphi = \sum_{j=1}^4 \alpha_j \psi_j, \quad \forall \varphi = \sum_{j=1}^4 \alpha_j \varphi_j \in N_+.
$$

From (2.9), (2.6) holds. To complete the proof of sufficiency it remains to show that $\mathcal{V} : N_+ \to N_-$ is isometric since $\dim N_+ = \dim N_- = 4$. For any $\varphi = \sum_{j=1}^4 \alpha_j \varphi_j \in N_+$, set $g = \varphi - \mathcal{V}\varphi$. From (2.10) it follows that

$$
\begin{aligned}
2i(\varphi, \varphi) - 2i(\mathcal{V}\varphi, \mathcal{V}\varphi) &= (i\varphi + i\mathcal{V}\varphi, \varphi - \mathcal{V}\varphi) - (\varphi - \mathcal{V}\varphi, i\varphi + i\mathcal{V}\varphi) \\
&= (g^{(4)}, g) - (g, g^{(4)}) \\
&= \overline{\alpha^\tau \mathbb{C}^\tau_{\phi_{1,4}}}(x) \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(x) \alpha|_0^l = 0.
\end{aligned}
$$

Thus the proof is complete. □

THEOREM 2.2. Let A be a selfadjoint operator defined by (1.1) and $\mathcal{B}_{\phi_{1,4}}$ the same as that in Theorem 2.1. Then A is nonnegative on $L^2(0,l)$ if and only if $\mathcal{B}_{\phi_{1,4}}$ is nonnegative on $\mathcal{C}^4$. In particular, A has SDE boundary conditions if and only if $\mathcal{B}_{\phi_{1,4}} = 0$.

Proof. For any $u = u_0 + \sum_{j=1}^4 \alpha_j \phi_j \in D(A)$, from (1.2) A is nonnegative if and only if

$$\begin{aligned}
&(u'''(x)\overline{u(x)} - u''(x)\overline{u'(x)})|_0^l \\
&= (\overline{u'''(x)}, \overline{u''(x)}, \overline{u'(x)}, \overline{u(x)}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u'''(x) \\ u''(x) \\ u'(x) \\ u(x) \end{pmatrix} \Bigg|_0^l \\
&= \overline{\alpha^\tau \mathcal{C}^\tau_{\phi_{1,4}}(x)} \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix} \mathcal{C}_{\phi_{1,4}}(x)\alpha \Bigg|_0^l \\
&= \overline{\alpha^\tau} \mathcal{B}_{\phi_{1,4}} \alpha \geq 0, \quad \forall \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\tau \in \mathcal{C}^4.
\end{aligned}$$

In particular, from (1.3) A has SDE boundary conditions if and only if

$$\begin{aligned}
&(u'''(x)\overline{u(x)} - u''(x)\overline{u'(x)})|_0^l \\
&= \overline{\alpha^\tau} \mathcal{B}_{\phi_{1,4}} \alpha = 0, \qquad \forall \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\tau \in \mathcal{C}^4.
\end{aligned} \tag{2.11}$$

From Theorem 2.1(i), (2.11) holds if and.only if $\mathcal{B}_{\phi_{1,4}} = 0$. Thus the proof is complete. □

3. Dual boundary condition operators. Set

$$\Gamma(u,v) = (u'''(x)\overline{v''(x)} - u''(x)\overline{v'''(x)} - u'(x)\overline{v(x)} + u(x)\overline{v'(x)})|_0^l, \quad \forall u, v \in H^4(0,l).$$

DEFINITION 3.1. Let A be a selfadjoint operator defined by (1.1). Define the operator $\hat{A}$ on $L^2(0,l)$ by

$$D(\hat{A}) = \{u \mid u \in H^4(0,l), \Gamma(u,v) = 0, \forall v \in D(A)\},$$
$$\hat{A}u = u^{(4)}(\cdot).$$

$\hat{A}$ is said to be the dual boundary condition operator of A.

EXAMPLE 3.1. Let us consider a beam with both ends fixed. The boundary conditions become

$$u(0) = u'(0) = 0, \qquad u(l) = u'(l) = 0.$$

It is easily checked that $u \in H^4(0,l)$ such that

$$\begin{aligned}
\Gamma(u,v) &= (u'''(x)\overline{v''(x)} - u''(x)\overline{v'''(x)} - u'(x)\overline{v(x)} + u(x)\overline{v'(x)})|_0^l \\
&= u'''(l)\overline{v''(l)} - u''(l)\overline{v'''(l)} - u'''(0)\overline{v''(0)} + u''(0)\overline{v'''(0)} \\
&= 0, \quad \forall v \in D(A),
\end{aligned}$$

if and only if

$$u''(0) = u'''(0) = 0, \qquad u''(l) = u'''(l) = 0.$$

Therefore $\hat{A}$ is the "beam" operator with both ends free.

EXAMPLE 3.2. Let us consider a cantilever beam with elastic forces applying at the free end, $x = l$. The boundary conditions become

$$u(0) = u'(0) = 0, \qquad u'''(l) - au(l) = 0, \qquad u''(l) + bu'(l) = 0,$$

where $a > 0$ and $b > 0$. $D(\hat{A})$ consists of $u \in H^4(0,l)$ such that

$$\begin{aligned}\Gamma(u,v) &= u'''(l)\overline{v''(l)} - u''(l)\overline{v'''(l)} - u'(l)\overline{v(l)} + u(l)\overline{v'(l)} - u'''(0)\overline{v''(0)} + u''(0)\overline{v'''(0)} \\ &= (-bu'''(l) + u(l)\overline{v'(l)} - (au''(l) + u'(l))\overline{v(l)} - u'''(0)\overline{v''(0)} + u''(0)\overline{v'''(0)} \\ &= 0, \quad \forall v \in D(A).\end{aligned} \tag{3.1}$$

It is easy to check that (3.1) holds if and only if $u \in H^4(0,l)$ such that

$$u''(0) = u'''(0) = 0, \qquad u'''(l) - \frac{1}{b}u(l) = 0, \qquad u''(l) + \frac{1}{a}u'(l) = 0. \tag{3.2}$$

Then $\hat{A}$ is the "beam" operator with the boundary conditions (3.2).

As concerns $\hat{A}$, we have the following

THEOREM 3.1. Let A be a selfadjoint operator defined by (1.1). Then

(i) the dual boundary condition operator $\hat{A}$ is selfadjoint;

(ii) $\hat{\hat{A}} = A$.

Proof. (i) Since A is selfadjoint, from Theorem 2.1, there are linearly independent elements $\phi_1, \phi_2, \phi_3, \phi_4$ in N such that (2.8) and (2.9) hold. Set

$$M = \{\varphi \mid \varphi \in N, \Gamma(\varphi, \phi_j) = 0, 1 \le j \le 4\}.$$

Then M is a linear subspace of N. From Proposition 2.1, it is easily checked that

$$D(\hat{A}) = \{u \mid u = u_0 + \varphi, u_0 \in H_0^4(0,l), \varphi \in M\}. \tag{3.3}$$

Now we will prove that $\dim M = 4$. Let $\varphi \in H^4(0,l)$. Since

$$\begin{aligned}\Gamma(\varphi, \phi_j) &= (\varphi'''(x)\overline{\phi_j''(x)} - \varphi''(x)\overline{\phi_j'''(x)} - \varphi'(x)\overline{\phi_j(x)} + \varphi(x)\overline{\phi_j'(x)})|_0^l \\ &= (\overline{\phi_j'''(x)}, \overline{\phi_j''(x)}, \overline{\phi_j'(x)}, \overline{\phi_j(x)}) \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} \varphi'''(x) \\ \varphi''(x) \\ \varphi'(x) \\ \varphi(x) \end{pmatrix} \Bigg|_0^l \\ &= \overline{\partial\phi_j^\tau} \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & -J & 0 & 0 \\ 0 & 0 & -J & 0 \\ 0 & 0 & 0 & J \end{pmatrix} \partial\varphi, \quad j = 1,2,3,4,\end{aligned}$$

it follows that $\varphi \in M$ if and only if $\varphi \in N$ such that

$$G\partial\varphi = 0, \tag{3.4}$$

where

$$G = (\overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(l)}, \overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(0)}) \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & -J & 0 & 0 \\ 0 & 0 & -J & 0 \\ 0 & 0 & 0 & J \end{pmatrix}$$

is a 4×8 matrix. From Proposition 2.4, the rank of the matrix

$$(\overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(l)}, \overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(0)})$$

equals 4. Thus the rank of G also equals 4. From the basic theory of linear algebra, the nullspace, formed of solutions to (3.4), has precisely 4 linearly independent elements. By Proposition 2.4, $\dim M = 4$.

Set

$$\alpha^j = \begin{pmatrix} 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \end{pmatrix} \partial\phi_j, \quad j = 1, 2, 3, 4, \tag{3.5}$$

where I_2 is the 2×2 unit matrix. Since $\partial\phi_1, \partial\phi_2, \partial\phi_3, \partial\phi_4$ are linearly independent in $\mathbb{C}^8$, so are $\alpha^1, \alpha^2, \alpha^3, \alpha^4$. From Proposition 2.4, there are $\varphi_j \in N$, for $j = 1, 2, 3, 4$, such that

$$\partial\varphi_j = \alpha^j, \quad j = 1, 2, 3, 4,$$

and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are linearly independent in N. By (3.5) and (2.10), we have that

$$\begin{aligned} \Gamma(\varphi_j, \phi_k) &= \overline{\partial\phi_k^{\tau}} \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & -J & 0 & 0 \\ 0 & 0 & -J & 0 \\ 0 & 0 & 0 & J \end{pmatrix} \partial\varphi_j = \overline{\partial\phi_k^{\tau}} \begin{pmatrix} 0 & J & 0 & 0 \\ J & 0 & 0 & 0 \\ 0 & 0 & 0 & -J \\ 0 & 0 & -J & 0 \end{pmatrix} \partial\phi_j \\ &= \overline{e_k^{\tau}} \left(\overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(l)} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(l) - \overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(0)} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \mathbb{C}_{\phi_{1,4}}(0) \right) e_j \\ &= 0, \quad k, j = 1, 2, 3, 4, \end{aligned}$$

where

$$e_1 = (1, 0, 0, 0)^{\tau}, \quad e_2 = (0, 1, 0, 0)^{\tau}, \quad e_3 = (0, 0, 1, 0)^{\tau}, \quad e_4 = (0, 0, 0, 1)^{\tau}.$$

Thus $\varphi_j \in M$, for $j = 1, 2, 3, 4$, and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ form a basis of M. In addition, by

(3.5), we have that

$$\begin{aligned}\mathcal{B}_{\varphi_{1,4}} &= \overline{\mathbb{C}^{\tau}_{\varphi_{1,4}}(l)}\begin{pmatrix}0 & 0\\ J & 0\end{pmatrix}\mathbb{C}_{\varphi_{1,4}}(l) - \overline{\mathbb{C}^{\tau}_{\varphi_{1,4}}(0)}\begin{pmatrix}0 & 0\\ J & 0\end{pmatrix}\mathbb{C}_{\varphi_{1,4}}(0)\\ &= (\overline{\partial\varphi_1}, \overline{\partial\varphi_2}, \overline{\partial\varphi_3}, \overline{\partial\varphi_4})^{\tau}\begin{pmatrix}0 & 0 & 0 & 0\\ J & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -J & 0\end{pmatrix}(\partial\varphi_1, \partial\varphi_2, \partial\varphi_3, \partial\varphi_4)\\ &= (\overline{\partial\phi_1}, \overline{\partial\phi_2}, \overline{\partial\phi_3}, \overline{\partial\phi_4})^{\tau}\begin{pmatrix}0 & -J & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & J\\ 0 & 0 & 0 & 0\end{pmatrix}(\partial\phi_1, \partial\phi_2, \partial\phi_3, \partial\phi_4)\\ &= \overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(l)}\begin{pmatrix}0 & -J\\ 0 & 0\end{pmatrix}\mathbb{C}_{\phi_{1,4}}(l) - \overline{\mathbb{C}^{\tau}_{\phi_{1,4}}(0)}\begin{pmatrix}0 & -J\\ 0 & 0\end{pmatrix}\mathbb{C}_{\phi_{1,4}}(0)\\ &= \overline{\mathcal{B}^{\tau}_{\phi_{1,4}}}.\end{aligned} \tag{3.6}$$

From Theorem 2.1, the combination of (3.3) and (3.6) yields that $\hat{A}$ is selfadjoint.

(ii) For any $u \in D(A)$, since

$$\Gamma(u, v) = -\overline{\Gamma(u, v)} = 0,$$

for any $v \in D(\hat{A})$, we have $u \in D(\hat{\hat{A}})$. Thus $D(A) \subset D(\hat{\hat{A}})$. Since A and $\hat{\hat{A}}$ are selfadjoint, we have that

$$A = \hat{\hat{A}}. \quad \square$$

Finally, we consider some dual properties between A and $\hat{A}$. From Theorem 2.2 and (3.6), the following theorem is immediate.

THEOREM 3.2. Let A be a selfadoint operator defined by (1.1). Then A is nonnegative if and only if $\hat{A}$ is nonnegative. In particular, A has SDE boundary conditions if and only if $\hat{A}$ has SDE boundary conditions.

THEOREM 3.3. Let A be a selfadjoint operator defined by (1.1) and let A have SDE boundary conditions. Let $\hat{A}$ be the dual boundary condition operator of A. Then

(i) $\sigma_p(A)\backslash\{0\} = \sigma_p(\hat{A})\backslash\{0\}$, where $\sigma_p(A)$ and $\sigma_p(\hat{A})$ are the point spectra of A and $\hat{A}$, respectively;

(ii) Let $\lambda \in \sigma_p(A)$ and $\lambda \neq 0$. If $u(\cdot)$ is an eigenfunction of A corresponding to the eigenvalue λ, then $u''(\cdot)$ is an eigenfunction of $\hat{A}$ corresponding to λ.

Proof. Let $\lambda \in \sigma_p(A), \lambda \neq 0$, and let u be an eigenfunction of A corresponding to λ. By Theorem 2.1, there are linearly independent elements $\phi_1, \phi_2, \phi_3, \phi_4$ in N such that (2.8) and (2.9) hold. Set

$$u = u_0 + \sum_{j=1}^{4}\alpha_j\phi_j, \quad u_0 \in H_0^4(0, l), \quad \alpha_j \in \mathbb{C}, \quad j = 1, 2, 3, 4.$$

Since A has SDE boundary conditions, for any $v = v_0 + \sum_{j=1}^4 \beta_j \phi_j \in D(A)$, from Theorem 2.2, we have

$$\begin{aligned}\Gamma(u'', v) &= [(u'')'''\overline{v''} - (u'')''\overline{v'''} - u'''\overline{v} + u''\overline{v'}]|_0^l \\ &= \lambda(u'\overline{v''} - u\overline{v'''})|_0^l - (u'''\overline{v} - u''\overline{v'})|_0^l \\ &= -\lambda\overline{\beta^\tau \mathcal{B}_{\phi_{1,4}}^\tau}\alpha - \overline{\beta^\tau}\mathcal{B}_{\phi_{1,4}}\alpha = 0,\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\tau$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^\tau$. Thus $u''(\cdot) \in D(\hat{A})$. In addition, since $u^{(4)}(\cdot) = \lambda u(\cdot)$, it is clear that $u''(\cdot))^{(4)} = \lambda u''(\cdot)$. Therefore, $\lambda \in \sigma_p(\hat{A})\backslash\{0\}$ and u'' is an eigenfunction of $\hat{A}$ corresponding to λ. By Theorem 3.1(ii), the conclusion is obtained. □

THEOREM 3.4. Let the selfadjoint operator A, defined by (1.1), have SDE boundary conditions, and let $\hat{A}$ be the dual boundary condition operator of A. Denote the nullspaces of A and $\hat{A}$ by $\mathbb{N}(A)$ and $\mathbb{N}(\hat{A})$, respectively. Let $L^2(0,l)$ have the following orthogonal direct sum decompositions:

$$L^2(0,l) = \mathbb{R}_0 \oplus \mathbb{N}(\hat{A}) = \hat{\mathbb{R}}_0 \oplus \mathbb{N}(A),$$

where $\mathbb{R}_0 = (\mathbb{N}(\hat{A}))^\perp$ and $\hat{\mathbb{R}}_0 = (\mathbb{N}(A))^\perp$. Then there is a unique, isometric, surjective, linear operator $Q : \mathbb{R}_0 \to \hat{\mathbb{R}}_0$ such that

$$\begin{aligned}A^{1/2}u &= -Q(u''), \quad \forall u \in D(A^{1/2}), \\ \hat{A}^{1/2}u &= -Q^{-1}(u''), \quad \forall u \in D(\hat{A}^{1/2}),\end{aligned}$$

where $A^{1/2}$ and $\hat{A}^{1/2}$ are the nonnegative square roots of A and $\hat{A}$, respectively, and $Q^{-1} : \hat{\mathbb{R}}_0 \to \mathbb{R}_0$ is the inverse of $Q : \mathbb{R}_0 \to \hat{\mathbb{R}}_0$.

Proof. Let $\varphi_n, \psi_n, \lambda_n$, and k be the same as those in Sec. 1. By Theorem 3.3, $\{\psi_n | n \geq k+1\}$ is the subspace of $L^2(0,l)$ formed of all the eigenfunctions of $\hat{A}$ corresponding to all the nonzero eigenvalues. We therefore obtain

$$\mathbb{R}_0 = \overline{\text{span}}\{\psi_n \mid n \geq k+1\}. \tag{3.7}$$

A similar argument yields

$$\hat{\mathbb{R}}_0 = \overline{\text{span}}\{\varphi_n \mid n \geq k+1\}. \tag{3.8}$$

By [2, Proposition 3.1 and Theorem 2.3], there is an isometric, surjective, linear operator $Q : \mathbb{R}_0 \to \hat{\mathbb{R}}_0$ such that

$$Q\psi_n = \varphi_n, \quad n \geq k+1; \tag{3.9}$$

$$A^{1/2}u = -Q(u''), \quad \forall u \in D(A^{1/2}).$$

A similar argument shows that there is an isometric, surjective, linear operator $\hat{Q} : \hat{\mathbb{R}}_0 \to \mathbb{R}_0$ such that

$$\hat{Q}\varphi_n = \psi_n, \quad n \geq k+1; \tag{3.10}$$

$$\hat{A}^{1/2}u = -\hat{Q}(u''), \quad \forall u \in D(\hat{A}).$$

From (3.9) and (3.10), it follows that

$$\hat{Q} = Q^{-1} : \hat{\mathbb{R}}_0 \rightarrow \mathbb{R}_0.$$

The proof is complete. $\square$

REMARK. It should be noted that Theorem 3.3 fails, in general, when A does not have SDE boundary conditions and this is the essence of the counterexample given in [2] to the result given in [1]; thus the result in [1] remains valid in the case of SDE boundary conditions.

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