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ON STIFFNESS MAXIMIZATION OF VARIABLE THICKNESS SHEET WITH UNILATERAL CONTACT[1]

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Abstract. The problem of maximizing the stiffness of a linearly elastic sheet, in unilateral contact with a rigid frictionless support, is considered. The design variable is the thickness distribution, which is subject to an isoperimetric volume constraint and upper and lower bounds. The bounds may vary over the domain of the sheet, and the lower one is allowed to be zero, hence giving the possibility of obtaining topology information about an optimal design.

By using saddle point theory, the existence of solutions, i.e., thickness functions and corresponding displacement states, is proved. In general, one cannot expect uniqueness of solutions, unless the lower bound is strictly positive, and the uniqueness of optimal states is shown in this case.

1. Introduction. Traditionally, one categorizes structural optimization into three major groups: sizing, shape, and topology optimization problems. The first deals with, e.g., choosing optimal thicknesses of a (two-dimensional) structure, the second involves picking a good shape of the boundary to the domain occupied by the structure, and the last one concerns holes in and connectivities of the domain. The problem considered in this paper, namely that of finding a thickness distribution in a linearly elastic sheet, such that a suitable objective functional is extremized, can clearly be put in the first category. However, by allowing the design variable to be equal to zero, one obtains in effect (also) a topology optimization. Moreover, if the design is a distributed parameter, as in the case of a thickness distribution in a sheet, a solution will also generate a three-dimensional shape.

The most crucial step (at least from an engineering point of view), when formulating a structural optimization problem, is to choose the objective functional, reflecting what is actually desired in the design. Partly because of the possibility of performing qualitative analysis, stiffness or compliance are common, and these have been dealt with to a large extent and by many authors; see, e.g., Prager and Taylor [1] and Hemp [2].

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In 1973 Rossow and Taylor [3] obtained numerical solutions for the maximum stiffness problem for a variable thickness sheet by the finite-element method. In 1981, Benedict [4] somewhat extended the work of Rossow and Taylor to include contact conditions. In the non-contact case, minimizing the work of the external loads is equivalent to maximizing the equilibrium potential energy (provided there are no nonzero prescribed displacements), as can be seen from Clapeyron's theorem [4]. Benedict postulated the latter quantity to be a measure of stiffness and the former to be a measure of compliance. As a consequence, minimizing compliance is not equivalent to maximizing stiffness when unilateral contact is present, but the optimality conditions to the problem formulated in [4], namely maximizing stiffness, are the same as for the traditional non-contact minimum compliance problem. These conditions show that optimum structures are optimal not only in the meaning of the objective functional, but also in a broader sense. For instance, the stress field tends to be very "democratic" due to an inherent property of uniform distribution of strain energy density, twice of which is an upper bound of the Mises equivalent stress [5]. Furthermore, a corresponding formulation for truss structures shows that an optimal design always has the same stress value in all (of the remaining) bars.

The consequences of postulating the equilibrium potential energy to be a measure of stiffness for contacting structures are not obvious, but maximizing stiffness, as it is defined in [4] and in this paper, is equivalent to minimizing the sum of displacements weighted by the applied forces and contact stress values weighted by the initial gaps. This indicates that one will obtain a stiff structure with "good" contact stresses, and this was also experienced by Benedict. Moreover, this feature is in accordance with the statement in [5] saying that if a (nonzero) displacement is prescribed, then the external work is to be maximized. The initial gaps play the role of prescribed displacements and "contact stress *values* weighted by the initial gaps" is minus the external work by the contact stresses.

Céa and Malanowski [6] and Bendsøe [7] have done mathematical research on problems with physical correspondences similar to the one under study. In [6] an existence proof was given for a problem that physically means maximizing stiffness of a transversally loaded membrane by changing the thickness distribution. Bendsøe considered essentially the same for plates. In these reports several side constraints on the design variable were enforced, always including a strictly positive lower bound on the thickness, hence preventing one from obtaining topology information through complete removal of material. They also always used upper bounds (on the thickness or its slopes) to ensure well-posedness, as will be done in this analysis. However, a main new feature of the present work is that the strictly positive lower bound will *not* be necessary.

In this paper, the problem of achieving a thickness distribution, in a plane linearly elastic sheet, for maximum stiffness is formulated, and the existence of such a thickness is proved. The sheet is subjected to unilateral contact with a frictionless rigid obstacle. The upper and lower bounds on the admissible thicknesses can vary over the domain, and the lower bound is allowed to be identically zero if desired (of course, the nonnegativity constraint will always be present to prevent physically absurd designs). This gives the possibility of non-design domains, purely sizing domains and domains where voids are allowed. Uniqueness is discussed and in cases when there is a strictly positive lower bound, uniqueness in the state (i.e., the displacements) is shown.

The paper is organized in the following way. Chapter 2 contains formulations of both the state and the optimization problem, assumptions and basic properties of the sets and functionals. The main existence result is proved in Chapter 3 and the uniqueness is discussed and proved in Chapter 4.

2. Formulation of the problem and basic properties.

2.1. *The state problem.* Let Ω be a bounded, open, and connected set in $\mathbf{R}^2$ with (uniformly) Lipschitzian boundary $\partial\Omega$. We consider a (sheet-like thin) linearly elastic body with elasticity constants $E_{ijkl} \in L^\infty(\Omega)$ obeying the usual symmetry and ellipticity features.[2] The body occupies a region $\overline{\Omega}_h \subset \mathbf{R}^3$, which is the closure of the following set:

$$\Omega_h = \{p = (x, y) \in \mathbf{R}^3 \mid x = (x_1, x_2) \in \Omega \subset \mathbf{R}^2, y \in (-h(x)/2, h(x)/2) \subset \mathbf{R}\}.$$

The subscript denotes the actual region's dependence on the given thickness function h, defined on Ω. Now the state problem terminology follows the one given by Kikuchi and Oden [8] for a plane Signorini problem with E_{ijkl} replaced by hE_{ijkl}. The boundary $\partial\Omega$ is partitioned as $\partial\Omega = \overline{\Gamma}_d \cup \overline{\Gamma}_t \cup \overline{\Gamma}_c$, where the Γ's are all relatively open in $\partial\Omega$, and the superposed bar denotes the closure in $\partial\Omega$. They are all pairwise disjoint and nonempty, and Γ_c is strictly contained in $\partial\Omega \setminus \Gamma_d$ and $\text{meas}(\Gamma_d) > 0$.

The body is subjected to body forces (force per unit area) $f = (f_1, f_2)$ acting over Ω and surface tractions (force per unit length) $t = (t_1, t_2)$ acting on the boundary part Γ_t of $\partial\Omega$.

The body is fixed along Γ_d, and Γ_c is the portion of the boundary that represents the candidate contact surface. The contact considered is unilateral contact with a rigid frictionless foundation, and the initial normal distance is g.

We suppose that the given functions $g \in H^{1/2}(\Gamma_c)$, $f \in (L^2(\Omega))^2$, and $t \in (L^2(\Gamma_t))^2$. The kinematically admissible displacements, $u = (u_1, u_2)$, are members of the following Hilbert space:

$$V = \{u = (u_1, u_2) \in (H^1(\Omega))^2 \mid u_i = 0 \text{ on } \Gamma_d,\ i = 1, 2\}.$$

More specifically, the set of admissible displacements is due to the non-penetration condition

$$\mathcal{U} = \{u \in V \mid u_N - g \leq 0 \text{ on } \Gamma_c\},$$

where $u_N = u_i n_i$ (n_i are the components of the outward unit normal vector of $\partial\Omega$). We equip V with the norm $\|\cdot\|_V$,

$$\|u\|_V = \left\{ \int_\Omega (u_{i,j} u_{i,j} + u_i u_i)\, d\Omega \right\}^{1/2} = \sqrt{(u, u)_V},$$

where $(\cdot, \cdot)_V$ is the inner product in V and $u_{i,j}$ means $\partial u_i / \partial x_j$. It is now clear that $\mathcal{U}$ is a nonempty closed and convex subset of V.

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[2] We use the summation convention where the indices take the values 1 and 2.

REMARK 2.1. The nonemptiness, convexity, and closedness of $\mathcal{U}$ in V are the only properties of $\mathcal{U}$ that will be utilized, and hence all the results will be valid if $\mathcal{U}$ is replaced by $\widetilde{\mathcal{U}} = V$, that is to say, the non-contact case is included in the present analysis. □

The external work functional $L(\cdot)$ is a bounded linear functional defined by

$$L(u) = \int_G f \cdot u \, d\Omega + \int_{\Gamma_t} t \cdot u \, d\Gamma.$$

The strain energy stored in the body is represented by a symmetric bilinear functional,

$$a_h(u, v) = \int_\Omega h E_{ijkl} u_{k,l} v_{i,j} \, d\Omega,$$

where h can be any (nonnegative) function in $L^\infty(\Omega)$. We can now define the total potential energy functional as

$$L^\infty(\Omega) \times (H^1(\Omega))^2 \ni (h, u) \mapsto J_h(u) \in \mathbf{R},$$

where

$$J_h(u) = \tfrac{1}{2} a_h(u, u) - L(u).$$

The restriction of the total potential energy functional to the case of unit thickness, i.e., $J_1(\cdot)$, is exactly the quadratic functional considered in [8], which has the standard properties as, e.g., strict convexity, weak lower semi-continuity, and coercivity. The first two carry over to $J_h(\cdot)$ for any nonnegative h in $L^\infty(\Omega)$, with the exception that convexity might be non-strict.

From Stampacchia's theorem [9] it follows that, in the case $h \geq \beta > 0$, there exists a unique u solving the state problem:

$$(\mathcal{V}) \quad u \in \mathcal{U} : a_h(u, \varphi - u) \geq L(\varphi - u) \quad \forall \varphi \in \mathcal{U}.$$

This can equivalently be formulated as a minimum of total potential energy for the displacement in equilibrium:

$$(\mathcal{M}) \quad u \in \mathcal{U} : J_h(u) \leq J_h(\varphi) \quad \forall \varphi \in \mathcal{U}.$$

If a solution u to the variational inequality is sufficiently regular, and if $h \geq \beta > 0$, then

$$\begin{aligned}
&\{h\sigma_{ij}(u)\}_{,j} + f_i = 0 \quad \text{in } \Omega, \\
&\sigma_{ij}(u) \equiv E_{ijkl} u_{k,l}, \\
&u_i = 0 \quad \text{on } \Gamma_d, \\
&\sigma_{ij}(u) n_j = \frac{1}{h} t_i \quad \text{on } \Gamma_t, \\
&\left.\begin{aligned} &\sigma_{Ti} = 0, \quad \sigma_N(u_N - g) = 0 \\ &\sigma_N \leq 0, \quad u_N - g \leq 0 \end{aligned}\right\} \quad \text{on } \Gamma_c, \\
&\sigma_N \equiv \sigma_{ij}(u) n_i n_j, \quad u_N = u_i n_i, \\
&\sigma_{Ti} \equiv \sigma_{ij} n_j - \sigma_N n_i,
\end{aligned}$$

among which the last four lines are the classical Signorini contact conditions.

2.2. *The set of permissible designs.* We need to define the set of permissible designs:

$$\mathcal{H} = \left\{ h \in L^\infty(\Omega) \;\middle|\; h_{\min}(x) \le h(x) \le h_{\max}(x) \text{ a.e. } x \in \Omega, \int_\Omega h(x)\, d\Omega = \mathrm{Vol} \right\}. \tag{1}$$

Here we have assumed that the two given bound functions belong to $L^\infty(\Omega)$ and

$$0 \le h_{\min} \le h_{\max} < +\infty \quad \text{a.e. in } \Omega.$$

Of course, the thickness is not allowed to be negative (hence $h_{\min} \ge 0$), and $h_{\max}$ can be thought of as being imposed for several reasons: from an engineering point of view, when a too thick sheet is impractical due to limited space; from a mathematical point of view, to prevent ill-posedness (possibly because of developing of ribs); or from a physical point of view, to ensure validity of a plane stress assumption.

Furthermore, we assume for equally natural reasons that

$$\int_\Omega h_{\min}\, d\Omega < \mathrm{Vol} \le \int_\Omega h_{\max}\, d\Omega$$

and that for some strictly positive constant γ,

$$h_{\max} \ge \gamma \quad \text{a.e. in } \Omega.$$

If desirable one can choose $h_{\max} = h_{\min}$ in a region where one wants a fixed thickness, $h_{\max} > h_{\min} > 0$ in a region where one wants a sizing optimization, and $h_{\min} = 0$ in a region where one wants to utilize the optimization fully, i.e., allow complete removal of material.

Define $\vartheta \in [0, 1)$ as

$$\vartheta := \frac{\int_\Omega h_{\max}\, d\Omega - \mathrm{Vol}}{\int_\Omega h_{\max}\, d\Omega - \int_\Omega h_{\min}\, d\Omega}$$

and h_0 as

$$h_0 = \vartheta h_{\min} + (1 - \vartheta) h_{\max}. \tag{2}$$

Then it follows that $h_0 \in \mathcal{H}$ and

$$h_0 \ge (1 - \vartheta)\gamma > 0 \quad \text{a.e. in } \Omega. \tag{3}$$

Collecting some of the above assumptions and properties, one can easily show the following auxiliary result.

LEMMA 2.1. The set $\mathcal{H}$ defined in (1) is nonempty, convex, bounded, and weakly* (and hence strongly) closed in $L^\infty(\Omega)$.

From (3) one understands that $a_{h_0}(\cdot,\cdot)$ is V-elliptic and hence $J_{h_0}(\cdot)$ is coercive, i.e.,

$$\lim_{\substack{u \in \mathcal{U} \\ \|u\|_V \to +\infty}} J_{h_0}(u) = +\infty. \tag{4}$$

Let $\beta := \operatorname{ess\,inf}_{x\in\Omega} h_{\min}(x)$. Then, if $\beta > 0$, $J_h(\cdot)$ is *strictly* convex for any h in $\mathcal{H}$.

2.3. *The optimal design problem.* The optimization problem in the focus of our attention in the remainder of this paper can (in a rather vague manner) be stated as: *Given a fixed amount of material, how should it be distributed in a domain $\Omega \subset \mathbf{R}^2$ in order that the (equilibrium) structure attains a maximum stiffness?*

As a measure of stiffness, we take the equilibrium potential energy; cf. Benedict [4], i.e., $\inf_{u\in\mathcal{U}} J_h(u)$ according to $(\mathcal{M})$. Among all designs with equal volume and between appropriate upper and lower bounds, we want to find one that maximizes $\Phi(h) \equiv \inf_{u\in\mathcal{U}} J_h(u)$, i.e., we want to solve

$$\text{(d)}\quad \text{Find } \tilde{h} \in \mathcal{H} : \Phi(\tilde{h}) \geq \Phi(h) \quad \forall h \in \mathcal{H}.$$

For reasons that soon will be more apparent, we also state

$$\text{(p)}\quad \text{Find } \tilde{u} \in \mathcal{U} : \Psi(\tilde{u}) \leq \Psi(u) \quad \forall u \in \mathcal{U}$$

where $\Psi(u) \equiv \sup_{h\in\mathcal{H}} J_h(u)$. From duality theory (see, e.g., Ekeland, Temam [10]), we have the following.

THEOREM 2.1. Suppose there exists a solution to the saddle-point problem

$$\begin{gathered}\text{(SJ)}\quad \text{Find } (\tilde{u}, \tilde{h}) \in \mathcal{U} \times \mathcal{H} : \\ J_h(\tilde{u}) \leq J_{\tilde{h}}(\tilde{u}) \leq J_{\tilde{h}}(u) \quad \forall (u,h) \in \mathcal{U} \times \mathcal{H}.\end{gathered}$$

Then $\tilde{u}$ solves (p) and $\tilde{h}$ solves (d) if and only if $(\tilde{u}, \tilde{h})$ solves (SJ).

From the above theorem it is obvious that if we have (the existence of) a solution to (SJ), then we also have (the existence of) a solution to our problem (d). The problems (p) and (d) are referred to as "primal" and "dual" problems. The problem (p) might seem uninteresting, but has actually been shown to be more practical to solve than, e.g., (d); see, e.g., Bendsøe, Ben-Tal [11] (and solutions to (d) can be obtained through multipliers in (p)).

The dual problem gives designs that maximize the stiffness, the primal one gives the state(s) in the sheet as a result of an optimal design, and clearly (SJ) gives both; if $(\tilde{u}, \tilde{h})$ solves (SJ), then $\tilde{h}$ is an optimal design and $\tilde{u}$ is the corresponding state. Moreover, $(\tilde{u}, \tilde{h})$ solves

$$\begin{gathered}\min_{\substack{u\in\mathcal{U}\\ h\in L^\infty(\Omega)}} L(u) - \int_{L_c} h\sigma_N(u) g \, d\Gamma \text{ subject to} \\ a_h(u, \varphi - u) \geq L(\varphi - u) \quad \forall \varphi \in \mathcal{U}, \\ h_{\min}(x) \leq h(x) \leq h_{\max}(x) \quad \text{a.e. } x \in \Omega, \\ \int_\Omega h(x)\, d\Omega = \text{Vol},\end{gathered}$$

which is on a more traditional form than (SJ). The appearance of $\sigma_N(u) \equiv E_{ijkl}u_{k,l}n_i n_j$, the contact stresses, is according to Clapeyron's theorem for unilateral contact, and the integral over Γ_c is well defined if $\sigma_N(u)$ belongs to $L^2(\Gamma_c)$. If not, one can generalize by replacing the integral by a suitable duality pairing [12]. The contact stresses are necessarily negative (with some notion of positivity), and hence the objective functional to be minimized is a sum of two terms, one of which is a weighted measure of contact stress magnitudes.

3. Existence of solutions. In this chapter a proof of existence of solutions to (SJ) will be provided. The key to the proof will be a saddle point theorem, mainly due to Ky Fan [13] and Sion [14], but put into the form shown in Theorem 3.1 below by Céa in [15]. Reference [10] contains several useful saddle-point theorems, more frequently used than the one stated below, but they all assume reflexivity.

Existence proofs of optimization problems similar to the problem (d) (see, e.g., Céa and Malanowski [6] and Bendsøe [7]) exclusively utilize the design-to-state mapping, F. If β were strictly positive, F would be well defined on $\mathcal{H}$. Moreover, in that case, F is continuous but noninjective, and in the case of variational inequality, also nondifferentiable. In the present case, F need not be well defined at all! However, by using saddle-point arguments one circumvents these problems.

THEOREM 3.1. Let V and W be two Hausdorff topological vector spaces, $\mathcal{U}$ be a convex compact subset of V, and $\mathcal{H}$ be a convex compact subset of W. Suppose that

$$J : \mathcal{U} \times \mathcal{H} \to \mathbf{R}$$

is a functional such that

(i) for every $u \in \mathcal{U}$ the functional $J.(u) : \mathcal{H} \ni h \mapsto J_h(u) \in \mathbf{R}$ is upper semi-continuous and concave,

(ii) for every $h \in \mathcal{H}$ the functional $J_h(\cdot) : \mathcal{U} \ni u \mapsto J_h(u) \in \mathbf{R}$ is lower semi-continuous and convex.

Then there exists a saddle point $(\tilde{u}, \tilde{h}) \in \mathcal{U} \times \mathcal{H}$ for J.

We note that V (defined in Chapter 2) and $L^\infty(\Omega)$ are Hausdorff topological vector spaces considered with the weak and weak* topology respectively; cf. [15]. Going back to the problem (SJ) defined in Chapter 2, we have

THEOREM 3.2. There exists at least one solution $(\tilde{u}, \tilde{h}) \in \mathcal{U} \times \mathcal{H}$ to the saddle-point problem (SJ).

Proof. Let $\mu > 0$ be given but sufficiently large for the set $\mathcal{U}_\mu = \{u \in \mathcal{U} \mid \|u\|_V \leq \mu\}$ to be nonempty.[3] $\mathcal{U}_\mu$ is convex, and closed and bounded in V's strong topology, and hence convex and compact in the weak topology of V. From Lemma 2.1, $\mathcal{H}$ is bounded and weakly* closed in $L^\infty(\Omega)$. Since the closed unit ball in a dual of a normed vector space is always weakly* compact (Alaoglu's theorem), this suffices to make $\mathcal{H}$ compact in the weak* topology of $L^\infty(\Omega)$. J is linear and weakly* continuous in h and convex and weakly lower semi-continuous in u, and consequently (i) and (ii) in Theorem 3.1 are satisfied. Therefore, there is a pair $(\tilde{u}_\mu, \tilde{h}_\mu) \in \mathcal{U}_\mu \times \mathcal{H}$ such that

$$J_h(\tilde{u}_\mu) \leq J_{\tilde{h}_\mu}(\tilde{u}_\mu) \leq J_{\tilde{h}_\mu}(u) \quad \forall (u, h) \in \mathcal{U}_\mu \times \mathcal{H} \tag{5}$$

and especially

$$J_{h_0}(\tilde{u}_\mu) \leq J_{\tilde{h}_\mu}(\tilde{u}_\mu) \leq J_{\tilde{h}_\mu}(u) \quad \forall u \in \mathcal{U}_\mu. \tag{6}$$

Setting $\Lambda(u) = \frac{1}{2}\|h_{\max}\|_{L^\infty(\Omega)}\|E_{ijkl}u_{i,j}u_{k,l}\|_{L^1(\Omega)} - L(u)$ one has that

$$J_h(u) \leq \Lambda(u) \quad \forall h \in \mathcal{H}.$$

[3] The case when $\mathcal{U}_\mu$ is empty for some strictly positive μ corresponds to a somewhat pre-stressed structure, i.e., g is negative somewhere on Γ_c.

Hence, in particular,

$$J_{\tilde{h}_\mu}(u) \leq \Lambda(u) \quad \forall \mu. \tag{7}$$

By (6) and (7),

$$J_{h_0}(\tilde{u}_\mu) \leq \Lambda(u) \quad \forall \mu,$$

that is to say, $J_{h_0}(\tilde{u}_\mu)$ has an upper bound independent of μ. Recalling (4), it then follows that $\tilde{u}_\mu$ is bounded in the V-norm, i.e.,

$$\|\tilde{u}_\mu\|_V \leq D \quad \forall \mu \tag{8}$$

for some constant $D > 0$. Take $\mu = 2D$ and set $\tilde{u} := \tilde{u}_{2D}$ and $\tilde{h} := \tilde{h}_{2D}$. For this μ, (5) is equivalent to

$$J_h(\tilde{u}) \leq J_{\tilde{h}}(\tilde{u}) \quad \forall h \in \mathcal{H} \tag{9}$$

and

$$J_{\tilde{h}}(\tilde{u}) \leq J_{\tilde{h}}(u) \quad \forall u \in \mathcal{U} \cap \widetilde{B}(0; 2D), \tag{10}$$

where $\widetilde{B}(0; 2D) = \{v \in V \mid \|v\|_V \leq 2D\}$. Let similarly $B(x; \varepsilon) = \{v \in V \mid \|v - x\|_V < \varepsilon\}$ be the open ball, with center x and radius ε, in V's strong topology. Then, from (8) and the triangle inequality, it is clear that $B(\tilde{u}; D) \subset \widetilde{B}(0; 2D)$, and therefore (10) implies

$$J_{\tilde{h}}(\tilde{u}) \leq J_{\tilde{h}}(u) \quad \forall u \in \mathcal{U} \cap B(\tilde{u}; D). \tag{11}$$

This says that the functional $J_{\tilde{h}}(\cdot)$ defined on $\mathcal{U}$ has a local minimum at $\tilde{u} \in \mathcal{U}$. Since $\mathcal{U}$ and $J_{\tilde{h}}(\cdot)$ are convex, as set and functional respectively, the local minimum is also global:

$$J_{\tilde{h}}(\tilde{u}) \leq J_{\tilde{h}}(u) \quad \forall u \in \mathcal{U}. \tag{12}$$

Putting (9) and (12) together, one gets

$$J_h(\tilde{u}) \leq J_{\tilde{h}}(\tilde{u}) \leq J_{\tilde{h}}(u) \quad \forall (u, h) \in \mathcal{U} \times \mathcal{H},$$

which is the desired result. □

Intuitively one might be worried about designs where material is removed in crucial regions where the structure must carry, e.g., surface tractions t (suppose $f = 0$). Such a design is indeed permissible (belongs to $\mathcal{H}$), but has no (finite) equilibrium potential energy since one can formally choose $u_n \in \mathcal{U}$ such that $a_h(u_n, u_n) = 0$ and $L(u_n) \to +\infty$ as $n \to +\infty$, and consequently $\Phi(h) = -\infty$. However, this cannot hold for an optimal design, as can be understood from Theorem 3.2. Naively speaking, such awkward designs are permissible, but sorted out in the optimization procedure, since in (d) Φ is maximized.

REMARK 3.1. In the case when $\beta > 0$, the result in the above theorem can be obtained in a way analogous to the more traditional manner in, e.g., [6] and [7]. For each h in $\mathcal{H}$ there is a unique $u \in \mathcal{U}$ satisfying $(\mathcal{V})$. To show the desired existence it suffices to verify the weakly* upper semi-continuity of the equilibrium potential energy as a functional of h. Let h_n, $h \in \mathcal{H}$ and u_n and u in $\mathcal{U}$ be the corresponding equilibrium states. By definition

$$J_{h_n}(u_n) - J_h(u) = \tfrac{1}{2} a_{h_n}(u_n, u_n) - \tfrac{1}{2} a_h(u, u) + L(u - u_n).$$

It is possible to obtain an upper bound of the last term by taking $h := h_n$ and $\varphi := u$ in $(\mathcal{V})$. This results in

$$J_{h_n}(u_n) - J_h(u) \leq a_{h_n}(u_n, u - u_n) + \tfrac{1}{2}a_{h_n}(u_n, u_n) - \tfrac{1}{2}a_h(u, u).$$

It can be straightforwardly verified that the right-hand side can be rewritten according to

$$J_{h_n}(u_n) - J_h(u) \leq \tfrac{1}{2}[-a_{h_n}(u_n - u, u_n - u) + a_{h_n - h}(u, u)].$$

The first term in the right-hand side is clearly nonpositive, and the last one approaches zero as $h_n \overset{w^*}{\to} h$ when $n \to +\infty$. Hence

$$\limsup_{n \to +\infty} J_{h_n}(u_n) \leq J_h(u). \qquad \square$$

4. Uniqueness of the state in a special case. Having dealt with existence it is natural to raise the question of uniqueness of solutions. If $(u, h) \in \mathcal{U} \times \mathcal{H}$ and $(\tilde{u}, \tilde{h}) \in \mathcal{U} \times \mathcal{H}$ are two solutions to (SJ), can one say that $u = \tilde{u}$ and/or $h = \tilde{h}$? The total potential energy is linear and hence not strictly concave in the design, and as a consequence one cannot expect the necessity of $h = \tilde{h}$. This was also numerically experienced for truss structures in, e.g., Klarbring, Petersson, and Rönnqvist [16].

If $\beta = 0$, there are many designs for which the potential energy is not strictly convex, and hence very likely for an optimal design, and one cannot expect uniqueness in u. This can be viewed as a capriciousness of the assigned displacements in regions where $h \geq h_{\min} = 0$ is active. (That the values of a u, part of a solution to (SJ), in regions where material is removed ($h = 0$), are *somewhat* constrained was explained in [16]. For instance, they cannot approach infinity.)

What about the uniqueness of u if $h_{\min} \geq \beta > 0$? The answer is given in the following.

THEOREM 4.1. Suppose $(u, h) \in \mathcal{U} \times \mathcal{H}$ and $(\tilde{u}, \tilde{h}) \in \mathcal{U} \times \mathcal{H}$ solve (SJ). Then, $u = \tilde{u}$, provided $\beta > 0$.

Proof. We show that Ψ (in the primal problem) is strictly convex, and then the theorem follows from Theorem 2.1.

Let $\theta \in \mathbf{R}$ and $u, v \in \mathcal{U}$ be arbitrary but such that $\theta \in (0, 1)$ and $u \neq v$. By definition

$$\Psi(\theta u + (1 - \theta)v) = \sup_{h \in \mathcal{H}} J_h(\theta u + (1 - \theta)v),$$

and since $\mathcal{H}$, due to Lemma 2.1, is compact in the weak* topology of $L^\infty(\Omega)$ and $J_\cdot(\theta u + (1 - \theta)v)$ is weakly* continuous, the supremum is attained at some $h^* \in \mathcal{H}$ (which depends on u, v, and θ). Moreover, $\beta > 0$ and then $J_{h^*}(\cdot)$ is strictly convex. Thus,

$$\Psi(\theta u + (1 - \theta)v) = J_{h^*}(\theta u + (1 - \theta)v) < \theta J_{h^*}(u) + (1 - \theta)J_{h^*}(v)$$

and, trivially,

$$\Psi(\theta u + (1 - \theta)v) < \theta\Psi(u) + (1 - \theta)\Psi(v);$$

so the theorem is proved. $\square$

The above result states that, in case $\beta > 0$, different optimal designs necessarily have the same state of displacements. Consider, for instance, the simple case of zero loads. Then *any* design $h \in \mathcal{H}$ is optimal and has zero displacements as its unique state solution.

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