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CONSERVATION LAWS AND CANONICAL FORMS IN THE STROH FORMALISM OF ANISOTROPIC ELASTICITY

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Abstract. We find and classify all first-order conservation laws in the Stroh formalism. All possible non-semisimple degeneracies are considered. The laws are found to depend on three arbitrary analytic functions. In some instances, there is an "extra" law which is quadratic in $\nabla \mathbf{u}$. Separable and inseparable canonical forms for the stored energy function are given for each type of degeneracy and they are used to compute the conservation laws. The existence of a real Stroh eigenvector is found to be a necessary and sufficient condition for separability. The laws themselves are stated in terms of the Stroh eigenvectors.

1. Introduction. Conservation laws play a major role in determining the properties of solutions to a system of partial differential equations. This is especially true in the theory of elasticity, where each conservation law corresponds to a "path-independent" integral. The "energy-momentum" tensor [1, 1951] that corresponds with Rice's "J-integral" [2, 1968] is the best-known example. It has been used to find energy release rates and stress intensity factors associated with a crack tip. In 1972, Knowles and Sternberg [3] used a restricted version of Noether's theorem [4], [5] to compute additional conservation laws for elasticity. Budiansky and Rice [6] gave energy release rate interpretations of these new laws. Noting that the version of Noether's theorem used in [3] would not yield all possible laws, Edelen [7] proposed in 1981 that "$\cdots$ a detailed cataloging of *all* invariance transformations and conservation laws in linear elasticity would seem a worthy task." Since then, Olver [8, 1984], [9, 1984], and [11, 1988] classified all conservation laws depending on up to one derivative for homogeneous planar bodies and for three-dimensional isotropic bodies. Caviglia and Morro [12, 1988] classified all laws that exist for any three-dimensional anisotropic elastic body. (But certain materials, such as a transversely isotropic one, do have additional laws [16, Chapter 6].) The Stroh formalism, [13], [14], and [15], deals with planar deformations of a three-dimensional body. Yeh, Shu, and Wu [17, 1993] classified the conservation laws for the cases when the material has three distinct Stroh eigenvalues and when the material is isotropic.

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Hatfield [16, 1994] independently obtained these results as well as results for all possible degeneracies of the material. An isotropic material, which is degenerate, easily decomposes into a planar and anti-planar part and is therefore far easier to deal with than a general degenerate anisotropic material. These degeneracies are worthy of analysis since recent work by Ting [18] shows that they do in fact occur in real materials. The results in [16] on the Stroh formalism are summarized here and the laws are given explicitly in terms of the generalized Stroh eigenvectors. An extension of Olver's [19] canonical forms for the Stroh formalism and an interesting condition for a separable form are given as well.

2. The Stroh formalism. We are concerned here with the linear theory of hyperelasticity (see [20]). We let $\{x^i\}$ denote the reference variables and $\{u^j\}$ denote the deformation variables. The equilibrium equations of elasticity arise from the variational problem (the summation convention will be in force except when specified)

$$I[\mathbf{u}] = \int_\Omega C_{ijkl} \cdot \frac{\partial u^i}{\partial x^j} \cdot \frac{\partial u^k}{\partial x^l} dx \tag{2.1}$$

where the elasticity tensor C_{ijkl} satisfies the symmetries

$$C_{ijkl} = C_{jikl} = C_{klij} \tag{2.2}$$

as well as the hypothesis of positive definiteness

$$C_{ijkl} \cdot e_{ij} \cdot e_{kl} > 0 \tag{2.3}$$

for any symmetric tensor $\mathbf{e} \neq 0$.

The Euler-Lagrange equations for (2.1), which we will also refer to as the equilibrium equations, are

$$C_{ijkl} \cdot u^k_{jl} = 0 \tag{2.4}$$

where we have used the notation $u^k_{jl} = (\partial^2 u^k)/(\partial x^j \partial x^l)$. We will also denote $u^k_l = \partial u^k / \partial x^l$. This notation will be used throughout.

With the aid of the *stress tensor* $\sigma_{ij} = C_{ijkl} \cdot u^k_l$, (2.4) can be written as

$$\mathrm{Div}(\sigma) = D_j(\sigma_{ij}) = 0 \tag{2.5}$$

where D_j denotes the taking of the total derivative (see [21]). Equation (2.5) is an example of a conservation law of (2.4). In general, a conservation is a divergence that vanishes on solutions of the system.

DEFINITION 2.1. Given the system of kth-order partial differential equations $\Delta^i[\mathbf{u}] = 0$, a conservation law is a p-tuple of smooth functions $P^i[\mathbf{u}]$ such that

$$\mathrm{Div}(\mathbf{P}) = D_i P^i[\mathbf{u}] = 0$$

whenever $\mathbf{u} = f(\mathbf{x})$ is a solution to $\Delta^i[\mathbf{u}] = 0$.

In the Stroh formalism we have $u^i_3 = 0$. We will restrict our attention to this case. We will interchangeably use the notations $(x, y) = (x^1, x^2) \in \mathbf{R}^2$ and $(u, v, w) = (u^1, u^2, u^3) \in \mathbf{R}^3$. Boldface symbols will denote vectors. By virtue of (2.5), we know that for any $\mathbf{u}$ that satisfies (2.4), there exist three *stress functions* $\Phi = (\phi^1(\mathbf{x}), \phi^2(\mathbf{x}), \phi^3(\mathbf{x}))$ such that

$$\sigma_{i1} = -\frac{\partial \phi^i}{\partial y}, \qquad \sigma_{i2} = \frac{\partial \phi^i}{\partial x}. \tag{2.6}$$

We define the following 3×3 matrices (see, for example, [22] or [23]):

$$\begin{aligned} Q &= Q_{ij} = C_{i1j1}, \\ T &= T_{ij} = C_{i2j2}, \\ R &= R_{ij} = C_{i1j2}. \end{aligned} \tag{2.7}$$

We note that Q and T are symmetric and positive definite. We define the following 6×6 matrices:

$$M_1 = \begin{bmatrix} -R^{\mathrm{T}} & I \\ -Q & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} T & 0 \\ R & I \end{bmatrix}. \tag{2.8}$$

We define $\mathbf{Y} = \left[\begin{smallmatrix} \mathbf{u} \\ \Phi \end{smallmatrix}\right] \in \mathbf{R}^6$. We can then express the relations (2.6) as

$$M_1 \cdot \frac{\partial \mathbf{Y}}{\partial x} = M_2 \cdot \frac{\partial \mathbf{Y}}{\partial y}. \tag{2.9}$$

Noting that M_2 is invertible, we multiply (2.9) by M_2^{-1} to get

$$N \frac{\partial \mathbf{Y}}{\partial x} = \frac{\partial \mathbf{Y}}{\partial y} \tag{2.10}$$

where

$$N = M_2^{-1} M_1 = \begin{bmatrix} -T^{-1} R^{\mathrm{T}} & T^{-1} \\ R T^{-1} R^{\mathrm{T}} - Q & -R T^{-1} \end{bmatrix}. \tag{2.11}$$

This matrix N is the same as $N(0)$ in Eq. (5.11) of [22] and it is called the *fundamental elasticity tensor*. It will be shown that the Jordan structure of this matrix essentially determines the nature of the conservation laws.

The characteristic polynomial of N, $s(\lambda)$, is a sixth-degree polynomial known as the "Stroh sextic". It is shown in [13] that since the strain energy is positive definite, so is $s(\lambda)$. Hence, its roots come in three complex conjugate pairs $p_\alpha = p_{\alpha 1} + \mathbf{i} p_{\alpha 2}$ where $\alpha = 1, \ldots, 6$; $p_\alpha = \overline{p_{\alpha+3}}$ and $\mathrm{Im}\{p_\alpha\} = p_{\alpha 2} > 0$ if $\alpha = 1, 2, 3$. The associated eigenvectors are $\mathbf{V}_\alpha \in \mathbf{C}^6$. If the matrix N is not semisimple, the material is said to be *degenerate.* An example of a degenerate material is an isotropic one. In this case the Stroh sextic has one triple pair of roots at $p = \pm\mathbf{i}$ and the eigenvectors of N span only a four-dimensional subspace of $\mathbf{C}^6$.

If the material is not degenerate, then we have the following general solution to (2.10) [13]:

$$\mathbf{Y} = \sum_{\alpha=1}^{3} \mathbf{V}_\alpha \cdot f_\alpha(z_\alpha) + \overline{\mathbf{V}_\alpha} \cdot \overline{f_\alpha(z_\alpha)} \tag{2.12}$$

where each f_α is an arbitrary complex-analytic function of its argument $z_\alpha = x + p_\alpha y$. The functions f_α are called the *Stroh functions*. If we write $\mathbf{V}_\alpha = [\begin{smallmatrix}\mathbf{A}_\alpha \\ \mathbf{B}_\alpha\end{smallmatrix}]$ with $\mathbf{A}_\alpha, \mathbf{B}_\alpha \in \mathbf{C}^3$, the vectors $\mathbf{A}_\alpha$ are called the *Stroh eigenvectors*. It is easy to show that they satisfy (no sum):

$$(Q + p_\alpha(R + R^{\mathrm{T}}) + p_\alpha^2 T)\mathbf{A}_\alpha = 0 \tag{2.13}$$

where p_α is a root of the Stroh sextic. The vectors $\mathbf{B}_\alpha$ satisfy

$$\mathbf{B}_\alpha = (R^{\mathrm{T}} + p_\alpha T)\mathbf{A}_\alpha = -\frac{1}{p_\alpha}(Q + p_\alpha R)\mathbf{A}_\alpha. \tag{2.14}$$

If we define the matrices $A = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3]$ and $B = [\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3]$, the solution (2.12) can be written as

$$\begin{aligned} \mathbf{u} &= A\mathbf{f} + \overline{A\mathbf{f}}, \\ \Phi &= B\mathbf{f} + \overline{B\mathbf{f}} \end{aligned} \tag{2.15}$$

where

$$\mathbf{f} = \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \end{bmatrix}.$$

It is not hard to see that if (2.13) and (2.14) hold, then

$$\mathbf{W}_\alpha = \begin{bmatrix} \mathbf{B}_\alpha \\ \mathbf{A}_\alpha \end{bmatrix}$$

is a left eigenvector of N with eigenvalue p_α (by "left eigenvector" we mean that $N^{\mathrm{T}}\mathbf{W}_\alpha = p_\alpha \mathbf{W}_\alpha$). If the eigenvectors are suitably normalized, this leads to the following relations due to Stroh ([14] and [15]):

$$\begin{aligned} A^{\mathrm{T}}B + B^{\mathrm{T}}A &= I, \\ A^{\mathrm{T}}\overline{B} + B^{\mathrm{T}}\overline{A} &= 0. \end{aligned} \tag{2.16}$$

Applying this to (2.15) we see that

$$\mathbf{f} = B^{\mathrm{T}}\mathbf{u} + A^{\mathrm{T}}\Phi. \tag{2.17}$$

We now give the analogous relations for the case when N is not semisimple. For a discussion of this, see [24]. Suppose there are only two independent pairs of eigenvectors $\mathbf{V}_1, \mathbf{V}_2, \overline{\mathbf{V}_1}, \overline{\mathbf{V}_2}$ with two generalized eigenvectors $\mathbf{V}_1^\#$ and $\overline{\mathbf{V}_1^\#}$ such that $N\mathbf{V}_1^\# = p_1\mathbf{V}_1^\# + \mathbf{V}_1$. We denote

$$\mathbf{V}_1^\# = \begin{bmatrix} \mathbf{A}_1^\# \\ \mathbf{B}_1^\# \end{bmatrix}.$$

The generalized Stroh eigenvector $\mathbf{A}_1^\#$ satisfies

$$(Q + p_1(R + R^{\mathrm{T}}) + p_1^2 T)\mathbf{A}_1^\# = \left(\frac{1}{p_1}Q - p_1 T\right)\mathbf{A}_1. \tag{2.18}$$

We also have

$$\mathbf{B}_1^{\#} = (p_1 T + R^{\mathrm{T}})\mathbf{A}_1^{\#} + T\mathbf{A}_1 = -\left(\frac{1}{p_1}Q + R\right)\mathbf{A}_1^{\#} + \frac{1}{p_1^2}Q\mathbf{A}_1. \tag{2.19}$$

The choice $\mathbf{A}_1^{\#}$ is not determined by (2.18). However, for reasons that will appear shortly, we will choose $\mathbf{A}_1^{\#}$ so that

$$(\mathbf{A}_1^{\#})^{\mathrm{T}}\mathbf{B}_1^{\#} = 0. \tag{2.20}$$

The general solution of (2.10) is given by

$$\begin{aligned} \mathbf{u} &= A\mathbf{f}^{\dagger} + \overline{A\mathbf{f}^{\dagger}}, \\ \Phi &= B\mathbf{f}^{\dagger} + \overline{B\mathbf{f}^{\dagger}}, \end{aligned} \tag{2.21}$$

where

$$\mathbf{f}^{\dagger} = \begin{bmatrix} f_1(z_1) \\ g_1(z_1) + \frac{\mathrm{i}}{2p_{12}}\overline{z_1}\cdot f_1'(z_1) \\ f_2(z_2) \end{bmatrix}, \qquad A = [\mathbf{A}_1^{\#}, \mathbf{A}_1, \mathbf{A}_2], \qquad B = [\mathbf{B}_1^{\#}, \mathbf{B}_1, \mathbf{B}_2].$$

If we define the matrices $A^{\dagger} = [\mathbf{A}_1, \mathbf{A}_1^{\#}, \mathbf{A}_2]^{\mathrm{T}}, B^{\dagger} = [\mathbf{B}_1, \mathbf{B}_1^{\#}, \mathbf{B}_2]^{\mathrm{T}}$, then the analogue of (2.16), which holds because of (2.20), is

$$\begin{aligned} A^{\dagger}B + B^{\dagger}A &= I, \\ A^{\dagger}\overline{B} + B^{\dagger}\overline{A} &= 0. \end{aligned} \tag{2.22}$$

The analogue of (2.17) is now

$$\mathbf{f}^{\dagger} = B^{\dagger}\mathbf{u} + A^{\dagger}\Phi. \tag{2.23}$$

Suppose there is only one eigenvalue $p = p_1 + \mathrm{i}p_2$, $p_2 > 0$ and one pair of independent eigenvectors $\mathbf{V}$ and $\overline{\mathbf{V}}$ with the generalized eigenvectors $\mathbf{V}^{\#}$ and $\mathbf{V}^{\#\#}$ such that $N\mathbf{V}^{\#} = p\mathbf{V}^{\#} + \mathbf{V}$ and $N\mathbf{V}^{\#\#} = p\mathbf{V}^{\#\#} + \mathbf{V}^{\#}$. Similar to the above, we write

$$\mathbf{V}^{\#\#} = \begin{bmatrix} \mathbf{A}^{\#\#} \\ \mathbf{B}^{\#\#} \end{bmatrix}$$

etc. We see that (2.13), (2.14) and (2.18), (2.19) hold for $\mathbf{V}$ and $\mathbf{V}^{\#}$ respectively. We also have

$$(Q + p(R + R^{\mathrm{T}}) + p^2 T)\mathbf{A}^{\#\#} = \left(\frac{1}{p}Q - pT\right)\mathbf{A}^{\#} - \frac{1}{p^2}Q\mathbf{A} \tag{2.24}$$

and

$$\mathbf{B}^{\#\#} = (pT + R^{\mathrm{T}})\mathbf{A}^{\#\#} + T\mathbf{A}^{\#}. \tag{2.25}$$

We will choose $\mathbf{A}^{\#}$ and $\mathbf{A}^{\#\#}$ so that simultaneously

$$\begin{aligned} (\mathbf{B}^{\#\#})^{\mathrm{T}}\mathbf{A}^{\#} + (\mathbf{A}^{\#\#})^{\mathrm{T}}\mathbf{B}^{\#} &= 0, \\ (\mathbf{B}^{\#\#})^{\mathrm{T}}\mathbf{A}^{\#\#} &= 0. \end{aligned} \tag{2.26}$$

The general solution of (2.10) is given by

$$\begin{aligned} \mathbf{u} &= A\mathbf{f}^\dagger + \overline{A\mathbf{f}^\dagger}, \\ \mathbf{\Phi} &= B\mathbf{f}^\dagger + \overline{B\mathbf{f}^\dagger}, \end{aligned} \tag{2.27}$$

where

$$\mathbf{f}^\dagger = \begin{bmatrix} f(z) \\ g(z) + \frac{\mathbf{i}}{2p_2}\overline{z}f'(z) \\ h(z) + \frac{\mathbf{i}}{2p_2}\overline{z}g'(z) - \frac{1}{8p_2^2}(\overline{z}^2 f''(z) + 2\overline{z}f'(z)) \end{bmatrix},$$

$$A = [\mathbf{A}^{\#\#}, \mathbf{A}^{\#}, \mathbf{A}], \qquad B = [\mathbf{B}^{\#\#}, \mathbf{B}^{\#}, \mathbf{B}].$$

Similar to the previous case, we define

$$A^\dagger = [\mathbf{A}, \mathbf{A}^{\#}, \mathbf{A}^{\#\#}]^{\mathrm{T}}, \qquad B^\dagger = [\mathbf{B}, \mathbf{B}^{\#}, \mathbf{B}^{\#\#}]^{\mathrm{T}}$$

and we have (provided we assure (2.26))

$$\begin{aligned} A^\dagger B + B^\dagger A &= I, \\ A^\dagger \overline{B} + B^\dagger \overline{A} &= 0. \end{aligned} \tag{2.28}$$

This yields

$$\mathbf{f}^\dagger = B^\dagger \mathbf{u} + A^\dagger \mathbf{\Phi}. \tag{2.29}$$

We make an observation about the degenerate cases. In the semisimple case, define

$$\begin{aligned} H &= 2\mathbf{i}AA^{\mathrm{T}}, \\ L &= -2\mathbf{i}BB^{\mathrm{T}}, \\ S &= \mathbf{i}(2AB^{\mathrm{T}} - I). \end{aligned} \tag{2.30}$$

These are real-valued and are called the Barnett-Lothe tensors [25]. In [25] it is shown that these tensors can be computed directly from the fundamental elasticity tensor using integrals and are well defined even for degenerate materials. As pointed out by Ting [24], we may alternatively define these tensors for degenerate material by using the generalized Stroh eigenvectors. If we choose the generalized Stroh eigenvectors to satisfy (2.20) or (2.26), we have

$$\begin{aligned} H &= 2\mathbf{i}AA^{\dagger}, \\ L &= -2\mathbf{i}BB^{\dagger}, \\ S &= \mathbf{i}(2AB^{\dagger} - I). \end{aligned} \tag{2.31}$$

3. Canonical forms. The elastic tensor has (in general) 21 components. If the material possesses elastic symmetry and the reference coordinates are taken in the correct frame relative to the symmetry, the elastic tensor simplifies. That is, if one makes the right orthogonal change of coordinates, the elastic tensor transforms into one of the twelve standard canonical forms of crystal symmetry. We wish to find canonical forms appropriate to the Stroh formalism. Rather than restricting our attention strictly to orthogonal changes of coordinates, we will widen the class of allowed changes of variables to the full linear group. Lodge [26, 1955] first explored this idea for fully three-dimensional elasticity. Since then, Olver gave canonical forms for planar elasticity and showed that all anisotropic materials are equivalent to an orthotropic material [27] (see also [28] and [29]). In [19], Olver treated planar deformations of three-dimensional bodies and gave canonical forms for separable and inseparable materials. In [16], these forms were extended to cover the various possible non-semisimple degeneracies. We note that we no longer treat $\mathbf{u}$ as a vector field on $\mathbf{R}^3$ such that $\partial \mathbf{u}/\partial x^3 = 0$, but instead as a function $\mathbf{u} : \mathbf{R}^2 \to \mathbf{R}^3$ (compare [30]). This has the advantage of simplifying things even further without losing any generality when it comes to conservation laws (the disadvantage being that the transformations will have no obvious physical interpretation).

Consider the integral (2.1) under the effects of a change of variables $(\tilde{x}, \tilde{\mathbf{u}}) = (L\mathbf{x}, S\mathbf{u})$ where $L \in \mathrm{GL}(2, \mathbf{R})$ and $S \in \mathrm{GL}(3, \mathbf{R})$. We have

$$I = \int_{\tilde{\Omega}} \widetilde{W}(\widetilde{\nabla \mathbf{u}}) d\tilde{\mathbf{x}} = \int_{L^{-1}\tilde{\Omega}} W(S\nabla \mathbf{u} L^{-1})|L| d\mathbf{x}.$$

This implies $W \to C_{i\alpha j\beta} S_{ir} S_{js} (L^{-1})_{\alpha\gamma} (L^{-1})_{\beta\delta} \cdot u^r_\gamma \cdot u^s_\delta \cdot |L|$ where the Greek indices run from 1 to 2 and the Latin from 1 to 3.

At this point, it is useful to make a simplification. We pass to the *symbol* $\mathcal{S} = C_{i\alpha j\beta} u^i u^j x^\alpha x^\beta$ and look for a canonical form for it. This has the advantage that instead of thinking "$C_{r\gamma s\delta} \to C_{i\alpha j\beta} S_{ir} S_{sj} (L^{-1})_{\alpha\gamma} (L^{-1})_{\beta\delta} |L|$", we may think "$C_{r\gamma s\delta} \to C_{i\alpha j\beta} S_{ir} S_{sj} L_{\alpha\gamma} L_{\beta\delta}$". There is of course a concern here: "Does a canonical form for the symbol give a canonical form for W?" This is seemingly a serious problem since the symbol does not uniquely determine W. However, the following lemma relieves our concern.

LEMMA 3.1. [31, Proposition 5] Let W_1 and W_2 be two stored energy functions (which are quadratic in $\nabla \mathbf{u}$). Then they determine the same Euler-Lagrange equations if and only if they have the same symbol.

We will define the Stroh eigenvectors to be vectors that satisfy (2.13). The effect of the above type of transformation on the eigenvectors is that they transform with S^{-1}. The Stroh eigenvalues will transform as linear fractional transformations given by L. Consequently, if we wish to show that certain Stroh eigenvectors are independent, it is sufficient to show that they are for the canonical forms.

Suppose there exists a transformation such that the symbol $\mathcal{S}$ takes the form

$$\mathcal{S} = C_{\gamma\alpha\delta\beta} u^\gamma u^\delta x^\alpha x^\beta + q(x, y) \cdot w^2$$

where q is binary quadratic and the Greek indices run from 1 to 2. Then we say that the symbol is *separable*. We note that if a material is separable, then the equilibrium equations decouple into a planar part and an anti-planar part. Any material with $x^3 = 0$ as a plane of symmetry is trivially separable. Whether or not the symbol is separable will be of critical importance for degenerate materials.

PROPOSITION 3.2. The symbol is separable if and only if one of the Stroh eigenvectors is a complex multiple of a real vector.

Proof. If the symbol is separable, then clearly $\mathbf{A} = [\,0\ 0\ 1\,]^{\mathrm{T}}$ will be a Stroh eigenvector of the separated form. After transforming back to the original form, $\mathbf{A}$ will be transformed to a real eigenvector. Suppose that $\mathbf{A} \in \mathbf{R}^3$ is a Stroh eigenvector with eigenvalue $p = p_1 + \mathbf{i}p_2$ with $p_2 \neq 0$. Then consideration of the real and imaginary parts of (2.13) shows that $Q\mathbf{A}, T\mathbf{A}$, and $(R + R^{\mathrm{T}})\mathbf{A}$ all lie in the same one-dimensional subspace of $\mathbf{R}^3$. If we let $\mathbf{s}_1$ and $\mathbf{s}_2$ be two independent vectors spanning the complement of that subspace, then the transformation given by the matrix $S = [\,\mathbf{s}_1\ \mathbf{s}_2\ \mathbf{A}\,]$ will separate $\mathcal{S}$. □

The following canonical forms are a refinement of Olver's [19]. The canonical form depends on whether or not the symbol is separable and on the Jordan structure of the matrix N. For proofs, see [19] and [16]. The Greek letters represent constants that depend on the elastic constants.

- Semisimple, separable:

$$\mathcal{S} = u^2x^2 + \alpha u^2y^2 + 2\beta uvxy + \alpha v^2x^2 + v^2y^2 + w^2(\gamma x^2 + 2\delta xy + \varepsilon y^2), \quad 0 < 1 < \alpha$$

- Semisimple, inseparable:

$$\begin{aligned}\mathcal{S} = u^2x^2 + \alpha u^2y^2 + 2\beta uvxy + \alpha v^2x^2 + v^2y^2 \\ + 2\gamma uwx^2 + 2\delta uwy^2 + 2\varepsilon vwx^2 + 2\theta vwy^2 + \rho w^2x^2 + \sigma w^2y^2\end{aligned}$$

- Non-semisimple, separable:

$$\mathcal{S} = u^2x^2 + \alpha u^2y^2 + 2(1-\alpha)uvxy + \alpha v^2x^2 + v^2y^2 + w^2(\gamma x^2 + 2\delta xy + \varepsilon y^2)$$

- Non-semisimple, inseparable:
  Either

$$\begin{aligned}\mathcal{S} = u^2x^2 + \alpha u^2y^2 + 2(1-\alpha)uvxy + \alpha v^2x^2 + v^2y^2 \\ + 2\gamma uwx^2 + 2\gamma uwy^2 + 2\varepsilon vwx^2 + 2\varepsilon vwy^2 + (\sigma + 2)w^2x^2 + \sigma w^2y^2\end{aligned} \tag{3.1}$$

  or

$$\begin{aligned}\mathcal{S} = u^2x^2 + u^2y^2 + 2\beta uvxy + v^2x^2 + v^2y^2 \\ + 2(\delta + 1)uwx^2 + 2\delta uwy^2 + \frac{1}{\beta^2}w^2x^2 + \frac{1}{\beta^2}w^2y^2\end{aligned} \tag{3.2}$$

If the matrix N is completely irreducible, then the symbol is inseparable and for (3.1) we have

$$\alpha = \frac{\varepsilon^2}{\varepsilon^2 + 2} \quad \text{and} \quad \gamma = 0$$

and for (3.2)

$$\delta = -\frac{1}{2}.$$

LEMMA 3.3. If the symbol is inseparable and the Stroh sextic has repeated eigenvalues, the matrix N is not semisimple. Furthermore, if the symbol is inseparable and there is only one tripled pair of complex-conjugate eigenvalues, the matrix N is completely irreducible.

COROLLARY 3.4. If the matrix N is semisimple, then the Stroh eigenvectors corresponding to the three roots with positive (resp., negative) imaginary parts are independent.

4. Conservation laws. We consider now the problem of finding conservation laws. We note that we are only concerned with *first-order* conservation laws. That is, the components of the law will be allowed to depend on $\mathbf{x}, \mathbf{u}$, and $\nabla\mathbf{u}$. We also note that there are certain laws that reveal no important information about the system.

DEFINITION 4.1. Let $P^\alpha[\mathbf{u}]$ form the components of a conservation law of the system of p.d.e.

$$\Delta^\beta[\mathbf{u}] = 0 \quad \beta = 1, \ldots, m. \tag{4.1}$$

The conservation law is called trivial if either

1) $P^\alpha[\mathbf{u}] = 0$, whenever $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is a solution to (4.1).
2) $D_\alpha P^\alpha \equiv 0$, for all $\mathbf{u} = \mathbf{f}(\mathbf{x})$.

A conservation law satisfying the first condition will be called a trivial law of the first kind, and a conservation law satisfying the second condition will be called a trivial law of the second kind. Trivial laws give an equivalence relation on conservation laws. We will classify laws only up to this equivalence relation.

DEFINITION 4.2. A conservation law of (4.1) is said to be in *characteristic form* if

$$D_\alpha(P^\alpha) = \Psi^\beta[\mathbf{u}] \cdot \Delta^\beta[\mathbf{u}].$$

The m-tuple of functions Ψ^β is called the *characteristic* of the conservation law. It forms the components of the generator of the variational symmetry which corresponds via Noether's theorem to the conservation law. An important fact is that if the system is totally nondegenerate [21], then each of its conservation laws is equivalent to a law that is in characteristic form. In the case of linear elasticity, the equilibrium equations are strongly elliptic and therefore the nondegeneracy conditions are met.

Let $P^i(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u})$ be the components of a conservation law of (2.4) in characteristic form. Then we have

$$\frac{\partial P^i}{\partial x^i} + \frac{\partial P^i}{\partial u^l} \cdot u^l_i + \frac{\partial P^i}{\partial u^l_j} \cdot u^l_{ji} = \Psi^k[\mathbf{u}] \cdot C_{krst} \cdot u^s_{rt}. \tag{4.2}$$

Since the L.H.S. is linear in second derivatives, so is the R.H.S. Consequently, we may assume that $\Psi^k = \Psi^k(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u})$. It is then immediately clear from examining the second derivative terms that

$$\frac{\partial P^i}{\partial u^s_j} + \frac{\partial P^j}{\partial u^s_i} = \Psi^k \cdot (C_{kisj} + C_{kjsi}). \tag{4.3}$$

Equation (4.3) proves the following critical theorem.

THEOREM 4.3. [8, Prop. 4.4] If $P^\alpha(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u})$ form the components of a conservation law of (2.5) with characteristic $\Psi^k(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u})$, then for each fixed $\mathbf{x}_0, \mathbf{u}_0$, the p-tuple $P^\alpha(\mathbf{x}_0, \mathbf{u}_0, \nabla\mathbf{u})$ is a conservation law with characteristic $\Psi^k(\mathbf{x}_0, \mathbf{u}_0, \nabla\mathbf{u})$.

The great value of this result is that it allows us to break the finding of all first-order laws into two steps. First, using (4.3), we find all laws that depend on $\nabla\mathbf{u}$ only. Once we know the general form of these laws, we allow it to depend on $\mathbf{x}$ and $\mathbf{u}$ and insert it into (4.2) and solve it. Along the way, we will be aided by two lemmas that help us to recognize the form of a trivial law of the second kind and a *Betti-reciprocity law*. A

Betti-reciprocity law is one that arises due to Betti's reciprocal theorem [20, Sec. 30]. It takes the explicit form

$$P^\alpha = C_{i\alpha j\beta} \cdot u^j_\beta \cdot \tilde{u}^i - C_{i\alpha l\beta} \cdot \tilde{u}^j_\beta \cdot u^i$$

where $\tilde{\mathbf{u}}$ is any other solution of the equilibrium equations.

LEMMA 4.4. Suppose that $D_1 P^1 + D_2 P^2 \equiv 0$. That is, $P^\alpha[\mathbf{u}]$ form the components of a trivial law of the second kind when there are two independent variables. Then there is a scalar-valued function $Q[\mathbf{u}]$ such that

$$\begin{aligned} P^1 &= -D_2 Q, \\ P^2 &= D_1 Q. \end{aligned}$$

LEMMA 4.5. [11, Proposition 1] Suppose $P^\alpha[\mathbf{u}]$ form the components of the conservation law of (2.5) such that the functions $P^\alpha[\mathbf{u}]$ are linear in $\mathbf{u}$ and its derivatives. Then it is equivalent to a Betti-reciprocity law.

Let $P^\alpha(\nabla\mathbf{u})$ be a conservation law that is independent of $\mathbf{x}$ and $\mathbf{u}$. Equation (4.3) can be written as

$$\begin{aligned} \frac{\partial P^1}{\partial u^i_1} &= \Psi^k \cdot C_{k1i1} = \Psi^k Q_{ki}, \\ \frac{\partial P^2}{\partial u^i_2} &= \Psi^k \cdot C_{k2i2} = \Psi^k T_{ki}, \\ \frac{\partial P^1}{\partial u^i_2} + \frac{\partial P^2}{\partial u^i_1} &= \Psi^k \cdot (C_{k1i2} + C_{k2i1}) = \Psi^k (R_{ki} + R_{ik}). \end{aligned} \tag{4.4}$$

We denote the vectors $\mathbf{\Pi}^i_j = [\partial P^i/\partial u^1_j, \partial P^i/\partial u^2_j, \partial P^i/\partial u^3_j]^{\mathrm{T}}$. We conclude from (4.4) that

$$\begin{aligned} \mathbf{\Psi} &= Q^{-1}\mathbf{\Pi}^1_1 = T^{-1}\mathbf{\Pi}^2_2, \\ \mathbf{\Pi}^1_2 + \mathbf{\Pi}^2_1 &= R^{\mathrm{T}} Q^{-1} \mathbf{\Pi}^1_1 + R T^{-1} \mathbf{\Pi}^2_2. \end{aligned} \tag{4.5}$$

If we think of P^1 and P^2 as functions of the six variables u^i_j, we can write

$$\nabla P^1 = \begin{bmatrix} \mathbf{\Pi}^1_1 \\ \mathbf{\Pi}^1_2 \end{bmatrix} \quad \text{and} \quad \nabla P^2 = \begin{bmatrix} \mathbf{\Pi}^2_1 \\ \mathbf{\Pi}^2_2 \end{bmatrix}.$$

The relations (4.5) take the matrix form

$$\begin{bmatrix} Q^{-1} & 0 \\ -R^{\mathrm{T}} Q^{-1} & I \end{bmatrix} \cdot \nabla P^1 = \begin{bmatrix} 0 & T^{-1} \\ -I & R T^{-1} \end{bmatrix} \cdot \nabla P^2. \tag{4.6}$$

Since the matrix on the L.H.S. is invertible, (4.6) can be written as

$$\nabla P^1 = M \cdot \nabla P^2$$

where

$$M = \begin{bmatrix} 0 & QT^{-1} \\ -I & (R+R^{\mathrm{T}})T^{-1} \end{bmatrix}. \tag{4.7}$$

We note that $M = -G^{\mathrm{T}}|_{v=0}$ where G is the matrix in Eq. (4.2) of [15]. Furthermore, it is easy to show that M is similar to $-N$. The general solution of an equation of type (4.7) is found in [32]. It will depend on the Jordan structure of the matrix M (and therefore, that of N). Since the roots of the Stroh sextic come in complex-conjugate pairs, there are three possibilities: the matrix is semisimple; there are exactly two real Jordan blocks; there is only one real Jordan block.

4.1. *The semisimple case.* We suppose that M is semisimple. The eigenvalues are the negatives of the Stroh eigenvalues. The right eigenvector of M corresponding to the eigenvalue $-p_j$ may be taken as

$$\mathbf{b}_j = \begin{bmatrix} -\frac{1}{p_j} Q\mathbf{A}_j \\ T\mathbf{A}_j \end{bmatrix}. \tag{4.8}$$

The left eigenvector is

$$\mathbf{a}_j = \begin{bmatrix} \mathbf{A}_j \\ p_j \mathbf{A}_j \end{bmatrix}. \tag{4.9}$$

Note that if the Stroh eigenvectors are normalized so that (2.16) holds, then we have

$$\mathbf{a}_i^{\mathrm{T}} \cdot \mathbf{b}_j = \delta_{ij}.$$

We denote

$$\nabla\mathbf{u} = \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix}$$

and define the complex variables

$$\begin{aligned} \eta &= \eta_1 + \mathbf{i}\eta_2 = \mathbf{b}_1^{\mathrm{T}} \cdot \nabla\mathbf{u}, \\ \xi &= \xi_1 + \mathbf{i}\xi_2 = \mathbf{b}_2^{\mathrm{T}} \cdot \nabla\mathbf{u}, \\ \zeta &= \zeta_1 + \mathbf{i}\zeta_2 = \mathbf{b}_3^{\mathrm{T}} \cdot \nabla\mathbf{u}. \end{aligned} \tag{4.10}$$

We apply the result [32, Theorem 5.2].

THEOREM 4.6. Suppose that M (hence N) is semisimple. Let P^1 and P^2 form the components of a conservation law that depends only on the first derivatives of $\mathbf{u}$. Then

$$\begin{aligned} P^1 &= K_1(\eta_1, \eta_2) + K_2(\xi_1, \xi_2) + K_3(\zeta_1, \zeta_2) + K_T, \\ P^2 &= L_1(\eta_1, \eta_2) + L_2(\xi_1, \xi_2) + L_3(\zeta_1, \zeta_2) + L_T, \end{aligned}$$

where

$$\begin{aligned} F(\eta) &= K_1 + \overline{p_1} L_1, \\ G(\xi) &= K_2 + \overline{p_2} L_2, \\ H(\zeta) &= K_3 + \overline{p_3} L_3 \end{aligned}$$

are complex-analytic functions and K_T, L_T form the components of a trivial law.

With the aid of (2.17), we find that

$$\eta = f', \qquad \xi = f_2', \qquad \zeta = f_3'. \tag{4.11}$$

We now proceed to compute the most general laws. To this end, we note that if the P^i's are of the form given in Theorem 4.6 but now depend on $\mathbf{x}$ and $\mathbf{u}$, then

$$D_i P^i = 2\,\mathrm{Re}\{D_{\overline{z_1}}F + D_{\overline{z_2}}G + D_{\overline{z_3}}H\} \tag{4.12}$$

where $D_{\overline{z_k}} = -\mathbf{i}/(2p_{k2}) \cdot (p_k D_x - D_y)$. Since the terms in (4.12) that depend on second derivatives of $\mathbf{u}$ vanish on solutions, we must have

$$\mathrm{Re}\left\{\begin{array}{c} F_{\overline{z_1}} + F_u \cdot u_{\overline{z_1}} + F_v \cdot v_{\overline{z_1}} + F_w \cdot w_{\overline{z_1}} \\ +G_{\overline{z_2}} + G_u \cdot u_{\overline{z_2}} + G_v \cdot v_{\overline{z_2}} + G_w \cdot w_{\overline{z_2}} \\ +H_{\overline{z_3}} + H_u \cdot u_{\overline{z_3}} + H_v \cdot v_{\overline{z_3}} + H_w \cdot w_{\overline{z_3}} \end{array}\right\} = 0 \tag{4.13}$$

where we denote $F_{\overline{z_1}} = \partial F/\partial \overline{z_1}$, $u_{\overline{z_1}} = D_{\overline{z_1}}(u)$, etc. We would like to express the first derivative terms of $\mathbf{u}$ that appear in (4.13) in terms of η, ξ, ζ, etc. The following notation is helpful. Let $\Sigma = \partial\Gamma/\partial u \cdot u_{\overline{z_1}} + \partial\Gamma/\partial v \cdot v_{\overline{z_1}} + \partial\Gamma/\partial w \cdot w_{\overline{z_1}}$. Then there are uniquely defined linear differential operators $\partial u_k = c_{k1}(\partial/\partial u) + c_{k2}(\partial/\partial v) + c_{k3}(\partial/\partial w)$ such that for $\Gamma_{u_k} \equiv \partial u_k \cdot \Gamma$ we have

$$\Sigma = \eta\Gamma_{u_1} + \xi\Gamma_{u_2} + \zeta\Gamma_{u_3} + \overline{\eta}\Gamma_{u_4} + \overline{\xi}\Gamma_{u_5} + \overline{\zeta}\Gamma_{u_6}.$$

We note that the u_k's should not be confused with u, v, and w and do not represent actual variables. Nevertheless, the *operators* ∂u_k are well defined. In a similar manner, we define ∂v_k corresponding with $D_{\overline{z_2}}$ and ∂w_k corresponding with $D_{\overline{z_3}}$.

LEMMA 4.7. The above defined operators have the following explicit representations:

$$\begin{aligned} \partial u_k &= \frac{\mathbf{i}}{2p_{12}}(p_k - p_1)\mathbf{A}_k \cdot \nabla, \\ \partial v_k &= \frac{\mathbf{i}}{2p_{22}}(p_k - p_2)\mathbf{A}_k \cdot \nabla, \\ \partial w_k &= \frac{\mathbf{i}}{2p_{32}}(p_k - p_3)\mathbf{A}_k \cdot \nabla, \end{aligned} \tag{4.14}$$

where $\nabla = [\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}]$.

Equation (4.13) takes the form

$$\mathrm{Re}\left\{\begin{array}{c} F_{\overline{z_1}} + \xi F_{u_2} + \zeta F_{u_3} + \overline{\eta}F_{u_4} + \overline{\xi}F_{u_5} + \overline{\zeta}F_{u_6} \\ +G_{\overline{z_2}} + \eta G_{v_1} + \zeta G_{v_3} + \overline{\eta}G_{v_4} + \overline{\xi}G_{v_5} + \overline{\zeta}G_{v_6} \\ +H_{\overline{z_3}} + \eta H_{w_1} + \xi H_{w_2} + \overline{\eta}H_{w_4} + \overline{\xi}H_{w_5} + \overline{\zeta}H_{w_6} \end{array}\right\} = 0. \tag{4.16}$$

Using Corollary 3.4 we conclude

$$\begin{aligned} F &= F_0(\eta, z_1) + f(\mathbf{u},\mathbf{x}) \cdot \eta + f_0(\mathbf{x},\mathbf{u}), \\ G &= G_0(\xi, z_2) + g(\mathbf{u},\mathbf{x}) \cdot \xi + g_0(\mathbf{x},\mathbf{u}), \\ H &= H_0(\zeta, z_3) + h(\mathbf{u},\mathbf{x}) \cdot \zeta + h_0(\mathbf{x},\mathbf{u}), \end{aligned} \tag{4.16}$$

where F_0, G_0, and H_0 are analytic functions of their arguments. In the treatment [17], the eigenvalues are assumed to be distinct, in which case Corollary 3.4 is already known to hold [14]. So here our result is somewhat more general. The F_0, G_0, and H_0 terms stand alone as conservation laws. Noting their form, we subtract them off and concentrate on the remaining terms. For a trivial law, $\widetilde{P}^1 = -D_y Q, \widetilde{P}^2 = D_x Q$ where $Q(\mathbf{x},\mathbf{u})$ is real-valued, we would have (as in (4.16))

$$\tilde{f} = -2\mathbf{i}p_{12}\partial\overline{u_4} \cdot Q, \qquad \tilde{g} = -2\mathbf{i}p_{22}\partial\overline{u_5} \cdot Q, \qquad \tilde{h} = -2\mathbf{i}p_{32}\partial\overline{u_5} \cdot Q$$

from which we may conclude:

LEMMA 4.8. Up to a trivial law, we may take f, g, and h to depend only on $\mathbf{x}$.

With the aid of Lemma 4.5, we conclude the complete result for the semisimple case.

THEOREM 4.9. Suppose that M (hence N) is semisimple. Then P^1 and P^2 form the components of a conservation law that depends on $\mathbf{x}, \mathbf{u}$, and $\nabla\mathbf{u}$ if and only if

$$\begin{aligned} P^1 &= K_1(\eta_1, \eta_2; \mathbf{x}) + K_2(\xi_1, \xi_2; \mathbf{x}) + K_3(\zeta_1, \zeta_2; \mathbf{x}) + K_T + K_R, \\ P^2 &= L_1(\eta_1, \eta_2; \mathbf{x}) + L_2(\xi_1, \xi_2; \mathbf{x}) + L_3(\zeta_1, \zeta_2; \mathbf{x}) + L_T + L_R, \end{aligned}$$

where

$$\begin{aligned} F(\eta, z_1) &= K_1 + \overline{p_1} L_1, \\ G(\xi, z_2) &= K_2 + \overline{p_2} L_2, \\ H(\zeta, z_3) &= K_3 + \overline{p_3} L_3 \end{aligned}$$

are complex-analytic functions, K_R, L_R form the components of a Betti reciprocity law, and K_T, L_T form the components of a trivial law.

4.2. *Non-semisimple, two real Jordan blocks.* We suppose that there are two complex-conjugate pairs of Stroh eigenvalues p_1 and p_2. Corresponding with p_1 there is also a generalized eigenvector

$$\mathbf{V}_1^\# = \begin{bmatrix} \mathbf{A}_1^\# \\ \mathbf{B}_1^\# \end{bmatrix}$$

of N satisfying (2.18) and (2.19). We find that the corresponding right generalized eigenvectors of M corresponding with the eigenvalue $-p_1$ may be taken as

$$\mathbf{b}_1 = \begin{bmatrix} -\frac{1}{p_1} Q\mathbf{A}_1 \\ T\mathbf{A}_1 \end{bmatrix}, \quad \mathbf{b}_1^\# = \begin{bmatrix} \frac{1}{p_1} Q\mathbf{A}_1^\# - \frac{\mathbf{i}\overline{p_1}}{2p_{12}p_1^2} Q\mathbf{A}_1 \\ -T\mathbf{A}_1^\# + \frac{\mathbf{i}}{2p_{12}} T\mathbf{A}_1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -\frac{1}{p_1} Q\mathbf{A}_2 \\ T\mathbf{A}_2 \end{bmatrix}. \tag{4.17}$$

(We have chosen this so as to make (4.21) as simple as possible.) For left generalized eigenvectors we take

$$\mathbf{a}_1 = \begin{bmatrix} -\mathbf{A}_1 \\ -p_1\mathbf{A}_1 \end{bmatrix}, \quad \mathbf{a}_1^{\#} = \begin{bmatrix} \mathbf{A}_1^{\#} + \frac{\mathbf{i}}{2p_{12}}\mathbf{A}_1 \\ p_1\mathbf{A}_1^{\#} + \frac{\overline{p_1}\mathbf{i}}{2p_{12}}\mathbf{A}_1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} \mathbf{A}_2 \\ p_2\mathbf{A}_2 \end{bmatrix}, \tag{4.18}$$

in which case we have

$$[\mathbf{a}_1\ \mathbf{a}_1^{\#}\ \mathbf{a}_2\ \overline{\mathbf{a}_1}\ \overline{\mathbf{a}_1^{\#}}\ \overline{\mathbf{a}_2}]^{\mathrm{T}} \cdot [\mathbf{b}_1^{\#}\ \mathbf{b}_1\ \mathbf{b}_2\ \overline{\mathbf{b}_1^{\#}}\ \overline{\mathbf{b}_1}\ \overline{\mathbf{b}_2}] = I. \tag{4.19}$$

We define the complex variables

$$\begin{aligned} \eta &= \eta_1 + \mathbf{i}\eta_2 = \mathbf{b}_1^{\mathrm{T}} \cdot \nabla\mathbf{u}, \\ \xi &= \xi_1 + \mathbf{i}\xi_2 = (\mathbf{b}_1^{\#})^{\mathrm{T}} \cdot \nabla\mathbf{u}, \\ \zeta &= \zeta_1 + \mathbf{i}\zeta_2 = \mathbf{b}_2^{\mathrm{T}} \cdot \nabla\mathbf{u}. \end{aligned} \tag{4.20}$$

Using (2.23) and (2.19) we find that

$$\operatorname{diag}\left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right] \cdot f^{\dagger} = \begin{bmatrix} \eta \\ -\xi \\ \zeta \end{bmatrix} \tag{4.21}$$

where $\mathbf{f}^{\dagger}$ is that of (2.21).

THEOREM 4.10. [32, Theorem 5.2] Suppose that M (hence N) is not semisimple and has exactly two real Jordan blocks. Let P^1 and P^2 form the components of a conservation law that depends only on the first derivatives of $\mathbf{u}$. Then

$$\begin{aligned} P^1 &= K_1(\eta_1, \eta_2, \xi_1, \xi_2) + K_2(\zeta_1, \zeta_2) + K_T, \\ P^2 &= L_1(\eta_1, \eta_2, \xi_1, \xi_2) + L_2(\zeta_1, \zeta_2) + L_T \end{aligned}$$

such that

$$\begin{aligned} F(\eta) + \xi\frac{\partial G}{\partial \eta} - \frac{\mathbf{i}}{2p_{12}}\overline{G(\eta)} &= K_1 + \overline{p_1}L_1, \\ H(\zeta) &= K_3 + \overline{p_3}L_3 \end{aligned}$$

where F, G, and H are complex-analytic functions and K_T, L_T form the components of a trivial law.

As in the previous subsection, we now assume the laws are in the form of Theorem 4.10 but allow the components to vary with $\mathbf{x}$ and $\mathbf{u}$. On solutions to 2.4 we must have

$$\operatorname{Re}\left\{D_{\overline{z_1}}\left(F(\eta) + \xi\frac{\partial G}{\partial \eta} - \frac{\mathbf{i}}{2p_{12}}\overline{G(\eta)}\right) + D_{\overline{z_2}}H(\zeta)\right\} = 0. \tag{4.22}$$

Since the terms involving second derivatives of $\mathbf{u}$ must already vanish on solutions, the remaining terms vanish identically. We now define differential operators similar to those of the previous subsection. In particular, we have

$$\frac{\partial \Gamma}{\partial u} \cdot u_{\overline{z_1}} + \frac{\partial \Gamma}{\partial v} \cdot v_{\overline{z_1}} + \frac{\partial \Gamma}{\partial w} \cdot w_{\overline{z_1}} = \eta\Gamma_{u_1} + \xi\Gamma_{u_1^{\#}} + \zeta\Gamma_{u_2} + \overline{\eta}\Gamma_{u_4} + \overline{\xi}\Gamma_{u_4^{\#}} + \overline{\zeta}\Gamma_{u_5} \tag{4.23}$$

and

$$\frac{\partial \Gamma}{\partial u} \cdot u_{\overline{z_2}} + \frac{\partial \Gamma}{\partial v} \cdot v_{\overline{z_2}} + \frac{\partial \Gamma}{\partial w} \cdot w_{\overline{z_2}} = \eta\Gamma_{v_1} + \xi\Gamma_{v_1^{\#}} + \zeta\Gamma_{v_2} + \overline{\eta}\Gamma_{v_4} + \overline{\xi}\Gamma_{v_4^{\#}} + \overline{\zeta}\Gamma_{v_5}. \tag{4.24}$$

LEMMA 4.11. The above defined operators have the following explicit representations:

$$\begin{aligned} \partial u_1 &= \frac{\mathbf{i}}{2p_{12}}\mathbf{A}_1 \cdot \nabla, \qquad \partial u_1^\# = 0, \qquad \partial u_2 = \frac{\mathbf{i}}{2p_{12}}(p_2 - p_1)\mathbf{A}_2 \cdot \nabla, \\ \partial u_4 &= \overline{\mathbf{A}_1^\#} \cdot \nabla, \qquad \partial u_4^\# = \overline{\mathbf{A}_1} \cdot \nabla, \qquad \partial u_5 = \frac{\mathbf{i}}{2p_{12}}(\overline{p_2} - p_1)\overline{\mathbf{A}_2} \cdot \nabla \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \partial v_1 &= \frac{\mathbf{i}}{2p_{22}}\left((p_1 - p_2)\mathbf{A}_1^\# + \left(\frac{p_1 - p_2}{2p_{12}}\mathbf{i} - 1\right)\mathbf{A}_1\right) \cdot \nabla, \\ \partial v_1^\# &= \frac{\mathbf{i}}{2p_{22}}(p_1 - p_2)\mathbf{A}_1 \cdot \nabla, \quad \partial v_2 = 0, \\ \partial v_4 &= \frac{\mathbf{i}}{2p_{22}}\left((\overline{p_1} - p_2)\overline{\mathbf{A}_1^\#} + \left(\frac{p_2 - \overline{p_1}}{2p_{12}}\mathbf{i} - 1\right)\mathbf{A}_1\right) \cdot \nabla, \\ \partial v_4^\# &= \frac{\mathbf{i}}{2p_{22}}(\overline{p_1} - p_2)\overline{\mathbf{A}_1} \cdot \nabla, \\ \partial v_5 &= \overline{\mathbf{A}_2} \cdot \nabla. \end{aligned} \tag{4.26}$$

COROLLARY 4.12. The operators $\partial u_2, \partial u_4^\#$, and ∂u_5 are independent if and only if the symbol is inseparable.

Proof. We use the canonical forms. If the symbol is inseparable, $p_1 \neq p_2$ or the matrix is completely irreducible. We therefore must show that $\overline{\mathbf{A}_1}, \mathbf{A}_2$, and $\overline{\mathbf{A}_2}$ are independent. For the form (3.1) we compute that $\overline{\mathbf{A}_1} = [\,\mathrm{i}\ 1\ 0\,]^{\mathrm{T}}$. By showing that the third component of $\mathbf{A}_2$ is nonzero and employing Proposition 3.2 we conclude the required independence. The argument for (3.2) is similar. If the symbol is separable, $\mathbf{A}_2 = [\,0\ 0\ 1\,]^{\mathrm{T}}$ and the operators cannot be independent.

COROLLARY 4.13. The operators $\{\partial u_2, \partial u_4, \partial u_4^\#, \partial u_5\}$ form a rank three set.

This means that if $f_{u_2} = f_{u_4} = f_{u_4^\#} = f_{u_5} = 0$, then $f = f(\mathbf{x})$ does not depend on $\mathbf{u}$.

COROLLARY 4.14. The operators in (4.26) form a rank three set.

Equation (4.22) can now be written as

$$\mathrm{Re}\left\{\begin{array}{c} F_{\overline{z_1}} + \eta F_{u_1} + \overline{\eta} F_{u_4} + \overline{\xi} F_{u_4^\#} + \zeta F_{u_2} + \overline{\zeta} F_{u_5} \\ +\xi(G_{\eta\overline{z_1}} + \eta G_{\eta u_1} + \overline{\eta} G_{\eta u_4} + \overline{\xi} G_{\eta u_4^\#} + \zeta G_{\eta u_2} + \overline{\zeta} G_{\eta u_5}) \\ -\dfrac{\mathbf{i}}{2p_{12}}(\overline{G}_{\overline{z_1}} + \eta \overline{G}_{u_1} + \overline{\eta}\overline{G}_{u_4} + \overline{\xi G}_{u_4^\#} + \zeta \overline{G}_{u_2} + \overline{\zeta G}_{u_5}) \\ +H_{\overline{z_2}} + \eta H_{v_1} + \overline{\eta} H_{v_4} + \xi H_{v_1^\#} + \overline{\xi} H_{v_4^\#} + \overline{\zeta} H_{v_5} \end{array}\right\} = 0. \tag{4.27}$$

Remembering that F and G are analytic functions of η and that H is an analytic function of ζ, we can use Corollaries 4.13 and 4.14 to show that

$$\begin{aligned} F &= F_0(z_1, \eta) - \frac{\mathbf{i}}{2p_{12}}\overline{z_1}\frac{\partial G_0}{\partial z_1} + f_1(\mathbf{x}) \cdot \eta^2 + f(\mathbf{x}, \mathbf{u}) \cdot \eta + f_0(\mathbf{x}, \mathbf{u}), \\ G &= G_0(z_1, \eta) + g(\mathbf{x}, \mathbf{u}) \cdot \eta, \\ H &= H_0(z_2, \zeta) + h(\mathbf{x}, \mathbf{u}) \cdot \zeta, \end{aligned} \tag{4.28}$$

where F_0, G_0, and H_0 are analytic functions of their arguments.

We recognize that the terms of (4.28) involving F_0, G_0, and H_0 stand alone as a class of conservation laws. We subtract them off and concentrate on the remaining terms. Inserting these into 4.27 and using Corollary 4.12 we find the key difference between the separable and inseparable cases. This difference will lead to an "extra" quadratic law in the separable case.

LEMMA 4.15. Up to a trivial law, we may take g and h to be independent of $\mathbf{u}$. Furthermore, we may take f to be independent of $\mathbf{u}$ provided the symbol is inseparable.

PROPOSITION 4.16. If the symbol is separable, there exists a quadratic law of the form

$$F_q = -\varphi_{u_1} \cdot \overline{z_1} \cdot \eta^2 + \varphi \cdot \eta \tag{4.29}$$

where φ is the unique (up to a real multiple) linear combination of u, v, and w such that

$$\varphi_{u_4^\#} = \varphi_{u_2} = \varphi_{u_5} = 0, \qquad \mathrm{Re}\{\varphi_{u_4}\} = 0.$$

THEOREM 4.17. Suppose that M (hence N) is not semisimple and has exactly two real Jordan blocks. Let P^1 and P^2 form the components of a conservation law that depends on $\mathbf{x}, \mathbf{u}$, and $\nabla \mathbf{u}$. Then

$$\begin{aligned} P^1 &= K_1(\eta_1, \eta_2, \xi_1, \xi_2) + K_2(\zeta_1, \zeta_2) + K_R + K_T, \\ P^2 &= L_1(\eta_1, \eta_2, \xi_1, \xi_2) + L_2(\zeta_1, \zeta_2) + L_R + L_T \end{aligned}$$

where (K_R, L_R) is a Betti-reciprocity law, (K_T, L_T) is a trivial law, and

$$\begin{aligned} K_1 + \overline{p_1} L_1 &= F(z_1, \eta) - \frac{\mathbf{i}}{2p_{12}} \left(\overline{z_1} \frac{\partial G}{\partial z_1} + \overline{G(z_1, \eta)} \right) + \xi \frac{\partial G}{\partial \eta} + F_q, \\ K_2 + \overline{p_2} L_2 &= H(z_2, \zeta) \end{aligned}$$

such that F, G, and H are analytic functions of their arguments and F_1 is of the form (4.29) which exists only if the symbol is separable.

4.3. *One Jordan block.* We deal with the case that there is one real Jordan block corresponding to one tripled Stroh eigenvalue pair $p = p_1 + \mathbf{i}p_1$ and its conjugate $\overline{p}$. We note that in this case, the symbol must be inseparable. The details are similar to the previous subsection. The generalized right eigenvectors of M may be taken as

$$\begin{aligned} \mathbf{b} &= \begin{bmatrix} -\frac{1}{p} Q\mathbf{A} \\ T\mathbf{A} \end{bmatrix}, \qquad \mathbf{b}^\# = \begin{bmatrix} \frac{1}{p} Q\mathbf{A}^\# - \frac{\mathrm{i}\overline{p}}{2p_2 p^2} Q\mathbf{A} \\ -T\mathbf{A}^\# + \frac{\mathrm{i}}{2p_2} T\mathbf{A} \end{bmatrix}, \\ \mathbf{b}^{\#\#} &= \begin{bmatrix} -\frac{1}{p} Q\mathbf{A}^{\#\#} + \frac{\mathrm{i}\overline{p}}{2p_2 p^2} \left(Q\mathbf{A}^\# - \frac{1}{p} Q\mathbf{A} \right) \\ T\mathbf{A}^{\#\#} - \frac{\mathrm{i}}{2p_2} T\mathbf{A}^\# \end{bmatrix}. \end{aligned} \tag{4.30}$$

The left-generalized eigenvectors may be taken as

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} \mathbf{A} \\ p\mathbf{A} \end{bmatrix}, \qquad \mathbf{a}^\# = \begin{bmatrix} -\mathbf{A}^\# - \frac{\mathrm{i}}{2p_2} \mathbf{A} \\ -p\mathbf{A}^\# - \frac{\overline{p}\mathrm{i}}{2p_2} \mathbf{A} \end{bmatrix}, \\ \mathbf{a}^{\#\#} &= \begin{bmatrix} \mathbf{A}^{\#\#} + \frac{\mathrm{i}}{2p_2} \mathbf{A}^\# - \frac{1}{4p_2^2} \mathbf{A} \\ p\mathbf{A}^{\#\#} + \frac{\mathrm{i}\overline{p}}{2p_2} \mathbf{A}^\# - \frac{\overline{p}}{4p_2^2} \mathbf{A} \end{bmatrix}. \end{aligned}$$

Using (2.22) we find that

$$[\mathbf{a} \quad \mathbf{a}^{\#} \quad \mathbf{a}^{\#\#} \quad \overline{\mathbf{a}} \quad \overline{\mathbf{a}^{\#}} \quad \overline{\mathbf{a}^{\#\#}}]^{\mathrm{T}} \cdot [\mathbf{b}^{\#\#} \quad \mathbf{b}^{\#} \quad \mathbf{b} \quad \overline{\mathbf{b}^{\#\#}} \quad \overline{\mathbf{b}^{\#}} \quad \overline{\mathbf{b}}] = I. \tag{4.31}$$

We then define the complex variables

$$\eta = \mathbf{b}^{\mathrm{T}} \cdot \nabla \mathbf{u}, \qquad \xi = (\mathbf{b}^{\#})^{\mathrm{T}} \cdot \nabla \mathbf{u}, \qquad \zeta = (\mathbf{b}^{\#\#})^{\mathrm{T}} \cdot \nabla \mathbf{u}. \tag{4.32}$$

We find that for the $\mathbf{f}^{\dagger}$ of (2.23), we have

$$\frac{\partial}{\partial z} \cdot \mathbf{f}^{\dagger} = \begin{bmatrix} \eta \\ -\xi \\ \zeta \end{bmatrix}. \tag{4.33}$$

THEOREM 4.18. [32, Theorem 5.2] Suppose the matrix M is completely irreducible. Let P^1 and P^2 form the components of a conservation law that depends only on $\nabla \mathbf{u}$. Then

$$P^1 = K + K_T,$$
$$P^2 = L + L_T,$$

where (K_T, L_T) is a trivial law and

$$K + \overline{p} L = F + \xi \frac{\partial G}{\partial \eta} - \frac{\mathbf{i}}{2p_2} \overline{G} + \xi^2 \frac{\partial^2 H}{\partial \eta^2} + 2\zeta \frac{\partial H}{\partial \eta} - \frac{\mathbf{i}}{p_2} \xi \overline{\frac{\partial H}{\partial \eta}} + \frac{1}{2p_2^2} \overline{H} \tag{4.34}$$

such that F, G, and H are analytic functions of η.

We define operators so that

$$D_{\overline{z}} \Gamma = \frac{\partial \Gamma}{\partial \overline{z}} + \eta \Gamma_{u_1} + \xi \Gamma_{u_1^{\#}} + \zeta \Gamma_{u_1^{\#\#}} + \overline{\eta} \Gamma_{u_4} + \overline{\xi} \Gamma_{u_4^{\#}} + \overline{\zeta} \Gamma_{u_4^{\#\#}}. \tag{4.35}$$

By using (4.31) we are able to show the following lemma.

LEMMA 4.19. The operators satisfying (4.35) have the explicit forms

$$\begin{aligned} \partial u_1 &= \left(\frac{\mathbf{i}}{2p_2} \mathbf{A}^{\#} - \frac{1}{4p_2^2} \mathbf{A} \right) \cdot \nabla, \qquad \partial u_1^{\#} = \frac{-\mathbf{i}}{2p_2} \mathbf{A} \cdot \nabla, \qquad \partial u_1^{\#\#} = 0, \\ \partial u_4 &= \overline{\mathbf{A}^{\#\#}} \cdot \nabla, \qquad \partial u_4^{\#} = -\overline{\mathbf{A}^{\#}} \cdot \nabla, \qquad \partial u_4^{\#\#} = \overline{\mathbf{A}} \cdot \nabla. \end{aligned} \tag{4.36}$$

By using the canonical forms I′ and II′, we conclude that the sets $\{\partial u_1^{\#}, \partial u_4^{\#}, \partial u_4^{\#\#}\}$ and $\{\partial u_4, \partial u_4^{\#}, \partial u_4^{\#\#}\}$ are both rank three. We allow the functions in (4.34) to vary with $\mathbf{x}$ and $\mathbf{u}$ and consider the equation (which holds on solutions)

$$\mathrm{Re}\{D_{\overline{z}}(P^1 + \overline{p} P^2)\} = 0. \tag{4.37}$$

Looking at the various powers of $\nabla \mathbf{u}$ we conclude that, up to a trivial law, we may take

$$\begin{aligned} F &= F_0(z, \eta) - \frac{\mathbf{i}}{2p_2} \overline{z} \frac{\partial G_0}{\partial z} - \frac{1}{2p_2^2} \left(\frac{\overline{z}^2}{2} \frac{\partial^2 H_0}{\partial z^2} + \overline{z} \frac{\partial H_0}{\partial z} \right) \\ &\quad + f_1(\mathbf{x}) \cdot \eta^2 + f(\mathbf{x}, \mathbf{u}) \cdot \eta + f_0(\mathbf{x}, \mathbf{u}), \\ G &= G_0(z, \eta) - \frac{\mathbf{i}}{p_2} \overline{z} \frac{\partial H_0}{\partial z} + g_1(\mathbf{x}) \cdot \eta^2 + g(\mathbf{x}) \cdot \eta, \\ H &= H_0(z, \zeta) + h(\mathbf{x}) \cdot \eta, \end{aligned} \tag{4.38}$$

where F_0, G_0, and H_0 are analytic functions of η and z.

We recognize that the terms in (4.38) involving F_0, G_0, and H_0 stand alone as a conservation law. We note its form and concentrate on the remaining terms. As in the doubled-root separable case, these terms will lead to an extra quadratic law as well as the Betti-reciprocity law.

PROPOSITION 4.20. If the matrix M is completely irreducible, there exists a quadratic law of the form (4.34) such that

$$F = (-\varphi_{u_1})\overline{z} \cdot \eta^2 + \varphi \cdot \eta, \qquad G = -\frac{1}{2}(\varphi_{u_1^\#})\overline{z} \cdot \eta^2, \qquad H = 0 \tag{4.39}$$

where φ is the (unique up to a real multiple) linear combination of u, v, and w such that

$$\varphi_{u_4^{\#\#}} = 0, \qquad \varphi_{u_4^\#} = 0, \qquad \mathrm{Re}\{\varphi_{u_4}\} = 0. \tag{4.40}$$

THEOREM 4.21. Suppose the matrix M is completely irreducible. Let P^1 and P^2 form the components of a conservation law that depends on $\mathbf{x}, \mathbf{u}$, and $\nabla \mathbf{u}$. Then

$$P^1 = K + K_R + K_T,$$
$$P^2 = L + L_R + L_T,$$

where (K_R, L_R) is a Betti-reciprocity law, (K_T, L_T) is a trivial law, and (4.34) holds such that

$$F = F_0(z,\eta) - \frac{\mathbf{i}}{2p_2}\overline{z}\frac{\partial G_0}{\partial z} - \frac{1}{2p_2^2}\left(\frac{\overline{z}^2}{2}\frac{\partial^2 H_0}{\partial z^2} + \overline{z}\frac{\partial H_0}{\partial z}\right) - (\varphi_{u_1})\overline{z} \cdot \eta^2 + \varphi \cdot \eta,$$
$$G = G_0(z,\eta) - \frac{\mathbf{i}}{p_2}\overline{z}\frac{\partial H_0}{\partial z} - \frac{1}{2}(\varphi_{u_1^\#})\overline{z} \cdot \eta^2,$$
$$H = H(z,\eta),$$

where F_0, G_0, and H_0 are analytic and φ satisfies (4.40).

5. Conclusions. We have computed all first-order conservation laws that exist in the Stroh formalism. The exact forms of the laws were found to depend on the canonical form of the symbol. The canonical forms depend on the separability of the symbol and on the degeneracy of the material. An interesting necessary and sufficient condition for separability is given by Proposition 3.2. The canonical forms were used to determine certain facts about the laws, but the laws themselves are expressed in terms of the generalized Stroh eigenvectors. In each case, the laws depend on three arbitrary analytic functions. In the separable, non-semisimple case and the completely irreducible case (which must be inseparable), we find that there is an "extra" law quadratic in $\nabla \mathbf{u}$. The applications of these laws may include the determination of stress intensity factors [33] and energy-release rates. Yeh et. al. have demonstrated an application to thermoelasticity [34]. The canonical forms may be of interest in the study of degenerate materials which, [18], do in fact occur in nature.

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