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ON THE COMPUTATION OF INFELD'S FUNCTION USED IN EVALUATING THE ADMITTANCE OF PROLATE SPHEROIDAL DIPOLE ANTENNAS

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Abstract. The Infeld function expressed in terms of the outgoing prolate spheroidal radial wave function and its derivative, and employed in the expression of the input self-admittance of prolate spheroidal antennas, has accurately been calculated by using a newly developed asymptotic expression for large degree n. This asymptotic power series has been derived by using a perturbation method with a perturbation parameter $\epsilon = 1/(\lambda_{1n} - h^2)$, where λ_{1n} is the spheroid's eigenvalue for the given parameter h of the spheroidal wave function.

1. Introduction. In the computation of the admittance of prolate spheroidal dipole antennas of finite feed gap length [1], [2], [3] we encounter the problem of the slow convergence of the input susceptance, expressed as an infinite sum of outgoing prolate spheroidal wave functions, and their first-order derivative. More specifically, each term of the susceptance includes the following function:

$$r_n(h,\xi) = \frac{(\xi^2-1)^{1/2} R_{1n}^{(4)}(h,\xi)}{\frac{d}{d\xi}[(\xi^2-1)^{1/2} R_{1n}^{(4)}(h,\xi)]}. \tag{1}$$

Infeld [1, p. 126] chose ζ_{n-1} to represent this function. In (1) $R_{1n}^{(4)}(h,\xi)$ is the prolate spheroidal radial function of order 1 and degree n of the fourth kind [4]. It satisfies the following differential equation [4]:

$$\frac{d}{d\xi}\left\{(\xi^2-1)\frac{d}{d\xi}R_{1n}^{(4)}(h,\xi)\right\} - \left(\lambda_{1n} - h^2\xi^2 + \frac{1}{\xi^2-1}\right) R_{1n}^{(4)}(h,\xi) = 0, \quad n = 1,2,3,4,\ldots, \tag{2}$$

where ξ is the spheroid's radial coordinate ($\xi \geq 1$), and λ_{1n} is the spheroid's eigenvalue for the given h parameter, i.e., $h = kF$ with k being the operating wavenumber and F the semi-interfocal distance of the spheroid.

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The Infeld function $r_n(h,\xi)$ can be computed from $R_{1n}^{(4)}(h,\xi)$ and its first-order derivative, which have recently been calculated with single [5] and double [6] precision accuracies. However, for large degree n it is useful to develop an asymptotic expression of $r_n(h,\xi)$. Infeld [1, p. 126] gave only the leading term of the function.

By introducing a perturbation parameter into the Riccati differential equation that the Infeld function satisfies, and by using the perturbation theory [7], its remaining asymptotic terms have been found. This asymptotic expression has been proved to be helpful in obtaining a high accuracy for large n.

2. Asymptotic expression of the Infeld function. First, by letting

$$P_n(h,\xi) = (\xi^2-1)^{1/2} R_{1n}^{(4)}(h,\xi), \tag{3}$$

we obtain from (2)

$$(\xi^2-1)\frac{d^2}{d\xi^2}P_n(h,\xi) - (\lambda_{1n} - h^2\xi^2)P_n(h,\xi) = 0. \tag{4}$$

Because of (4), the Infeld function given by (1) satisfies the following Riccati differential equation:

$$\frac{d}{d\xi}r_n(h,\xi) + r_n^2(h,\xi)\left[\frac{\lambda_{1n}}{\xi^2-1} - \frac{h^2\xi^2}{\xi^2-1}\right] = 1. \tag{5}$$

Equation (5) was also derived by Infeld.

Now, if we introduce the perturbation parameter

$$\epsilon = \frac{1}{\lambda_{1n} - h^2}, \tag{6}$$

and use the following transformation:

$$s_n(h,x) = \left(\frac{\lambda_{1n} - h^2}{\xi^2-1}\right)^{1/2} r_n(h,\xi), \qquad x = \frac{\sqrt{\xi^2-1}}{\xi}, \tag{7}$$

(5) is reduced to

$$\sqrt{\epsilon}x(1-x^2)\frac{ds_n(h,x)}{dx} + \sqrt{\epsilon}s_n(h,x) + xs_n^2(h,x)\left(1 - h^2\frac{x^2}{1-x^2}\epsilon\right) - x = 0. \tag{8}$$

In (8) if $\epsilon \to 0$ we obtain $s_n^2(h,x) = 1$ and choose $s_n(h,x) = -1$ to have a physical meaning [1]. This is also the only leading term obtained by Infeld.

Therefore, by employing a perturbation theory [7], the function $s_n(h,x)$ can be assumed to be expanded in rational powers of ϵ with the weighting functions $h_m(h,x)$ to be found, as follows:

$$s_n(h,x) = \sum_{m=0}^{\infty} \epsilon^{m/2} h_m(h,x). \tag{9}$$

The above asymptotic expansion was extensively discussed by Kevorkian and Cole [7].

The substitution of (9) into (8) yields

$$\sum_{m=0}^{\infty}\left[x(1-x^2)\ \frac{dh_m(h,x)}{dx}+h_m(h,x)\right]\epsilon^{(m+1)/2}$$
$$+x\left[\sum_{m=0}^{\infty}\sum_{l=0}^{m}h_l(h,x)h_{m-l}(h,x)\left(1-\frac{h^2x^2}{1-x^2}\epsilon\right)\epsilon^{m/2}-1\right]=0. \tag{10}$$

By comparing like rational powers of ϵ in (10) we find the relationship among the weighting functions $h_m(h,x)$ as follows:

$$h_0^2(h,x)=1, \tag{11}$$

$$2xh_{m+1}(h,x)=x(1-x^2)\frac{dh_m(h,x)}{dx}+h_m(h,x)+\sum_{l=1}^{m}xh_l(h,x)h_{m+1-l}(h,x)$$
$$-h^2\frac{x^3}{1-x^2}\sum_{l=0}^{m-1}h_l(h,x)h_{m-1-l}(h,x),\quad m=1,2,3,\ldots. \tag{12}$$

The recursion relation (12) shows that we can find $h_{m+1}(h,x)$ provided that the weighting functions $h_k(h,x)$ with $k=0,1,2,\ldots,m$ are known. From Eq. (11) we obtain two solutions for $h_0(h,x)$, of which only one solution has a physical meaning, i.e., $h_0(h,x)=-1$, which is Infeld's only term. If we commence with his term, by using (12) the closed forms of $h_1(h,x),h_2(h,x),\ldots,h_6(h,x)$ are given by

$$h_1(h,x)=-\frac{1}{2x},\qquad h_2(h,x)=-\frac{1}{4}+\frac{1}{8x^2}-\frac{h^2}{2}\frac{x^2}{1-x^2},$$
$$h_3(h,x)=-\frac{1}{8x^3}(1-x^2)-h^2\frac{x}{1-x^2},$$
$$h_4(h,x)=\frac{25}{128}\frac{1}{x^4}-\frac{9}{32x^2}+\frac{3}{32}+\frac{h^2}{8}\frac{1}{1-x^2}\left(-\frac{9}{2}-5x^2-3h^2\frac{x^4}{1-x^2}\right),$$
$$h_5(h,x)=-\frac{13}{32}\frac{1}{x^5}+\frac{23}{32}\frac{1}{x^3}-\frac{5}{16}\frac{1}{x}-\frac{5}{8}h^2x-h^2\frac{x}{1-x^2}\left(\frac{3}{8}+\frac{1}{8x^2}+\frac{5}{8}x^2+\frac{3}{2}\frac{h^2x^2}{1-x^2}\right), \tag{13}$$

$$h_6(h,x)=\frac{1073}{1024}\frac{1}{x^6}-\frac{1123}{512}\frac{1}{x^4}+\left(\frac{339}{256}+\frac{h^2}{8}\right)\frac{1}{x^2}-\left(\frac{23}{128}+\frac{h^2}{2}\right)-\frac{5}{8}h^2x^2$$
$$+\frac{h^2}{1-x^2}\left(\frac{5}{256}\frac{1}{x^2}-\frac{11}{64}-\frac{13}{64}x^2-\frac{5}{8}x^4\right)$$
$$-h^4(1-x^2)^2\frac{x^2}{64}(70x^2+161)-\frac{5}{16}\frac{h^6x^6}{(1-x^2)^3}.$$

The generation of higher-order weighting functions $h_m(h,x)$ is more tedious. However, in the appendix it is shown that there also exists a recursion relation among the expansion coefficients of $h_m(h,x)$, which facilitates their systematic computation. Therefore, the final form of the Infeld function, with the use of Eqs. (6), (7), and (9), is given by

$$r_n(h,\xi) = \sqrt{\xi^2 - 1} \sum_{m=0}^{\infty} (\lambda_{1n} - h^2)^{-(m+1)/2} h_m(h,x), \qquad x = \sqrt{\xi^2 - 1}/\xi. \tag{14}$$

Equation (14) has been derived under the asymptotic condition, i.e., $x\sqrt{\lambda_{1n} - h^2} > 1$.

3. Numerical results. Table 1 shows the convergence characteristics of the asymptotic series of (14) in terms of its number of terms N for $\xi = 1.05, h = 2$ and $n = 15, 17$, and 19. The computed values are verified with those calculated with a double precision accuracy [6]. It is shown from the table that we have obtained about 5 to 6 significant figures of accuracy with only 7 terms of the asymptotic series for $x\sqrt{\lambda_{1n} - h^2} > 3$. Better results can be obtained by increasing n or N. For thin spheroids, i.e., $\xi = 1.005$, Table 2 gives the computed values of $r_n(h,\xi)$ with 7 terms for $n = 50, 51, \ldots, 63$ to ensure about six significant figures of accuracy.

TABLE 1. Convergence characteristics of the asymptotic series of (14) in terms of its number of terms N for $\xi = 1.05$ and $h = 2$ for large degree n. The true values [6] of $r_n(h,\xi)$ are -2.290737×10^{-2}, -2.005182×10^{-2}, -1.782769×10^{-2} for $n = 15, 17, 19$, respectively.

N	$n = 15$	$n = 17$	$n = 19$
1	-2.07528×10^{-2}	-1.83623×10^{-2}	-1.64671×10^{-2}
2	-2.29588×10^{-2}	-2.00893×10^{-2}	-1.78560×10^{-2}
3	-2.28812×10^{-2}	-2.00356×10^{-2}	-1.78173×10^{-2}
4	-2.29114×10^{-2}	-2.00541×10^{-2}	-1.78292×10^{-2}
5	-2.29052×10^{-2}	-2.00507×10^{-2}	-1.78273×10^{-2}
6	-2.29084×10^{-2}	-2.00523×10^{-2}	-1.78281×10^{-2}
7	-2.29067×10^{-2}	-2.00516×10^{-2}	-1.78278×10^{-2}

TABLE 2. Values of the Infeld function $r_n(h,\xi)$ computed from (14) using 7 terms for $\xi = 1.005$ and $h = 2$ with large n ($n = 50, 51, \ldots, 63$)

n	$r_n(h,\xi)$	n	$r_n(h,\xi)$
51	-2.12694×10^{-3}	50	-2.17263×10^{-3}
53	-2.04104×10^{-3}	52	-2.08311×10^{-3}
55	-1.96177×10^{-3}	54	-2.00063×10^{-3}
57	-1.88839×10^{-3}	56	-1.92439×10^{-3}
59	-1.82027×10^{-3}	58	-1.85371×10^{-3}
61	-1.75686×10^{-3}	60	-1.78801×10^{-3}
63	-1.69770×10^{-3}	62	-1.72678×10^{-3}

Appendix. From Eqs. (11), (12), and (13) the weighting function $h_m(h,x)$ is expanded as follows:

$$h_m(h,x) = x^{-m}\sum_{k=0}^{\infty} a_{mk}(h)x^{2k}, \tag{A1}$$

with its derivative given by

$$\frac{dh_m(h,x)}{dx} = \sum_{k=0}^{\infty}(2k-m)a_{mk}(h)x^{2k-m-1}. \tag{A2}$$

The substitution of (A1) and (A2) into the recursion relation (12) yields after some algebra:

$$\begin{aligned} 2\sum_{k=0}^{\infty} a_{m+1,k}(h)x^{2k-m} &= \sum_{k=0}^{\infty}(2k-m)a_{mk}(h)x^{2k-m} - \sum_{k=0}^{\infty}(2k-n)a_{mk}(h)x^{2k-m+2} \\ &\quad + \sum_{k=0}^{\infty} a_{mk}(h)x^{2k-m} + \sum_{s=0}^{\infty}\sum_{k=0}^{s}\sum_{l=1}^{m} a_{lk}(h)a_{m+1-l,s-k}(h)x^{2s-m} \\ &\quad - h^2\sum_{t=0}^{\infty}\sum_{s=0}^{t}\sum_{k=0}^{s}\sum_{l=0}^{m-1} a_{lk}(h)a_{m-1-l,s-k}(h)x^{2t-m+4}. \end{aligned} \tag{A3}$$

By comparing like powers of x, the recursion relation among the expansion coefficients $a_{ml}(h)$ is obtained as follows:

$$\begin{aligned} a_{m+1,l}(h) &= \tfrac{1}{2}(2l-m+1)a_{ml}(h) - \tfrac{1}{2}(2l-m-2)a_{m,l-1}(h) \\ &\quad + \frac{1}{2}\sum_{k=0}^{l}\sum_{r=1}^{m} a_{rk}(h)a_{m+1-r,l-k}(h) \\ &\quad - \frac{h^2}{2}\sum_{r=0}^{l-2}\sum_{k=0}^{r}\sum_{s=0}^{m-1} a_{sk}(h)a_{m-1-s,r-k}(h). \end{aligned} \tag{A4}$$

If we begin with the expansion coefficient of $h_0(h,x)$, i.e., $a_{0,l}(h) = -\delta_{0,l}$, the coefficients $a_{ml}(h)$ $(m = 1,2,\dots;\ l = 0,1,2,\dots)$ of $h_m(h,x)$ can easily be generated from (A4).

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