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ON EXISTENCE OF PERIODIC ORBITS FOR THE FITZHUGH NERVE SYSTEM

By

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Abstract. Applying bifurcation theory, we construct various phase portraits of the FitzHugh differential system and describe the set of parameters for which this system has periodic solutions.

1. Introduction. In the paper [1] FitzHugh proposed the following system of differential equations as a model of nerve conduction in the squid giant axon:

$$\begin{aligned} \dot{x} &= y - x^3/3 + x + \mu, \\ \dot{y} &= \rho(a - x - by). \end{aligned} \tag{1}$$

The system (1) was investigated by several authors (see [2] and references therein).

The existence of periodic solutions is a central problem in the investigation of system (1). The goal of this paper is to state a condition under which a system equivalent to (1) has nontrivial periodic solutions.

Following [1, 2], we suppose that the parameters in (1) satisfy the conditions

$$\mu, a \in R, \quad 0 < b < 1, \quad \text{and } \rho > 0. \tag{2}$$

Let x_μ be a real root of the equation

$$x_\mu^3/3 - x_\mu + x_\mu/b - a/b - \mu = 0,$$

i.e., x_μ is the x-coordinate of the steady state of the system (1). The steady state is unique because $b < 1$. Suppose that inequalities

$$0 < \rho b < 1 \tag{3}$$

are valid. Let us make the following change of coordinates

$$u = x - x_\mu, \qquad v = y + \rho b x - \rho b x_\mu + x_\mu/b - a/b,$$

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and take variables $\beta = 1/b - 1$, $\eta = x_\mu$, and $\eta_0 = \sqrt{1-\rho b}$ as new parameters.

We obtain the system

$$\begin{aligned} \dot{u} &= v - (u^3/3 + \eta u^2 + (\eta^2 - \eta_0^2)u), \\ \dot{v} &= (\eta_0^2 - 1)(u^3/3 + \eta u^2 + (\eta^2 + \beta)u), \end{aligned} \tag{4}$$

where

$$\beta > 0, \quad \eta \in R, \quad \text{and } 0 < \eta_0 < 1; \tag{5}$$

see [3, 4].

The system (4) is equivalent to the system (1) and the first one is a Liénard-type system. Taking into account the latter reason, J. Sugie obtained the following result in [2].

THEOREM 1.1 [2]. Suppose that assumptions (5) are satisfied. Further, suppose that either

$$\eta_0^2 \leq \eta^2, \qquad \eta^4 - 4\eta^2\eta_0^2 + \eta_0^4 + 2\beta\eta^2 - 4\beta\eta_0^2 + 4\beta^2 \geq 0; \tag{6}$$

or

$$2(\eta_0^2 + \beta)^3 < \eta^2(\eta^2 + 3\beta)^2. \tag{7}$$

Then the system (4) has no nonconstant periodic solutions.

The region (6) or (7) in (η^2, η_0^2)-space ($\beta = \text{const}$) is shown by the shaded area in Fig. 1. Its boundary consists of the straight line segment OA and of the curve Q which

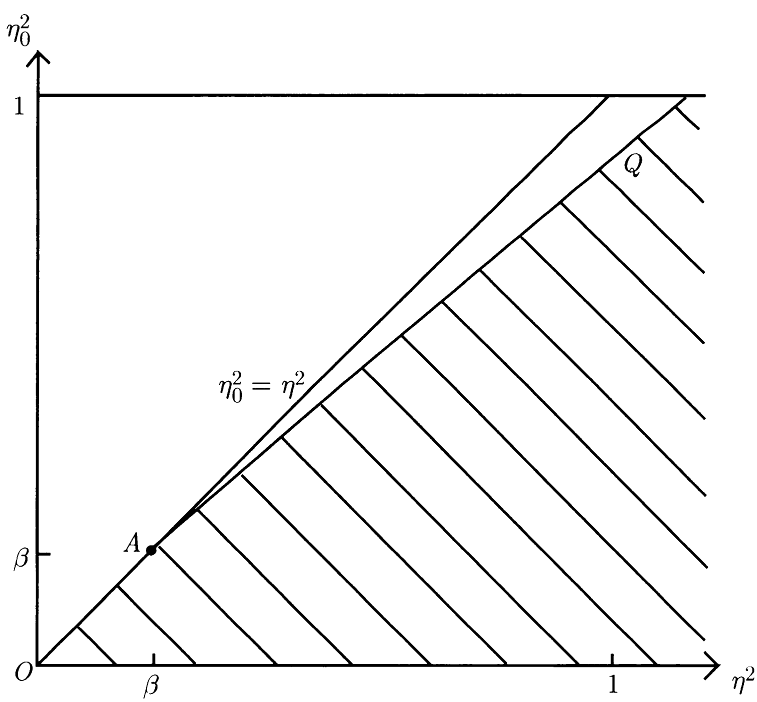

FIG. 1

is given by the relations

$$\eta^2 \geq \beta; \qquad 2(\eta_0^2 + \beta)^3 = \eta^2(\eta^2 + 3\beta)^2. \tag{8}$$

For all $\beta > 0$ the curve Q is upwards convex, monotonically increasing, and tangent to the bisector $\eta_0^2 = \eta^2$ at the point $A(\beta;\beta)$. A concrete view of the curve depends on the value of β. In Fig. 1 the curve Q is plotted for $\beta = 0.2$.

In the present paper we are going to define exactly the position of the curvilinear part of the boundary of the parameters' values region in which the periodic solutions do not exist.

2. Bifurcation set for the system (4). We describe the various phase portraits of (4) as the parameters β, η, and η_0 vary according to (5). This description is based on bifurcation theory [5, 6, 7]. We take [6] as a standard for the presentation of results.

It is convenient to construct the bifurcation set in $(\eta^2, \eta_0^2, \beta)$ parametric space: the set of points for which the system (4) is structurally unstable. We describe the bifurcation set in the three-dimensional parametric space if we describe its typical two-dimensional sections by a plane $\beta = \text{const}$, $\beta > 0$. Such a section for $\beta < 1$ is sketched in Fig. 2.

First, we note that in the whole parametric space there is a single fixed point $u = v = 0$. This fact follows from the inequality $\beta > 0$. The fixed point is a sink if $\eta_0^2 < \eta^2$ (region I), and it is a source surrounded by a stable cycle if $\eta_0^2 > \eta^2$ (region 1). The existence of such a cycle is proved in [3].

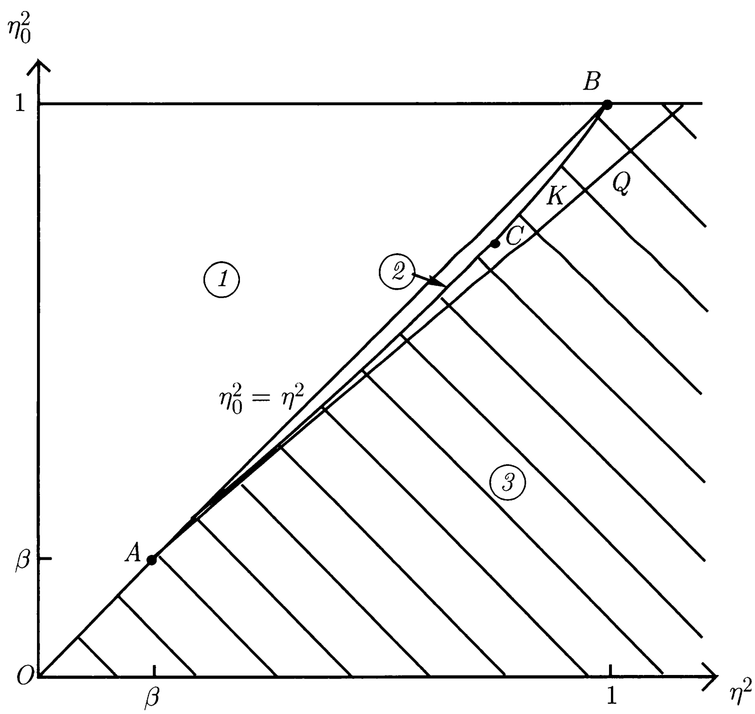

FIG. 2

Structural instability occurs in three distinct ways in system (4):

(i) the Andronov-Hopf bifurcation;

(ii) the generalized Andronov-Hopf bifurcation;

(iii) the double-cycle bifurcation.

(i) The Andronov-Hopf bifurcation can be obtained by conventional linear analysis. Let Δ and σ be the determinant and the trace, respectively, of the Jacobian matrix of the system (4) at the point $O(0;0)$:

$$\Delta = (1 - \eta_0^2)(\eta^2 + \beta); \qquad \sigma = \eta_0^2 - \eta^2. \tag{9}$$

We see that $\sigma = 0$, $\Delta > 0$ on the bisector $\eta_0^2 = \eta^2$. So, on the bisector the fixed point becomes nonhyperbolic, having a pair of pure imaginary eigenvalues with nonzero imaginary part: an Andronov-Hopf bifurcation occurs. A limit cycle surrounding an equilibrium point emerges from the equilibrium in this bifurcation. The direction of the bifurcation and the stability of the cycle is determined by the sign of the first Lyapunov value l_1 which we calculate following [7, p. 198]:

$$l_1 = \pi(\eta^2 - \beta)(1 - \eta_0^2)/(4\Delta^{3/2}). \tag{10}$$

If $\eta^2 < \beta$ then $l_1 < 0$, and the Andronov-Hopf bifurcation is supercritical; a stable cycle emerges upcrossing the bisector $\eta_0^2 = \eta^2$. If $\eta^2 > \beta$ then $l_1 > 0$, and the bifurcation is subcritical; an unstable cycle emerges downcrossing the bisector (interior to a stable cycle already existing).

(ii) Further, the generalized Andronov-Hopf bifurcation of codimension two is observed. There is a complex focus with $l_1 = 0$ at the point $A(\beta;\beta)$. In this case, the stability of the focus is determined by the sign of the second Lyapunov value l_2 where

$$l_2 = -5\pi/(144\beta\sqrt{2\beta(1-\beta)}); \tag{11}$$

see [7, p. 199]. We have $l_2 < 0$ for all parameters; so the complex focus is stable. The point $A(\beta;\beta)$ specifies a border between soft and sharp loss of stability.

(iii) It is known that near the point A there exists a double-cycle bifurcation curve K tangent to the bisector at A, on which a pair of closed orbits, one an attractor and the other a repellor, coalesce and vanish. Points of the curve K correspond to double cycles (stable outside, because $l_2 < 0$).

Let us note that the phase portraits in region I near AB and near the η^2-axis are not topologically equivalent: the former has two limit cycles while the latter does not have limit cycles at all (this fact follows, say, from the result of J. Sugie). We come to the conclusion that there exists a bifurcation curve which corresponds to the global codimension-one double-cycle bifurcation and divides region I into two subregions 2 and 3 and coincides with the curve K mentioned above. This conclusion is in good agreement with our numerical investigation.

The curve K is tangent to the bisector $\eta^2 = \eta_0^2$ at the point $A(\beta;\beta)$ and separates the bisector and curve Q from (8).

It is impossible to give explicit formulas for K except for its segment from a neighborhood of the point A. We shall return to this segment later.

It is possible to make some estimates of the position of the curve K. For example, we know that K lies between the bisector and the curve Q. As we see below this estimate is very good for points located near the point A.

Further, the next claim is easily proved from a standard phase plane analysis.

Suppose that assumptions (5) are satisfied. Further, suppose that

$$(\eta_0^2 - \eta^2)^2 + 4(\eta_0^2 - 1)(\eta^2 + \beta) \geq 0, \quad \eta^2 \geq 1, \quad \text{and } \beta < 1. \tag{12}$$

Then, the system (4) has no nonconstant periodic solutions.

Inequalities (12) imply merely that a single sink $O(0;0)$ is a stable node. The claim follows immediately from the fact that, in this case, the angle

$$\{(u,v) \in R^2 \colon u \geq 0,\ (\eta^2 - \eta_0^2)u/2 \leq v\}$$

is the ω-invariant set of the system (4).

From the claim results the conclusion that the double-cycle curve K cannot intersect the horizontal line $\eta_0^2 = 1$ at points whose η^2-coordinates are more than 1. On the other hand, it is obvious that the curve K cannot intersect the bisector $\eta_0^2 = \eta^2$. Because of this, we conclude that the point $B(1;1)$ is the second endpoint of the double cycle curve K.

We found the other points of the curve K except its endpoints A and B numerically. We used the LINBAS program authored by A. Chibnik in our investigation.

The LINBAS program is dedicated for investigation of the bifurcations of periodic solutions of differential systems with several parameters. The program constructs bifurcation curves in the parametric space; see [8].

As an example, for $\beta = 0.2$ our calculations lead to the table:

$\beta = 0.2$	η^2	η_0^2 (curve K)	η_0^2 (curve Q)
	0.2	0.2	0.2
	0.3	0.295863	0.295289
	0.4	0.387085	0.384804
	0.5	0.476718	0.471287
	0.6	0.566814	0.555953
	0.7	0.659984	0.639431
	0.8	0.761365	0.722087
	0.9	0.876840	0.804149
	0.99	0.987519	0.877622
	1.0		0.885767
	1.1		0.967043
	1.14065		1.00000

For comparison, in the third column of the table we exemplify the η_0^2-coordinates of points of the curve Q for $\beta = 0.2$ and for corresponding values of η^2.

From the table one can see that the curves K and Q are very close to each other when η^2 is not too far from β.

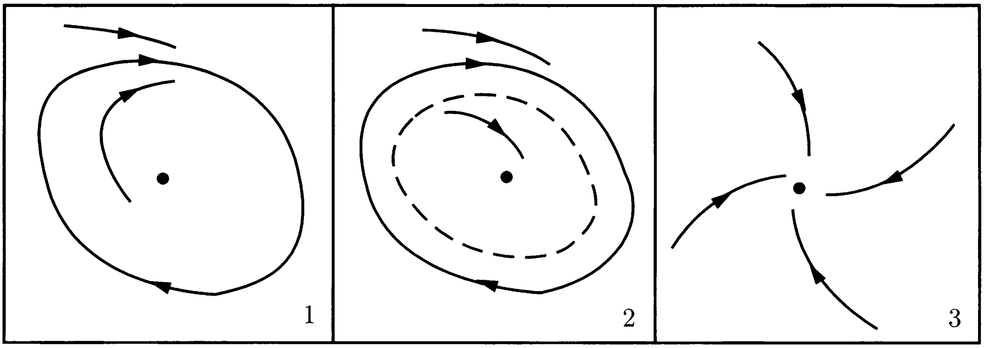

FIG. 3

In Fig. 2, K is plotted for $\beta = 0.2$.

The bisector and the curve K divide the plane of parameters η^2, η_0^2 into three regions: Fig. 2. Corresponding phase portraits are sketched schematically in Fig. 3 (an unstable cycle is dotted).

Periodic solutions in the system (4) for $\beta < 1$ are absent if the parameters are from the region 3 (shaded in Fig. 2). Thus, for $\beta < 1$ the actual boundary of the region in which oscillations do not exist consists of two parts: the straight line segment OA and the double-cycle curve K.

Some points from the regions 2 and 3 were found in [2] with the help of computer simulation. In Example 4.3 of [2] the point $C(.8;.75)$ was found numerically as a point whose coordinates do not satisfy (6) or (7) for $\beta = 0.2$, but nevertheless system (4) has no periodic solutions by corresponding parameters' values. The point $C(.8;.75)$ is located between the curves K and Q.

If $\beta \geq 1$ then the situation is trivial. In this case, the curve K is absent in Fig. 2; and the whole bifurcation set consists of the Andronov-Hopf bifurcation curve, i.e., of the bisector $\eta_0^2 = \eta^2$ only. The system (4) has phase portraits as in the regions 1 and 3.

3. The asymptotic formula for the double-cycle curve. The asymptotic formula is valid for the segment of K that adjoins to the point A:

$$l_1^2\sqrt{4\Delta - \sigma^2} - 8\pi l_2\sigma = 0 \quad (l_1 l_2 < 0), \tag{13}$$

where Δ, σ, l_1, and l_2 are given by (9)–(11); see [7, p. 199]. After substitution of (9)–(11) into (13) we obtain the asymptotic expansion

$$\eta_0^2 = \beta + (\eta^2 - \beta) - \frac{9}{80\beta}(\eta^2 - \beta)^2 + o(\eta^2 - \beta)^2, \quad \eta^2 \geq \beta. \tag{14}$$

We used REDUCE to obtain (14).

For the points of the curve Q located near the point A, we have from (8):

$$\eta_0^2 = \beta + (\eta^2 - \beta) - \frac{1}{8\beta}(\eta^2 - \beta)^2 + o(\eta^2 - \beta)^2, \quad \eta^2 \geq \beta. \tag{15}$$

If we compare (14) and (15) and take into account the results of our numerical simulations we conclude that the curves K and Q are very near each other in a neighborhood of the point A. Therefore, as regards analytic formulas that are valid in this neighborhood we agree with a final remark in [2]: "Although Example 4.3 suggests that a better result than Theorem 1.1 exists it would be difficult to achieve a satisfactory one."

On the other hand, near the line $\eta_0^2 = 1$ the curves K and Q are distinguished noticeably. So, it is possible to improve the result of J. Sugie for such parameters' values. For example, the conditions (12) give some points in the parametric space where cycles are absent and which do not belong to the region (6) or (7).

REMARK. It is proved in [4] that system (4) has exactly one asymptotically stable limit cycle, if parameters satisfy the conditions:

$$\eta_0^2 \geq 4\eta^2; \qquad \text{or} \quad \eta_0^2 \geq 6\beta \quad \text{and} \quad \eta_0^2 \geq \eta^2. \tag{16}$$

As already noted, limit cycles in system (4) are absent, if conditions (6), (7), or (12) hold.

Naturally, assumptions (5) are taken to be satisfied in all cases.

If the conditions (5)–(7), (12), or (16) are not satisfied then there may exist some more even numbers of limit cycles (stable and unstable cycles coupled). Hence, we investigated the system (4) " with an accuracy of even numbers of limit cycles". Such an investigation is a commonplace in the qualitative theory of planar differential systems.

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