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THE SINGULAR LIMIT OF A HYPERBOLIC SYSTEM AND THE INCOMPRESSIBLE LIMIT OF SOLUTIONS WITH SHOCKS AND SINGULARITIES IN NONLINEAR ELASTICITY

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Abstract. Discontinuous solutions with shocks for a family of almost incompressible hyperelastic materials are studied. An almost incompressible material is one whose deformations are not *a priori* constrained but whose stress response reacts strongly (of order ε^{-1}) to deformations that change volume. The material class considered is isotropic and admits motions that are self-similar, exhibit cavitation, and are energy minimizing. For the initial-value problem when considering the entire material, the solutions converge (as ε tends to zero) to an isochoric solution of the limit (incompressible) system with the corresponding arbitrary hydrostatic pressure being the singular limit of the pressures in the almost incompressible materials. The shocks, if they exist, disappear: their speed tends to infinity and their strength tends to zero.

1. Introduction to the problem. In this article we give support for the following conjecture in mechanics: An incompressible nonlinear elastic material can be regarded as the limit of a family of almost incompressible materials; materials whose deformations are not *a priori* constrained but whose stress response reacts strongly to deformations that change volume. This family will consist of compressible materials all sharing a basic constitutive relation for the stress modulo an extra pressure term of order $1/\varepsilon$. The arbitrary hydrostatic pressure resulting in the incompressible case is actually a singular limit of the almost incompressible pressures that depend exclusively on the motion. Such almost incompressible materials were originally discussed by Spencer [11] and such a limiting relationship was noted in Truesdell and Noll [12, p. 122].

The idea of an incompressible limit has been well-studied for fluids using relevant solutions in smooth (Sobolev) spaces: Ebin [5] and Klainerman and Majda [7], [8]. Hence, very general results can be obtained using a priori estimates. For elastic solids, the arguments for fluids have been extended by Schochet [10] again working with solutions (motions) with L^p derivatives; the idea being that the equations of motion can be written

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as a symmetric hyperbolic system to which one can apply the machinery of functional analysis. More recently Charrier et al [3] have considered static solutions for almost incompressible materials and passed to the limit. Their work was based on calculus of variations arguments of Ball and also dealt with solutions in Sobolev spaces. It is however very common in nonlinear elasticity to encounter jumps in velocity and deformation gradient (so-called shocks) and hence the Sobolev space setting is not applicable for solutions of the equations of motion and their derivatives. In general, if one wishes to work with solutions (motions) with shocks, the natural space is the set of functions whose first derivatives are of the class BV, the space consisting of L^1 functions whose distributional derivatives are finite Borel measures. As yet there has been no analytical demonstration of the incompressible limit when shocks are present. In order to provide evidence of the previously stated conjecture one should demonstrate the incompressible limit in a situation where shocks are needed to establish local existence, and in a situation where there are some numerical or physical experiments verifying that shocks do indeed exist. Our task is made possible by considering an isotropic material class that admits radially symmetric motions. These motions depend only on the ratio of the Euclidean spatial norm to time and exhibit cavitation: a spherical cavity forms at the origin and propagates with a fixed speed. This enables us to study the behavior of the solutions via ordinary differential equations. These motions were introduced by the Spectors [9] who were in turn motivated by the paper of Ball [1] pertaining to statics. They are very natural motions to consider. In statics, Ball [1] considered these materials and reached the startling conclusion that the trivial static deformation was not stable but, rather, radially symmetric deformations with cavities were energy minimizers and stable. The motions that appear in this article are dynamic versions of the deformations of Ball. In [9] it was shown that these motions have no greater energy than the trivial static motion. Moreover, if a shock exists (there is numerical evidence that it does), the energy is less than that of the trivial motion.

2. Constitutive assumptions. Working in material (Lagrangian) coordinates, we deal with materials occupying $\Omega := \mathbf{R}^3$. The motion of such a body is described by a function $u : \mathbf{R}^3 \times \mathbf{R}^+ \to \mathbf{R}^3$ where $u(x,t)$ is the position of the material point x at time t. The gradient $\nabla u(\cdot,t)$ of u with respect to x is called the deformation gradient and will frequently be denoted by F.

The materials in consideration are hyperelastic, isotropic, and homogeneous. Namely, the Piola-Kirchhoff stress tensor produced in response to a given motion with deformation gradient F is given by

$$S(F) = \frac{\partial W}{\partial F},$$

where $W(F)$ is the stored energy function of the material. A hyperelastic material is isotropic if W is a symmetric function of the eigenvalues of $(FF^{\mathrm{T}})^{1/2}$.

Here we consider the following stored energy function,

$$W = \Phi(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}\sum \lambda_i^2 + h(\lambda_1\lambda_2\lambda_3), \tag{2.1}$$

where λ_i are the eigenvalues of $(FF^{\mathrm{T}})^{1/2}$. We assume $h \in C^3(\mathbf{R}^+, \mathbf{R}^+)$ has the following properties:

$$h''(v) > 0, \quad h'''(v) < 0, \quad \lim_{v \to 0^+} h'(v) = -\infty, \quad \lim_{v \to +\infty} h'(v) = +\infty, \quad h'(H) = 0, \quad (2.2)$$

for some $H > 0$. The function h depends exclusively on the determinant of the deformation gradient and thus responds to changes in volume. As will be noted later, the condition $h'' > 0$ implies that the equations of motion are hyperbolic or, equivalently, the stored energy function is rank one convex. The condition, $h''' < 0$ was used by the Spectors. We observe that it implies the failure of a growth condition which would force deformations with finite energy to be continuous and, hence, $u(\cdot, t)$ as a map from any bounded subset of $\mathbf{R}^3$ to $\mathbf{R}^3$ would be continuous and cavitation, which as we shall see is essential in obtaining nontrivial solutions, would be ruled out. Following Ball [1], a material is strong if

$$W(F) \geq C(|F|^p + 1) \quad \text{for some } p > 3.$$

By the Sobolev Embedding theorem, any deformation of a bounded region with finite energy would be continuous. A weaker condition is

$$\frac{W(F)}{|F|^3} \to \infty \quad \text{as } |F| \to \infty. \tag{2.3}$$

By L'Hôpital's rule, we see that

$$\lim_{\det \to \infty} \frac{h(\det)}{\det^3} = \lim_{\det \to \infty} \frac{h''(\det)}{6 \det}$$

is not infinite; hence, (2.3) cannot be satisfied and the material is not strong.

Now suppose the material was incompressible, that is, the volume of any part of the material remains unchanged by deformations and hence the deformation gradient must have constant determinant (not necessarily 1!). In such materials (*cf.* Gurtin [6]), the constitutive relation gives the stress modulo a hydrostatic pressure that does no work during the motion. This arbitrary hydrostatic pressure is found by solving the equations of motion, which will include the kinematic constraint of incompressibility. We note that in general this pressure function is not uniquely determined by the equation of motion (see Gurtin [6, p. 117]). The full Piola-Kirchhoff stress for the incompressible material is

$$S_I(x,t) = \frac{\partial \Phi}{\partial F} + p(x,t) F^{-\mathrm{T}}.$$

In (2.1) the $h(\det)$ will only contribute a constant hydrostatic pressure of magnitude $h'(\det)$ and thus could be absorbed into the arbitrary pressure term.

For materials that are not incompressible the constitutive relation gives the full stress. We create the family of almost incompressible materials by augmenting Φ with Ψ^ε where

$$\Psi^\varepsilon = \frac{(\lambda_1 \lambda_2 \lambda_3 - C)^2}{2\varepsilon}, \quad C \in \mathbf{R},\ C > 0 \tag{2.4}$$

and considering

$$\Phi^\varepsilon = W_\varepsilon(F) = \Phi + \Psi^\varepsilon. \tag{2.5}$$

The specific form of Ψ_ε is chosen because firstly it does not change the material type in question and secondly for small ε it produces a large penalty in the stress response for deformations that change volume. The possibility of the addition of other such terms in forcing out the incompressible limit is discussed in Remark 2 of Sec. 6. The Piola-Kirchhoff stress for such a material is given by

$$\begin{aligned} S_\varepsilon(F) &= \frac{\partial \Phi}{\partial F} + \frac{\partial \Psi_\varepsilon}{\partial \det F}\frac{\partial \det F}{\partial F} \\ &= \frac{\partial \Phi}{\partial F} + \frac{\det F - C}{\varepsilon}(\det F)F^{-\mathrm{T}}. \end{aligned} \tag{2.6}$$

In this paper, we prove that the arbitrary pressure in the incompressible case is a singular limit of the "constitutive pressures" in the compressible materials. We consider the stress in our material family and investigate the limit as ε tends to zero. The arbitrary pressure will actually be the singular limit of

$$\frac{\det F - H}{\varepsilon}(\det F) \quad (C = H)$$

where, in the limit, the dependence on $\det F$ is lost. Specifically we show

$$p(x,t) = H \cdot \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \det \nabla u_\varepsilon(x,t) \right).$$

In Sec. 3, we derive the equations describing motions of the materials in question. We also give some discussion to function spaces and weak solutions. In Sec. 4, the ordinary differential equations that our special type of solutions must satisfy are derived together with the jump conditions for a radial shock. We then use the work of the Spectors to prove existence for both the almost incompressible and incompressible systems. We discuss convergence matters in Secs. 5 and 6.

3. Equations of motion and admissible solutions. For an elastic material with Piola-Kirchhoff stress tensor S the equation of motion, which describes conservation of linear momentum, is

$$u_{tt}(x,t) = \operatorname{div} S(\nabla u(x,t)).^1 \tag{3.1}$$

Thus for an almost incompressible material

$$u^\varepsilon_{tt}(x,t) = \operatorname{div}\left(S_1(\nabla u^\varepsilon(x,t)) + \frac{\det \nabla u^\varepsilon - C}{\varepsilon}(\operatorname{adj} \nabla u^\varepsilon)^{\mathrm{T}} \right), \tag{3.2}$$

[1] System (3.1) is hyperbolic (*cf.* Dafermos [4]) if the acoustic tensor is positive definite. The acoustic tensor $Q(F,v)$ is given in terms of the stored energy function, W, by $Q_{ij}(F,v) = \sum_{\alpha,\beta} \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}} v_\alpha v_\beta$. For the stored energy function, W_ε, a direct computation gives $Q(F,v) = \mathbf{I} + (\det F)^2(h''(\det F) + 1/\varepsilon)F^{-\mathrm{T}}v \otimes F^{-\mathrm{T}}v$, which is positive definite for each $v \in \mathbf{R}^3$.

where

$$S_1(\nabla u^\varepsilon) = \frac{\partial \Phi}{\partial \nabla u^\varepsilon}.$$

On the other hand, for the incompressible material with basic constitutive relation (2.1), the equations of motion are

$$u_{tt} = \operatorname{div}(S_1(\nabla u) + p(x,t)(\nabla u)^{-\mathrm{T}}) \quad \text{and} \quad \partial_t \det \nabla u = 0.$$

To relate the two sets of equations, we note that we can naturally write (3.2) as a system with $p_\varepsilon(x,t) = (\det \nabla u^\varepsilon - C)/\varepsilon$ also treated as a dependent variable: i.e., consider

$$\begin{gathered} u^\varepsilon_{tt} = \operatorname{div}(S_1(\nabla u^\varepsilon) + p^\varepsilon(x,t) \det \nabla u^\varepsilon (\nabla u^\varepsilon)^{-\mathrm{T}}), \\ \partial_t(\varepsilon p^\varepsilon(x,t) - \det \nabla u^\varepsilon) = 0. \end{gathered} \tag{3.3}$$

Informally, we think of the second equation as a "forcing" equation or a balance equation between time rates of change in large pressure deviations and small volume deviations. The second equation implies

$$p^\varepsilon = \frac{\det \nabla u^\varepsilon - C_\varepsilon(x)}{\varepsilon}.$$

For our special type of solutions, that is, solutions that depend only on the ratio of spatial norm to time (*cf.* (3.8)), $C_\varepsilon(x)$ will be forced to be a constant C_ε. In fact, physical considerations will imply that it is independent of ε and equals H; thus for our class of solutions, (3.2) and (3.3) become equivalent.

Formally letting $\varepsilon = 0$ in (3.3), we obtain

$$\begin{gathered} u_{tt} = \operatorname{div}(S_1(\nabla u) + p(x,t) \det \nabla u (\nabla u)^{-\mathrm{T}}), \\ \partial_t \det \nabla u = 0. \end{gathered}$$

These are the equations of motion for an incompressible material with basic constitutive relation (2.1) modulo the $\det \nabla u$ term in the first equation; thus the pressure convergence is modulo a constant (*cf.* Theorem 3.3). The second equation together with appropriate initial conditions will imply $\det \nabla u$ is constant (independent of space and time). We consider the following problems for some fixed $T > 0$:

$$\begin{gathered} u^\varepsilon_{tt}(x,t) = \operatorname{div}(S_1(\nabla u^\varepsilon) + p^\varepsilon(x,t) \det \nabla u^\varepsilon (\nabla u^\varepsilon)^{-\mathrm{T}}), \\ \partial_t(\varepsilon p^\varepsilon - \det \nabla u^\varepsilon) = 0, \\ u(x,0) = \lambda_\varepsilon, \qquad u_t(x,0) = 0 \end{gathered} \tag{3.4}$$

on Ω $(:= \mathbf{R}^3) \times [0,T]$, and

$$\begin{gathered} u_{tt}(x,t) = \operatorname{div}(S_1(\nabla u) + p(x,t)(\nabla u)^{-\mathrm{T}}), \\ \partial_t \det \nabla u = 0, \\ u(x,0) = \lambda, \qquad u_t(x,0) = 0 \end{gathered} \tag{3.5}$$

on $\Omega \times [0,T]$. The natural question arises as to whether solutions of (3.4) are close to those of (3.5) for small ε and λ_ε close to λ. For the solutions we construct, we show that the answer is affirmative.

Following [1] and [9], we give some discussion as to what we mean by a solution. Since we deal with the material occupying all of $\mathbf{R}^3$, we adjust as follows. Define

$$D_p(\mathbf{R}^3) = \{Y \in W^{1,p}_{\text{loc}}(\mathbf{R}^3, \mathbf{R}^3) \mid \det \nabla Y(x) > 0 \text{ for a.e. } x \in \mathbf{R}^3\}$$

to be the set of admissible deformations of the material. For $T > 0$, $u : [0,T] \to D_p(\mathbf{R}^3)$ is called a motion if

$$\text{(i) } u \in C^0([0,T], W^{1,p}_{\text{loc}}(\mathbf{R}^3, \mathbf{R}^3)) \quad \text{and} \quad \text{(ii) } u \in C^1([0,T], L^p_{\text{loc}}(\mathbf{R}^3, \mathbf{R}^3)).$$

We look for weak solutions to (3.4) and (3.5) in which $u^\varepsilon(x,t)$ (resp. $u(x,t)$) is a motion. For example, for (3.4) this amounts to requiring that for every $t \in [0,T]$, $S_\varepsilon(\nabla u^\varepsilon(\cdot,t)) \in L^1_{\text{loc}}(\mathbf{R}^3)$, and for every $\psi \in C_0^\infty(\mathbf{R}^3 \times (0,T), \mathbf{R}^3), \theta \in C_0^\infty(\mathbf{R}^3 \times (0,T), \mathbf{R})$, we have

$$\int_0^{\mathrm{T}} \int_{\mathbf{R}^3} [\nabla\psi \cdot S_\varepsilon(\nabla u^\varepsilon) - \psi_t \cdot u^\varepsilon_t] \, dx \, dt = 0,$$
$$\int_0^{\mathrm{T}} \int_{\mathbf{R}^3} \theta_t \cdot (\varepsilon p^\varepsilon - \det \nabla u^\varepsilon) \, dx \, dt = 0,$$

where $S_\varepsilon(\nabla u^\varepsilon) = S_1 + p_\varepsilon(x,t) \det \nabla u^\varepsilon (\nabla u^\varepsilon)^{-\mathrm{T}}$. $N \cdot M = \operatorname{trace}(NM^{-\mathrm{T}})$ is the standard scalar product of $n \times n$ matrices N and M.

We work within the class of radially symmetric solutions:

$$u(x,t) = \frac{r(R,t)}{R} x, \tag{3.6}$$

where $R = |x|$, the standard Euclidean norm. We state the following lemmas of Ball [1] and the Spectors [9], noting that in their cases, Ω is a ball of fixed radius. However, working in the natural spaces of $W^{1,p}_{\text{loc}}$ and L^p_{loc} makes their results pertinent.

LEMMA 3.1. Consider a radial deformation $f(x) = \frac{r(R)}{R}x$. Then $f \in D_p$ iff for every $\rho > 0$, r is absolutely continuous on $(0,\rho), r'(\frac{r}{R})^2 > 0$ almost everywhere, and

$$\int_0^\rho \left[|r'(R)|^p + \left| \frac{r(R)}{R} \right|^p \right] R^2 \, dR < \infty.$$

In this case, the weak derivatives of f are given by

$$\nabla f(x) = \frac{r(R)}{R}\mathbf{I} + \left[r'(R) - \frac{r(R)}{R} \right] \frac{x}{R} \otimes \frac{x}{R} \quad \text{a.e. } x \in \mathbf{R}^3. \tag{3.7}$$

We look for solutions having the following form:

$$u^\varepsilon(x,t) = \begin{cases} \frac{\phi_\varepsilon(s)}{s} x & \text{if } t > 0, \\ \lambda_\varepsilon x & \text{if } t = 0, \end{cases} \tag{3.8}$$

where

$$s = \frac{|x|}{t},$$

and $\phi_\varepsilon(s) \in PC^1$, the space of piecewise C^1 functions.

LEMMA 3.2. Let $p < 3$. Let u^ε be as above and assume there exists some $\sigma_\varepsilon > 0$ such that $\phi_\varepsilon(s) = \lambda_\varepsilon s$ for $s > \sigma_\varepsilon$. If $\phi_\varepsilon \in PC^1([0,\infty),[0,\infty))$ and $\dot{\phi}_\varepsilon > 0$ a.e. then u is a motion.

We note that Lemma 3.2 also holds if for some constant $\lambda, \dot{\phi} > 0, \phi - \lambda s \to 0$, and $\frac{d}{ds}\phi \to \lambda$. These hypotheses will pertain to our limit solution (*cf.* Theorem 4.5).

Our analysis of the motions will be based on analysis of the ϕ_ε that will all satisfy $\phi_\varepsilon(0) = a$ for some fixed $a > 0$ (thus a will be a parameter for sets of converging solutions). The resulting solution u^ε will be discontinuous; for $t > 0$, a cavity will form at $x = \mathbf{0}$ and expand with speed a, i.e., $\lim_{x\to 0} |u_t^\varepsilon(x,t)| = a$. Solutions with $a = 0$ would be forced to be trivial static deformations. As previously discussed there is strong evidence to indicate that motions with cavities have lower energy. In order to obtain existence of the u^ε via the ϕ_ε, we must allow for a possible jump discontinuity in $\dot{\phi}_\varepsilon(s)$.[2] We will see that the resulting ordinary differential equation that ϕ_ε must comply with may not have solutions for all s. Extending a solution for all s, which in view of (3.8) is essential, may require a jump in the first derivative of ϕ_ε. This will produce a discontinuity in u_t^ε on a sphere that propagates with speed equal to the position of the discontinuity in $\dot{\phi}_\varepsilon$.

We now state the main result of this paper.

THEOREM 3.3. For every $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ converging to $H^{1/3}$ and dynamic weak solutions $u^\varepsilon, p^\varepsilon$ of (3.4). The u^ε converge pointwise and uniformly on compact subsets of $\mathbf{R}^3 \times [0,T]$ to a solution of (3.5) for $\lambda = H^{1/3}$ with the associated arbitrary hydrostatic pressure $p(x,t)$ in (3.5) being the limit, modulo a constant, of the $p^\varepsilon(\det \nabla u^\varepsilon)$. That is,

$$p^\varepsilon(\det \nabla u^\varepsilon) \to \frac{p(x,t)}{H}$$

pointwise on $\mathbf{R}^3 \times (0,T]$ and uniformly on compact subsets.

An immediate consequence is the following.

COROLLARY 3.4. For every $(x,t) \in \mathbf{R}^3 \times (0,T]$, the specific volume $v_\varepsilon(x,t) = \det \nabla u^\varepsilon$ is differentiable with respect to the parameter ε at $\varepsilon = 0$ and

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \det \nabla u^\varepsilon(x,t) = \frac{p(x,t)}{H}.$$

4. Solutions to the compressible and incompressible equations. For u a radially symmetric motion, i.e., $u(x,t) = \frac{r(R,t)}{R}x$, and $t \in [0,T]$,

$$\nabla u(x,t) = \frac{r(R,t)}{R}\mathbf{I} + \left[r_R(R,t) - \frac{r(R,t)}{R}\right]\frac{x}{R} \otimes \frac{x}{R} \quad \text{for a.e. } x \in \mathbf{R}^3.$$

We look for a solution to (3.4), for some appropriate λ_ε, of the form

$$u^\varepsilon(x,t) = \begin{cases} \frac{\phi_\varepsilon(s)}{s}x & \text{if } t > 0, \\ \lambda_\varepsilon x & \text{if } t = 0, \end{cases}$$

$$p^\varepsilon(x,t) = \tilde{p}^\varepsilon(s) \quad \text{where } s = \frac{|x|}{t}.$$

[2]In fact, there are numerical experiments verifying that ϕ_ε is not C^1 and hence shocks do indeed exist.

We first solve the second equation of (3.4) to obtain

$$\tilde{p}^{\varepsilon} = \frac{\det \nabla u^{\varepsilon} - C_{\varepsilon}}{\varepsilon} = \frac{\dot{\phi}_{\varepsilon}(\frac{\phi_{\varepsilon}}{s})^2 - C_{\varepsilon}}{\varepsilon}, \tag{4.1}$$

where $\cdot$ denotes $\frac{d}{ds}$. Let $v_{\varepsilon}(s) = \det \nabla u^{\varepsilon} = \dot{\phi}_{\varepsilon}(\frac{\phi_{\varepsilon}}{s})^2$, the specific volume.

To reduce the first equation of (3.4) to an ordinary differential equation for ϕ_{ε} we follow [9]. Our pressure creating function will have the form

$$h(v) + \frac{(v - C_{\varepsilon})^2}{2\varepsilon}. \tag{4.2}$$

We briefly describe the steps. For simplicity first consider a generic u, stored energy function Φ, and related stress S (we leave out index ε). We then use the specific form of Φ_{ε} with (4.2) incorporated.

By (3.7), the eigenvalues of ∇u are $r_R(R,t)$, $\frac{r(R,t)}{R}$, $\frac{r(R,t)}{R}$. A direct calculation gives

$$(\operatorname{adj} \nabla u)^{\mathrm{T}} = \left[r_R \frac{r}{R}\mathbf{I} - \left(r_R \frac{r}{R} - \left(\frac{r}{R}\right)^2 \right) \frac{x}{R} \otimes \frac{x}{R} \right]. \tag{4.3}$$

Using the notation of [9], let

$$\widehat{\Phi}_i(R,t) = \Phi,_i \left(r_R(R,t), \frac{r(R,t)}{R}, \frac{r(R,t)}{R} \right), \tag{4.4}$$

where $\Phi,_i$ is the derivative of Φ with respect to the ith variable component. It follows that (*cf.* Ball [1, p. 568]),

$$S(\nabla u) = \widehat{\Phi}_2 \mathbf{I} + (\widehat{\Phi}_1 - \widehat{\Phi}_2) \frac{x}{R} \otimes \frac{x}{R}. \tag{4.5}$$

Equation (3.1) reduces to

$$\frac{\partial}{\partial R}(R^2 \widehat{\Phi}_1) - 2R\widehat{\Phi}_2 = R^2 r_{tt}. \tag{4.6}$$

If $\frac{r(R,t)}{R} = \frac{\phi(s)}{s}$ for $s = \frac{R}{t}$ then (4.6) reduces to

$$\frac{d}{ds}(s^2 \widehat{\Phi}_1(s)) - 2s\widehat{\Phi}_2(s) = s^4 \ddot{\phi}.$$

Here we have used $\widehat{\Phi}_i(s)$ to denote $\Phi,_i(\dot{\phi}(s), \frac{\phi(s)}{s}, \frac{\phi(s)}{s})$. Carrying out the differentiation and using the specific form of Φ^{ε} for each ε, we obtain

$$\ddot{\phi}_{\varepsilon} = \frac{\frac{2}{s}(\dot{\phi}_{\varepsilon} - \frac{\phi_{\varepsilon}}{s})(1 + \dot{\phi}_{\varepsilon}(\frac{\phi_{\varepsilon}}{s})^3 (h''(v_{\varepsilon}(s)) + \frac{1}{\varepsilon}))}{s^2 - 1 - (\frac{\phi_{\varepsilon}}{s})^4 (h''(v_{\varepsilon}(s)) + \frac{1}{\varepsilon})}. \tag{4.7}$$

As previously discussed, our solutions will exhibit a jump in u_t^{ε} on a propagating spherical surface $\Sigma(t)$, namely a radial shock. In order for the motion $u^{\varepsilon}, p^{\varepsilon}$ to be an integral (weak) solution to (3.4) the jump and speed of the radial shock, $\mathfrak{s}$ must comply with the Rankine-Hugoniot jump conditions. For (3.4) these are (*cf.* [12])

$$[[S_0 \cdot n]] + [[p^\varepsilon \operatorname{adj} \nabla u^\varepsilon \cdot n]] + \mathfrak{s}[[u_t^\varepsilon]] = 0 \quad \text{on } \Sigma(t),$$
$$[[\varepsilon p^\varepsilon - \det \nabla u^\varepsilon]] = 0,$$

where $[[u_t^\varepsilon]]$ denotes the jump in u_t^ε across $\Sigma(t)$. Using the fact that u has radial form (3.6), (4.3), and (4.5) we obtain

$$[[\widehat{\Phi}_1^\varepsilon]] + \left[\left[p^\varepsilon \left(\frac{r^\varepsilon(R)}{R}\right)^2\right]\right] + \mathfrak{s}[[r_t^\varepsilon]] = 0 \quad \text{on } \Sigma(t).$$

Writing $\Sigma(t)$ as $\mathfrak{s}t\Sigma$ where Σ denotes the unit sphere in $\mathbf{R}^3$ and using (3.8) we have

$$[[\widehat{\Phi}_1^\varepsilon(\mathfrak{s})]] + \left[\left[\frac{1}{\varepsilon}\dot{\phi}_\varepsilon(\mathfrak{s})\left(\frac{\phi_\varepsilon(\mathfrak{s})}{\mathfrak{s}}\right)^4\right]\right] = \mathfrak{s}^2[[\dot{\phi}_\varepsilon]].$$

More precisely,

$$\left(\mathfrak{s}^2 - 1 - \frac{1}{\varepsilon}\left(\frac{\phi_\varepsilon(\mathfrak{s})}{\mathfrak{s}}\right)^4\right)(\dot{\phi}_{\varepsilon_+}(\mathfrak{s}) - \dot{\phi}_{\varepsilon_-}(\mathfrak{s})) = \left(\frac{\phi_\varepsilon(\mathfrak{s})}{\mathfrak{s}}\right)^2 (h'(v_{\varepsilon_+}) - h'(v_{\varepsilon_-})). \tag{4.8}$$

Note that because of the sole dependence of the u^ε on $|x|/t$, $\mathfrak{s}$ takes on the dual role of speed and position of the shock.

At this stage we should give some thought to what C is in (2.4) (or C_ε is in (4.1)) and why we choose it to be H. We will solve for $\phi(\phi_\varepsilon)$ with initial conditions $\phi(0) = a$, for some fixed $a > 0$. This has the effect, on the corresponding motion, of creating a cavity at $x = \mathbf{0}$ that expands with speed a. We require the radial component of the Cauchy stress on this cavity to be zero: that is, the cavity should consist of a traction-free surface. We assume that the extra constitutive pressure in the almost incompressible case has no effect on the cavity. That is,

$$\tilde{p}^\varepsilon(0) = \frac{v_\varepsilon(0) - C_\varepsilon}{\varepsilon} = 0,$$

and hence $v_\varepsilon(0) = C_\varepsilon$.

To compute the radial component of the Cauchy stress we proceed as follows. The relation between the Cauchy stress T_ε and the Piola-Kirchhoff stress S_ε is

$$T_\varepsilon(\nabla u^\varepsilon) = S_\varepsilon(\nabla u^\varepsilon)(\nabla u^\varepsilon)^{\mathrm{T}} \det \nabla u_\varepsilon.$$

We wish to compute $T_\varepsilon \cdot \frac{x}{R} \cdot \frac{x}{R}$. For u^ε satisfying (3.8) we find with the aid of (3.7) and (4.5) that the radial component of the stress is

$$T_\varepsilon(s) = \left(\frac{s}{\phi_\varepsilon}\right)^2 \widehat{\Phi}_1^\varepsilon\left(\dot{\phi}_\varepsilon, \frac{\phi_\varepsilon}{s}, \frac{\phi_\varepsilon}{s}\right). \tag{4.9}$$

Using the specific form of $\widehat{\Phi}_1^\varepsilon$,

$$T_\varepsilon(s) = v_\varepsilon(s)\left(\frac{\phi_\varepsilon}{s}\right)^{-4} + h'(v_\varepsilon(s)) + \left(\frac{v_\varepsilon(s) - C_\varepsilon}{\varepsilon}\right). \tag{4.10}$$

Taking into account the properties of h (*cf.* (2.2)) and that $\lim_{s\to 0^+} T_\varepsilon(s) = 0$, we have $C_\varepsilon = H$. If the assumption that $\tilde{p}^\varepsilon(0) = 0$ seems unmotivated, we could simply use $C = H$ from the start and choose appropriate solutions of (3.4 ii): that is, regard $\tilde{p}^\varepsilon(0) = 0$ as a boundary condition for p^ε on the cavity. In view of the traction-free cavity and the properties of h (*cf.* (2.2)), we would then obtain $v_\varepsilon(0) = H$ for every ε.

We now proceed towards a solution to (4.7). Equation (4.7) has a regular singular point at $s = 0$. We wish to solve with initial conditions

$$\phi_\varepsilon(0) = a, \qquad \dot{\phi}_\varepsilon = 0, \qquad T_\varepsilon(0) = 0. \tag{4.11}$$

The traction-free cavity condition is responsible for the latter two conditions. All three are necesssary to establish existence and uniqueness to (4.7) with its singularity at $s = 0$. With these initial conditions, it is natural to write (4.7) as an equivalent first-order system with dependent variables T_ε and ϕ_ε. Thus consider

$$\begin{aligned} \dot{T}_\varepsilon(s) &= P_\varepsilon(T_\varepsilon(s), \phi_\varepsilon(s), s), \\ \dot{\phi}_\varepsilon(s) &= Q_\varepsilon(T_\varepsilon(s), \phi_\varepsilon(s), s), \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} P_\varepsilon(T_\varepsilon(s), \phi_\varepsilon(s), s) &= \frac{s^3}{\phi_\varepsilon^2} R_\varepsilon - 2\left[Q_\varepsilon^2 \frac{s^2}{\phi_\varepsilon^3} - \frac{1}{\phi_\varepsilon}\right], \\ R_\varepsilon(T_\varepsilon(s), \phi_\varepsilon(s), s) &= 2(sQ_\varepsilon - \phi_\varepsilon)\frac{s^3 + Q_\varepsilon \phi_\varepsilon^3[h''(\hat{v}_\varepsilon(T_\varepsilon, \frac{s}{\phi_\varepsilon})) + \frac{1}{\varepsilon}]}{s^6 - s^4 - \phi_\varepsilon^4[h''(\hat{v}_\varepsilon(T_\varepsilon, \frac{s}{\phi_\varepsilon})) + \frac{1}{\varepsilon}]}, \\ Q_\varepsilon(T_\varepsilon(s), \phi_\varepsilon(s), s) &= \left\{\frac{s}{\phi_\varepsilon}\right\}^2 \hat{v}_\varepsilon\left(T_\varepsilon, \frac{s}{\phi_\varepsilon}\right). \end{aligned}$$

The system (4.12) is obtained by differentiating (4.10) and using (4.7) several times together with the following definition of $\hat{v}_\varepsilon$. The function $\hat{v}_\varepsilon$ is a $C^2(\mathbf{R}^2, \mathbf{R}^+)$ function that will play a crucial role in the passage to the limit. The idea is that, for any ε, given a value of the Cauchy stress, ϕ, and s, we want to be able to recover the specific volume $v(s)$ and hence $\dot{\phi}(s)$, which together with ϕ and s give that value of the stress. We manipulate the particular algebraic structure of the equation that relates the Cauchy stress to v, ϕ, s, and ε. Specifically, define $\widehat{E}_\varepsilon : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ as follows:

$$\widehat{E}_\varepsilon(v, w) = vw^4 + h'(v) + \frac{1}{\varepsilon}(v - H).$$

For each $w \in \mathbf{R}$, $\widehat{E}_\varepsilon(\cdot, w)$ is surjective (*cf.* (2.2)) and hence there exists $\hat{v}_\varepsilon : \mathbf{R}^2 \to \mathbf{R}^+$ such that for every $E, w \in \mathbf{R}$,

$$\widehat{E}_\varepsilon(\hat{v}_\varepsilon(E, w), w) = E.$$

Also, (2.2) implies $\widehat{E}_\varepsilon \in C^2(\mathbf{R}^+ \times \mathbf{R}, \mathbf{R})$ and the derivative of $\widehat{E}_\varepsilon$ with respect to v is greater than 0. Thus by the Implicit Function Theorem, $\hat{v}_\varepsilon \in C^2(\mathbf{R}^2, \mathbf{R}^+)$. By construction we note that if $T_\varepsilon(s), \phi_\varepsilon(s)$ are as in (4.10), then

$$\hat{v}_\varepsilon\left(T_\varepsilon(s), \frac{s}{\phi_\varepsilon(s)}\right) = v_\varepsilon(s) = \left(\frac{\phi_\varepsilon(s)}{s}\right)^2 \dot{\phi}_\varepsilon(s).$$

The initial conditions corresponding to (4.12) are

$$\phi_\varepsilon(0) = a, \qquad T_\varepsilon(0) = 0. \tag{4.13}$$

The proofs of the following propositions follow from the respective propositions in [9]. We make a few remarks for each proof.

PROPOSITION 4.1. For every $\varepsilon > 0$ there exists a unique solution to (4.12) and (4.13) and hence to (4.7), (4.11) on some interval $[0, s_{0_\varepsilon}]$. Moreover on $[0, s_{0_\varepsilon}]$,

$$\dot{\phi}_\varepsilon(s) > 0, \qquad \ddot{\phi}_\varepsilon(s) > 0, \qquad s^2 - 1 - \left(\frac{\phi_\varepsilon}{s}\right)^4 \left(h''(v_\varepsilon(s)) + \frac{1}{\varepsilon}\right) < 0, \tag{4.14}$$

$$\dot{\phi}_\varepsilon(s) < \frac{\phi_\varepsilon(s)}{s}, \qquad \dot{v}_\varepsilon(s) \geq 0, \qquad v_\varepsilon(0) = H. \tag{4.15}$$

Proof. The first part follows from the standard existence and uniqueness theorem in ordinary differential equations. The second inequality of (4.15) follows from differentiating $v_\varepsilon(s)$ and using (4.7). □

PROPOSITION 4.2. The solution can be extended uniquely to a maximal interval of existence $[0, s_{M_\varepsilon})$ where $s_{M_\varepsilon} < \infty$ is precisely where the denominator of (4.7) is 0. Moreover, we have

$$\phi_\varepsilon(s) \in C^1([0, s_{M_\varepsilon}], \mathbf{R}) \cap C^2([0, s_{M_\varepsilon}), \mathbf{R}),$$

and (4.14), (4.15) hold for all $s \in [0, s_{M_\varepsilon})$.

Proof. Continuation of the solution is by the usual argument. Note that the inequalities $\dot{\phi}_\varepsilon > 0$ and $\ddot{\phi}_\varepsilon > 0$ imply that both ϕ_ε and $\dot{\phi}_\varepsilon$ must tend to some limit as s approaches s_{M_ε}. The first inequality of (4.15) follows by uniqueness and the fact that for any $c > 0$, cs is a solution to (4.7). This inequality implies that ϕ_ε and $\dot{\phi}_\varepsilon$ must tend to finite limits as s approaches s_{M_ε}. Finally, $s_{M_\varepsilon} < \infty$ follows from assuming otherwise and concluding with the aid of the first inequality of (4.15), that $(\frac{\phi_\varepsilon}{s})^4(h''(v_\varepsilon(s)) + \frac{1}{\varepsilon})$ is bounded from below and above on $[s_{0_\varepsilon}, \infty)$. This contradicts the fact that (4.14 iii) holds on $[0, s_{M_\varepsilon})$. □

PROPOSITION 4.3. For every $\varepsilon > 0$, either

$$\dot{\phi}_\varepsilon(s_{M_\varepsilon}) = \frac{\phi_\varepsilon(s_{M_\varepsilon})}{s_{M_\varepsilon}},$$

or there exists $s_{J_\varepsilon} < s_{M_\varepsilon}$ such that $\dot\phi_\varepsilon(s_{J_\varepsilon}) < \frac{\phi_\varepsilon(s_{J_\varepsilon})}{s_{J_\varepsilon}}$ and the Rankine-Hugoniot jump conditions are satisfied at s_{J_ε} for $\dot\phi_{\varepsilon_+} = \frac{\phi_\varepsilon(s_{J_\varepsilon})}{s_{J_\varepsilon}}$ and $\dot\phi_{\varepsilon_-} = \dot\phi_\varepsilon(s_{J_\varepsilon})$. That is,

$$\left(s_{J_\varepsilon}^2 - 1 - \frac{1}{\varepsilon}\left(\frac{\phi_\varepsilon(s_{J_\varepsilon})}{s_{J_\varepsilon}}\right)^4\right)(\dot\phi_{\varepsilon_+}(s_{J_\varepsilon}) - \dot\phi_{\varepsilon_-}(s_{J_\varepsilon})) \\ = \left(\frac{\phi_\varepsilon(s_{J_\varepsilon})}{s_{J_\varepsilon}}\right)^2 (h'(v_{\varepsilon_+}) - h'(v_{\varepsilon_-})).$$

Proof. Assume that equality fails and hence $\dot\phi_\varepsilon(s_{M_\varepsilon}) < \frac{\phi_\varepsilon(s_{M_\varepsilon})}{s_{M_\varepsilon}}$. Let $\lambda_{M_\varepsilon} = \frac{\phi_\varepsilon(s_{M_\varepsilon})}{s_{M_\varepsilon}}$. Define

$$F(s) := -\left(s^2 - 1 - \frac{1}{\varepsilon}\left(\frac{\phi_\varepsilon(s)}{s}\right)^4\right)\left(\frac{\phi_\varepsilon(s)}{s} - \dot\phi_\varepsilon(s)\right) \\ + \left(\frac{\phi_\varepsilon(s)}{s}\right)^2\left(h'\left(\left(\frac{\phi_\varepsilon(s)}{s}\right)^3\right) - h'\left(\left(\frac{\phi_\varepsilon(s)}{s}\right)^3 \dot\phi_\varepsilon(s)\right)\right).$$

We apply the Intermediate Value Theorem to F on the interval $[0, s_{M_\varepsilon}]$. Here it is crucial that the third derivative of h is always negative. □

Define $\lambda_\varepsilon = \frac{\phi_\varepsilon(s_{M_\varepsilon})}{s_{M_\varepsilon}}$ in the first case of Prop. 4.3 and $\lambda_\varepsilon = \frac{\phi_\varepsilon(s_{J_\varepsilon})}{s_{J_\varepsilon}}$ in the second. Define

$$s_\varepsilon = \begin{cases} s_{M_\varepsilon} & \text{if } \dot\phi_\varepsilon(s_{M_\varepsilon}) = \frac{\phi_\varepsilon(s_{M_\varepsilon})}{s_{M_\varepsilon}}, \\ s_{J_\varepsilon} & \text{otherwise.} \end{cases} \tag{4.16}$$

We use this λ_ε to construct a ϕ_ε defined and continuous for all $s \geq 0$ and piecewise $C^2[0,\infty]$ as our candidate for a solution of (3.4) via (3.8). Precisely, define

$$\phi_\varepsilon(s) = \begin{cases} \text{previous } \phi_\varepsilon(s) & \text{if } s < s_\varepsilon, \\ \lambda_\varepsilon s & \text{if } s \geq s_\varepsilon. \end{cases}$$

Note the abuse of notation regarding the definition of ϕ_ε. Since

$$H \leq v_\varepsilon(s) = \dot\phi_\varepsilon(s)\left(\frac{\phi_\varepsilon(s)}{s}\right)^2 \leq \left(\frac{\phi_\varepsilon(s)}{s}\right)^3, \text{ we have } \lambda_\varepsilon \geq H^{1/3}. \tag{4.17}$$

Applying the Mean Value Theorem to (4.8), we obtain

$$s_{J_\varepsilon}^2 = 1 + \left(\frac{\phi_\varepsilon(s_{J_\varepsilon})}{s_{J_\varepsilon}}\right)^4\left(h''(c_\varepsilon) + \frac{1}{\varepsilon}\right), \tag{4.18}$$

where $c_\varepsilon \in (v_\varepsilon(s_{J_\varepsilon}), \lambda_\varepsilon^3)$. In view of (4.17), (4.18), and Proposition 4.2, we have

$$s_\varepsilon \to \infty \quad \text{as } \varepsilon \to 0.$$

THEOREM 4.4. Let

$$u^\varepsilon(x,t) = \begin{cases} \frac{\phi_\varepsilon(s)}{s}x & \text{if } t > 0, \\ \lambda_\varepsilon x & \text{if } t = 0, \end{cases}$$

$$p^\varepsilon(x,t) = \begin{cases} \frac{\dot\phi_\varepsilon(\frac{\phi_\varepsilon}{s})^2 - H}{\varepsilon} & \text{if } t > 0, \\ \frac{\lambda_\varepsilon^3 - H}{\varepsilon} & \text{if } t = 0. \end{cases}$$

Then for each $\varepsilon > 0$, $(u^\varepsilon(x,t), p^\varepsilon(x,t))$ is a weak solution of (3.4).

Proof. It is clear that this choice of p^ε gives a weak solution to the second equation. With this choice, we are in the situation of the Spectors; so their proof applies. We make a few points. By Lemma 3.2, u^ε is a motion. Let

$$I_\delta = \int_\alpha^{\mathrm{T}} \int_{B_\rho \setminus B_\delta} [\nabla\psi \cdot S_\varepsilon(\nabla u^\varepsilon) - \psi_t \cdot u_t^\varepsilon]\, dx\, dt = 0,$$

where the support of ψ is contained in $B_\rho \times [\alpha, T]$ for some $\alpha > 0$ and $\rho > 0$. We show $S_\varepsilon(\nabla u^\varepsilon(\cdot,t)) \in L^1_{\mathrm{loc}}(\mathbf{R}^3)$ by using (4.4), the condition of zero traction on the cavity, and the fact that $s\widehat{\Phi}_2^\varepsilon \in L^1((0,s_\varepsilon),\mathbf{R})$. By Lebesgue dominated convergence, it suffices to show $I_\delta \to 0$ as $\delta \to 0$. This is done by breaking up the region of integration into pre and post shock regions and then integrating by parts and using a form of the divergence theorem. We use the Rankine-Hugoniot jump conditions to obtain no contribution on the shock itself, and of course use the fact that in regions of smoothness, u_ε via ϕ_ε actually solves the equations. It then turns out that the I_δ is equal to an integral whose integrand is bounded by the radial component of the Cauchy stress at $s = \delta/t$ and it is thus the zero traction on the cavity that allows us to conclude $I_\delta \to 0$ since the integrand tends to zero uniformly on $[\alpha, T]$. $\square$

Let us pause to note a few properties of the motion u^ε. After $t = 0$, a spherical cavity and a spherical shock form at $x = \mathbf{0}$ and propagate with speeds a and s_ε respectively. At any time $t > 0$, if $|x| > s_\varepsilon t, u^\varepsilon(x,t) = \lambda_\varepsilon x$; that is, material points remain fixed in relation to $t = 0$. Moreover, let $\rho_\varepsilon = s_\varepsilon T$. If $|x| > \rho_\varepsilon$, $u^\varepsilon(x,t) = \lambda_\varepsilon x$ (x is left fixed) for all $t \in [0,T]$. As was mentioned in the introduction, these motions have been shown to minimize energy in the occurrence of shocks. The cavitation occurs as a reaction to the fixed displacement properties of the motion in order to dissipate as much energy as possible; see [1] and [9].

Finally, we consider the incompressible system (3.5). The second equation of (3.5) implies

$$\det \nabla u(x,t) = \lambda^3 \quad \text{or} \quad \dot\phi\left(\frac{\phi}{s}\right)^2 = \lambda^3.$$

For the first equation, we note that

$$S_1(\nabla u) = S_0(\nabla u) + h'(\det \nabla u)\,\mathrm{adj}\,\nabla u,$$

where $S_0(\nabla u) = \frac{\partial\Phi^0}{\partial F}, \Phi^0 = \frac{1}{2}\sum\lambda_i^2$. Since $\det\nabla u$ is constant for isochoric motions, the second component of S_1 is of the form (constant) $(\nabla u)^{-\mathrm{T}}$. We absorb this into the arbitrary hydrostatic pressure and find the reduction of

$$u_{tt} = \mathrm{div}[S_0(\nabla u) + p(x,t)(\nabla u)^{-\mathrm{T}}]$$

for a solution with form (3.8) and $p(x,t) = \tilde{p}(s)$. Using the same method as before, and noting that for a radial motion of the form (3.6),

$$(\nabla u)^{-\mathrm{T}} = \frac{1}{\lambda^3}\left[r_R \frac{r}{R}\mathbf{I} + \left(\left(\frac{r}{R}\right)^2 - r_R\frac{r}{R}\right)\frac{x}{R}\otimes\frac{x}{R}\right], \tag{4.19}$$

we obtain the following system for $\phi(s)$ and $\tilde{p}(s)$:

$$\ddot{\phi}(s) = \frac{\frac{2}{s}(\dot{\phi}(s) - \frac{\phi(s)}{s}) + \frac{\dot{\tilde{p}}(s)}{\lambda^3}(\frac{\phi(s)}{s})^2}{s^2-1},$$
$$\dot{\phi}\left(\frac{\phi}{s}\right)^2 = \lambda^3.$$

We choose $\lambda = H^{1/3}$.

These equations are readily solved with initial conditions $\phi(0) = a$, $\dot{\phi}(0) = 0$, and $T_I(0) = 0$, where $T_I(s)$ is the radial component of the Cauchy stress at s, i.e.,

$$T_I(s) = v(s)\left(\frac{\phi}{s}\right)^{-4} + \tilde{p}(s) = H\left(\frac{\phi}{s}\right)^{-4} + \tilde{p}(s).$$

Note that $T_I(0) = 0$ implies $\tilde{p}(0) = 0$. The solution is

$$\phi(s) = (Hs^3 + a^3)^{\frac{1}{3}},$$
$$\tilde{p}(s) = \int_0^s \left\{\frac{2a^3H}{\tau^4}\right\}\left[\frac{H\tau^2 + \frac{a^3}{\tau^3}}{(H + \frac{a^3}{\tau^3})^{\frac{7}{3}}}\right] d\tau. \tag{4.20}$$

Let

$$p_* = \lim_{s\to+\infty} \tilde{p}(s).$$

Note that $p_* < \infty$.

THEOREM 4.5. Let

$$u(x,t) = \begin{cases} \frac{\phi(s)}{s}x & \text{if } t > 0, \\ H^{\frac{1}{3}}x & \text{if } t = 0, \end{cases}$$
$$p(x,t) = \begin{cases} \tilde{p}(s) & \text{if } t > 0, \\ p_* & \text{if } t = 0. \end{cases}$$

Then $(u(x,t), p(x,t))$ is a weak solution of (3.5).

Proof. The proof is similar to that of Theorem 4.4 and in fact, given the absence of shocks, considerably easier. □

5. Incompressible limit. We now proceed with investigating the behavior of the u^ε, via the ϕ_ε, as ε tends to 0. We have the following:

THEOREM 5.1. $\phi_\varepsilon(s)$ converges to $\phi(s)$ pointwise on $[0,\infty]$, uniformly on compact subsets, where

$$\phi(s) = (Hs^3 + a^3)^{\frac{1}{3}},$$

and $\tilde{p}_\varepsilon(s)$ converges to $\tilde{p}(s)/H$ where $\tilde{p}(s)$ is the associated hydrostatic pressure with the above isochoric motion for the equations of incompressibility, i.e., $\tilde{p}(s)$ is given by (4.20).

COROLLARY 5.2. λ_ε converges to $H^{1/3}$.

COROLLARY 5.3. The shock strength $\dot{\phi}_\varepsilon(s_{J_\varepsilon}) - \frac{\phi_\varepsilon(s_{J_\varepsilon})}{s_{J_\varepsilon}}$ approaches 0 as ε tends to 0.

The proof of the following lemma is standard in the theory of ordinary differential equations; for example, see Birkhoff and Rota [2].

LEMMA 5.4. Suppose $\alpha, s_0 > 0$. Consider the systems

$$\dot{x}_\varepsilon = F_\varepsilon(x_\varepsilon, s) \quad \text{and} \quad \dot{y} = F(y,s), \tag{5.1}$$

together with $x_\varepsilon(0) = x_0 = y(0)$. Assume existence and uniqueness to both initial-value problems on $[0, s_0]$. Let

$$A = \{(x,s) \in \mathbf{R}^n \times \mathbf{R} \mid s \in [0,s_0],\ |x - y(s)| \le \alpha\}.$$

Suppose $F_\varepsilon(x,s)$ is Lipchitz continuous on A and converges uniformly to $F(x,s)$ on A. Then $x_\varepsilon(s)$ converges uniformly to $y(s)$ on $[0,s_0]$.

Proof of Theorem 5.1. Recall the system involving $T_\varepsilon, \phi_\varepsilon$ which was the basis for existence and uniqueness of (4.11), (4.12):

$$\dot{\phi}_\varepsilon = Q_\varepsilon(T_\varepsilon, \phi_\varepsilon, s) \qquad \dot{T}_\varepsilon = P_\varepsilon(T_\varepsilon, \phi_\varepsilon, s).$$

Now consider the system

$$\begin{aligned}\dot{\phi} = Q(T,\phi,s) &= \left(\frac{s}{\phi}\right)^2 H,\\ \dot{T} = P(T,\phi,s) &= \frac{s^3}{\phi^2}\left(2(sQ-\phi)\left(-\frac{Q}{\phi}\right)\right) - 2\left(\frac{Q^2 s^2}{\phi^3} - \frac{1}{\phi}\right),\end{aligned}$$

together with the initial conditions $\phi(0) = a$ and $T(0) = 0$. This has the unique solution

$$\phi(s) = (Hs^3 + a^3)^{\frac{1}{3}},$$

$$\begin{aligned} T(s) &= H\left(\frac{\phi}{s}\right)^{-4} + \int_0^s \frac{2a^3(H\tau^2 + \frac{a^3}{\tau^3})}{\tau^4(H + \frac{a^3}{\tau^3})^{\frac{7}{3}}}\,d\tau \\ &= H\left(\frac{\phi}{s}\right)^{-4} + \frac{\tilde{p}(s)}{H}, \end{aligned} \tag{5.3}$$

where $\tilde{p}(s)$ is given by (4.20). Let $s_0 > 0$. Choose ε sufficiently small such that $s_\varepsilon > s_0$ and let

$$A = \{(\psi, E, s) \mid s \in [0,s_0], |\phi(s) - \psi|^2 + |T(s) - E|^2 \le \frac{a}{2}\}.$$

For $(\psi, E, s) \in A$, we show $\hat{v}_\varepsilon(E, \frac{s}{\psi})$ is bounded uniformly in ε. Recall $\hat{v}_\varepsilon$ is the function defined in Sec. 4 that satisfies

$$\varepsilon E = \varepsilon \hat{v}_\varepsilon \left(E, \frac{s}{\psi}\right)\left(\frac{\psi}{s}\right)^{-4} + \varepsilon h'\left(\hat{v}_\varepsilon\left(E, \frac{s}{\psi}\right)\right) + \hat{v}_\varepsilon\left(E, \frac{s}{\psi}\right) - H. \tag{5.4}$$

The left-hand side of (5.4) is uniformly bounded for ε small and $(E, \psi, s) \in A$. Suppose $\hat{v}_\varepsilon(E, s/\psi)$ is not uniformly bounded on A. For every $B > H$, there would exist ε and $(E, \psi, s) \in A$ such that $\hat{v}_\varepsilon(E, s/\psi) > B$. However, for $\hat{v}_\varepsilon > H$, $h'(\hat{v}_\varepsilon) > 0$ and the right-hand side of (5.4) can be made arbitrarily large. Contradiction. Similarly, one can show that $\hat{v}_\varepsilon(E, s/\psi)$ is uniformly bounded away from zero on A. It follows that

$$\hat{v}_\varepsilon\left(E, \frac{s}{\psi}\right) \to H$$

uniformly on A. Hence, $Q_\varepsilon \to Q$ and $P_\varepsilon \to P$ uniformly on A. Applying Lemma 5.4 we have $\phi_\varepsilon(s) \to \phi(s)$ and $T_\varepsilon(s) \to T(s)$ uniformly on $[0, s_0]$, and thus for ε small we have $(s, \phi_\varepsilon(s), T_\varepsilon(s)) \in A$ for all $s \in [0, s_0]$. Hence

$$v_\varepsilon(s) = \hat{v}_\varepsilon\left(T_\varepsilon(s), \frac{s}{\phi_\varepsilon(s)}\right) \to H$$

uniformly on $[0, s_0]$ and in view of (5.3) and (4.10) with $C_\varepsilon = H$,

$$\tilde{p}_\varepsilon(v_\varepsilon(s)) \to \frac{\tilde{p}(s)}{H}$$

uniformly on $[0, s_0]$. □

Proof of Corollary 5.2. $\frac{\phi(s)}{s} = (H + \frac{a^3}{s^3})^{1/3}$ approaches $H^{1/3}$ as $s \to \infty$. Let $\delta > 0$. Choose s_0 such that $\frac{\phi(s_0)}{s_0} - H^{1/3} < \frac{\delta}{2}$. Choose ε sufficiently small such that $s_\varepsilon > s_0 + 1$. We have

$$\frac{\phi_\varepsilon(s)}{s} \to \frac{\phi(s)}{s}$$

pointwise on $(0, s_0]$. Choose ε further small such that

$$\left|\frac{\phi_\varepsilon(s_0)}{s_0} - \frac{\phi(s_0)}{s_0}\right| < \frac{\delta}{2},$$

and hence

$$\left|\frac{\phi_\varepsilon(s_0)}{s_0} - H^{\frac{1}{3}}\right| < \delta.$$

Since $\frac{\phi_\varepsilon(s)}{s} \geq \lambda_\varepsilon$ and $\lambda_\varepsilon \geq H^{1/3}$,

$$H^{\frac{1}{3}} \leq \lambda_\varepsilon \leq \frac{\phi_\varepsilon(s_0)}{s_0} \quad \text{and so} \quad \lambda_\varepsilon - H^{\frac{1}{3}} < \delta. \quad \square$$

Proof of Corollary 5.3. Suppose shock strength (SS_ε) does not approach zero as $\varepsilon \to 0$. There exists a sequence $\varepsilon_n \to 0$ such that

$$SS_{\varepsilon_n} = \dot{\phi}_{\varepsilon_n}(s_{J_{\varepsilon_n}}) - \frac{\phi_{\varepsilon_n}(s_{J_{\varepsilon_n}})}{s_{J_{\varepsilon_n}}} = \dot{\phi}_{\varepsilon_n}(s_{J_{\varepsilon_n}}) - \lambda_{\varepsilon_n} \to \alpha < 0.$$

By Corollary 5.2, there exists a constant $b > 0$ such that

$$H^{\frac{1}{3}} - \dot{\phi}_{\varepsilon_n}(s_{J_{\varepsilon_n}}) > b \quad \text{or} \quad \dot{\phi}_{\varepsilon_n}(s_{J_{\varepsilon_n}}) < H^{\frac{1}{3}} - b$$

for n sufficiently large. Choose such an n. Let β be the s coordinate of the intersection of the line through $(0, a)$ with slope $H^{1/3} - b$ and the line through $(0, 0)$ with slope $H^{1/3}$. By (4.14 ii), $\dot{\phi}_{\varepsilon_n}(s) < H^{1/3} - b$ for $s < s_{J_{\varepsilon_n}}$ and by (4.17), $\lambda^3_{\varepsilon_n} > H^{1/3}$. Thus by the construction of ϕ_{ε_n} and the Mean Value Theorem, $s_{J_{\varepsilon_n}} \leq \beta$ for all n sufficiently large. Contradiction. □

Theorem 5.1 and Corollary 5.2 imply that $u^\varepsilon(x, t)$ converges pointwise to $u(x, t)$ for $(x, t) \in \mathbf{R}^3 \times [0, T]$ and converges uniformly on compact subsets of $\mathbf{R}^3 \times (0, T]$. To prove the uniform convergence on compact subsets of $\mathbf{R}^3 \times [0, T]$, we need the following:

PROPOSITION 5.5. $\frac{\phi_\varepsilon(s)}{s}$ converges uniformly to $\frac{\phi(s)}{s}$ on $[1, \infty)$.

Proof. We note that as s tends to infinity, $\frac{\phi(s)}{s}$ tends to $H^{1/3}$. Let $\delta > 0$. Choose $s_0 > 1$ such that $\frac{\phi(s)}{s} - H^{1/3} < \frac{\delta}{2}$ for all $s \geq s_0$. There exists ε_1 such that

$$\frac{\phi_\varepsilon(s_0)}{s_0} - \frac{\phi(s_0)}{s_0} < \frac{\delta}{2} \quad \text{for } \varepsilon < \varepsilon_1.$$

Thus $\frac{\phi_\varepsilon(s_0)}{s_0} - H^{1/3} < \delta$ for $\varepsilon < \varepsilon_1$. The inequality $\dot{\phi}_\varepsilon < \frac{\phi_\varepsilon(s)}{s}$ implies $\frac{\phi_\varepsilon(s)}{s}$ is a decreasing function of s. So for $\varepsilon < \varepsilon_1$ and $s > s_0$ we have

$$H^{\frac{1}{3}} < \frac{\phi_\varepsilon(s)}{s} < \frac{\phi_\varepsilon(s_0)}{s_0} < \delta + H^{\frac{1}{3}} \quad \text{and} \quad H^{\frac{1}{3}} < \frac{\phi(s)}{s} < \delta + H^{\frac{1}{3}},$$

which in turn imply that

$$\left| \frac{\phi_\varepsilon(s)}{s} - \frac{\phi(s)}{s} \right| < \delta.$$

By Theorem 5.1, there exists ε_2 such that if $\varepsilon < \varepsilon_2$

$$\left| \frac{\phi_\varepsilon(s)}{s} - \frac{\phi(s)}{s} \right| < \delta \quad \text{for } s \in [1, s_0].$$

Now choose $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$. For $s \in [1, \infty]$, we have

$$\left| \frac{\phi_\varepsilon(s)}{s} - \frac{\phi(s)}{s} \right| < \delta. \quad \square$$

Finally, let Λ be a compact subset of $\mathbf{R}^3 \times [0, T]$. Without loss of generality, assume $\Lambda = B_\rho \times [0, T]$ where B_ρ is the ball of radius ρ in $\mathbf{R}^3$. $\Lambda = \Lambda_1 \cup \Lambda_2$ where

$$\Lambda_1 = \{(x, t) \in \Lambda \mid |x| > t\} \quad \text{and} \quad \Lambda_2 = \{(x, t) \in \Lambda \mid |x| \leq t\}.$$

On Λ_1, we have $s = \frac{|x|}{t} > 1$. If $t = 0$, $|u^\varepsilon - u| = |\lambda_\varepsilon - H^{1/3}|\,|x|$ and hence Corollary 5.2 implies uniform convergence on $\{(x, t) \in \Lambda_1 \mid t = 0\}$.

If $t > 0$, we have

$$|u^\varepsilon - u| = \left| \frac{\phi_\varepsilon(s)}{s} - \frac{\phi(s)}{s} \right| |x|,$$

and hence uniform convergence on $\{(x,t) \in \Lambda_1 \mid t > 0\}$ follows from Proposition 5.5. On Λ_2, we have $s \le 1$. For $\{(x,t) \in \Lambda_2 \mid t > 0\}$,

$$|u^\varepsilon - u| = \left| \frac{\phi_\varepsilon(s)}{s} - \frac{\phi(s)}{s} \right| |x| = t|\phi_\varepsilon(s) - \phi(s)| < T|\phi_\varepsilon(s) - \phi(s)|,$$

and uniform convergence on Λ_2 follows from Theorem 5.1. The proof of Theorem 3.3 is now complete.

6. Remarks.

1. One unresolved problem is the behavior of p^ε at $t = 0$. One expects

$$p^\varepsilon(x,0) \to \frac{p(x,0)}{H} = \frac{p_*}{H},$$

that is,

$$\frac{\lambda_\varepsilon^3 - H}{\varepsilon} \to \frac{p_*}{H}.$$

This is not clearly true or false from the analysis given so far. Our method of addressing this problem is as follows. We are interested in the convergence of

$$\frac{v_\varepsilon(s) - H}{\varepsilon} \tag{6.1}$$

for large s. (6.1) is continuous everywhere except possibly at s_ε where it may encounter a jump to its constant state $\frac{\lambda_\varepsilon^3 - H}{\varepsilon}$. By Corollary 5.3 this jump tends to zero. By Prop. 4.2, (4.17), (4.18), and the definition of s_ε, we have $s_\varepsilon = O(\frac{1}{\sqrt{\varepsilon}})$, and hence one should examine the limiting behavior of the function in (6.1) evaluated at $1/\sqrt{\varepsilon}$. By (4.10) it suffices to examine the convergence of $T_\varepsilon(1/\sqrt{\varepsilon})$ and $\phi_\varepsilon(1/\sqrt{\varepsilon})$. To prove convergence one could rescale system (4.12) with $\tau = \sqrt{\varepsilon} s$ and show that solutions for the resulting system in τ converge on compact sets.

2. We now address the possibility of additional terms of the form

$$\frac{|\det F - H|^q}{q\varepsilon}. \tag{6.2}$$

The equation of motion becomes

$$u_{tt}^\varepsilon(x,t) = \operatorname{div}\left(S_1(\nabla u^\varepsilon(x,t)) + \frac{|\det \nabla u^\varepsilon - H|^q}{q\varepsilon} (\operatorname{adj} \nabla u^\varepsilon)^{\mathrm{T}} \right). \tag{6.3}$$

If $q \ne 2$, we are not able, as we were for $q = 2$, to simply write (6.3) as an equivalent system in terms of u^ε and the extra pressure term resulting from the addition of (6.2).

Note that even if the absolute values were not present in (6.2) and we considered an extra equation for the additional pressure of the form

$$\partial_t[(\varepsilon p^\varepsilon)^{\frac{1}{q}} - \det \nabla u^\varepsilon] = 0,$$

we would find that the jump conditions for a piecewise smooth solution would be different. Thus we abandon this somewhat formal route and consider the behavior of solutions to (6.3) as ε tends to zero.

Suppose $0 < q \leq 1$. The radial component of the Cauchy stress for a solution of the form (3.10) is

$$T_\varepsilon(s) = v_\varepsilon(s)\left(\frac{\phi_\varepsilon}{s}\right)^{-4} + h'(v_\varepsilon(s)) + \left(\frac{|v_\varepsilon(s) - H|^{q-1}}{\varepsilon}\right)\mathrm{H}_H,$$

where H_H is the Heaviside function with jump discontinuity at $s = H$. Since $q - 1 \leq 0$, we see that for any ε and choice of $\lim_{s\to 0^+} v_\varepsilon(s)$, $\lim_{s\to 0^+} T_\varepsilon \neq 0$. Thus the crucial boundary requirement (traction-free cavity) is not attainable.

Suppose $q > 2$. Recall that the joining of our solution ϕ_ε to a line was possible given that the third derivative of the extra forcing term had no positive contribution (*cf.* Proposition 4.3). Thus the cases with $q > 2$ are left unexplored.

Suppose $1 < q < 2$. In these cases, all of our analytical results hold. That is, by adding terms of the form (6.2) with $1 < q < 2$, we obtain solutions that converge to (4.20)—the motion converges to u and

$$\left(\frac{|v_\varepsilon(s) - H|^{q-1}}{\varepsilon}\right)\mathrm{H}_H \text{ converges to } \tilde{p}(s).$$

We make the following comments on the important steps in the argument. In the ordinary differential equation, (4.7), $1/\varepsilon$ is replaced with

$$\frac{q-1}{\varepsilon|v_\varepsilon(s) - H|^{2-q}}.$$

Joining the solution to a shock is possible because the derivative of the above with respect to v_ε is negative for $v_\varepsilon > H$ (we are again able to show $\dot{v}_\varepsilon > 0$ and hence $v_\varepsilon > H$ throughout the motion). The inequality

$$\left(\frac{\phi_\varepsilon}{s}\right)^4 \frac{1}{(v_\varepsilon(s) - H)^{2-q}} > \left(\frac{\phi_\varepsilon}{s}\right)\frac{v_\varepsilon(s)}{(v_\varepsilon(s) - H)^{2-q}} > \left(\frac{\phi_\varepsilon}{s}\right)v_\varepsilon(s)^{1+q}$$

implies $s_{M_\varepsilon}, s_{J_\varepsilon}$ tend to infinity as $\varepsilon \to 0$ and hence s_ε tends to infinity. In passing to the limit, we again have $\hat{v}_\varepsilon$ converging uniformly to H on a compact neighborhood of the limit solution. Together with Lemma 5.4, this implies the convergence of solutions.

Finally we remark that if we deleted the absolute value signs and considered $q = 1$, our solutions would compress so rapidly that in the limit we would obtain a solution with vanishing determinant of the deformation gradient, that is, a solution of the form

$$u(x,t) = \left(\frac{a^3}{s^3}\right)x = a^3 t.$$

3. The shocks satisfy the Lax admissibility condition. This means that shock speed must lie strictly between the characteristic speeds. For our system the characteristic speeds are the positive square roots of the eigenvalues of the acoustic tensor (see Sec. 3) where the vector v is the normal to the radial shock, i.e., x/R (see [9] for more details).

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