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ON CONSTANDA'S MATRIX OF NONUNIQUENESS IN A THEORY OF PLATES IN ASYMMETRIC ELASTICITY

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Abstract. We apply the direct method to the solution of the Dirichlet problem in the bending of micropolar plates and show that a necessary and sufficient condition for solubility is the nonsingularity of a special constant matrix constructed for any smooth closed boundary curve.

Introduction. In the application of the direct method to the solution of the Dirichlet problem for the two-dimensional Laplace equation, there are certain boundary curves (of logarithmic capacity one [1]) for which the corresponding homogeneous Fredholm integral equation of the first kind has nonzero solutions. This leads ultimately to the breakdown of the method [2].

In this paper, proceeding as in [3], we show how a similar situation occurs when we apply the direct method to the solution of the Dirichlet problem for the equations of bending of micropolar plates [4]. We identify the particular boundary curves using a generalisation of the concept of capacity in which Robin's constant is replaced by a constant (3×3)-matrix whose nonsingularity is used to express a necessary and sufficient condition for the solubility of the Dirichlet problem by the direct method.

Preliminary results. Throughout the paper we assume that, unless otherwise stated, Greek and Latin indices take the values $1, 2$ and $1, 2, 3$ respectively, we sum over repeated indices, $\mathcal{M}_{p\times q}$ is the space of $(p \times q)$-matrix functions, $H^{(i)}$ are the columns of a matrix $H \in \mathcal{M}_{p\times q}$, and E_n is the unit matrix in $\mathcal{M}_{n\times n}$. Finally, if X is a space of scalar functions and $Q \in \mathcal{M}_{p\times q}$, then $Q \in X$ means that every component of Q belongs to X.

We consider a homogeneous and isotropic micropolar plate occupying the region $S \times [-h_0/2, h_0/2]$, where $S \subset \mathbf{R}^2$ is a domain bounded by a simple closed C^2-curve ∂S and $h_0 = \text{const} << \operatorname{diam} S$ is the thickness. In the absence of body forces and couples and of forces and couples on the faces, the equilibrium equations for bending can be written

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in the form [4]

$$L(\partial x)u(x)=0, \tag{1}$$

where $x=(x_1,x_2)$ is a generic point in S, $u=(u_1,\dots,u_5)^{\mathrm{T}}$ is a vector characterizing the displacements and microrotations, $L(\partial x)=(\partial/\partial x_1,\partial/\partial x_2)$ is the matrix partial differential operator defined by $L(\xi_1,\xi_2)=$

$$\begin{pmatrix} \Delta_3+h^2(\lambda+\mu)\xi_1^2, & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 & 0 & \kappa \\ h^2(\lambda+\mu)\xi_1\xi_2 & \Delta_3+h^2(\lambda+\mu)\xi_2^2 & -\mu\xi_2 & -\kappa & 0 \\ \mu\xi_1 & \mu\xi_2 & \Delta_1 & -\kappa\xi_2 & \kappa\xi_1 \\ 0 & -\kappa & \kappa\xi_2 & \Delta_2+(\alpha+\beta)\xi_1^2 & (\alpha+\beta)\xi_1\xi_2 \\ \kappa & 0 & -\kappa\xi_1 & (\alpha+\beta)\xi_1\xi_2 & \Delta_2+(\alpha+\beta)\xi_2^2 \end{pmatrix}, \tag{2}$$

$\lambda,\mu,\kappa,\alpha,\beta,\gamma$ are the elastic constants of the material, $\Delta=\xi_1^2+\xi_2^2$, $\Delta_1=(\mu+\kappa)\Delta$, $\Delta_2=\gamma\Delta-2\kappa$, $\Delta_3=(\mu+\kappa)(h^2\Delta-1)$, and $h^2=h_0^2/12$. Along with L we consider the traction boundary operator $T(\partial x)$ defined by $T(\xi_1,\xi_2)=$

$$\begin{pmatrix} h^2(\mu_1\xi_\alpha n_\alpha+\lambda_1\xi_1 n_1) & h^2(\mu\xi_1 n_2+\lambda\xi_2 n_1) & 0 & 0 & 0 \\ h^2(\lambda\xi_1 n_2+\mu\xi_2 n_1) & h^2(\mu_1\xi_\alpha n_\alpha+\lambda_1\xi_2 n_2) & 0 & 0 & 0 \\ \mu n_1 & \mu n_2 & \mu_1\xi_\alpha n_\alpha & -\kappa n_2 & \kappa n_1 \\ 0 & 0 & 0 & \gamma\xi_\alpha n_\alpha+\alpha_1\xi_1 n_1 & \beta\xi_1 n_2+\alpha\xi_2 n_1 \\ 0 & 0 & 0 & \alpha\xi_1 n_2+\beta\xi_2 n_1 & \gamma\xi_\alpha n_\alpha+\alpha_1\xi_2 n_2 \end{pmatrix}$$

where $\lambda_1=\lambda+\mu$, $\mu_1=\mu+\kappa$, $\alpha_1=\alpha+\beta$, $n=(n_1,n_2)^{\mathrm{T}}$ is the unit outward normal to ∂S.

The boundary integral equation method for the solution of the Dirichlet (u prescribed on ∂S) and Neumann (Tu prescribed on ∂S) problems for (1) is based on the representation of solutions as single and double layer elastic potentials defined, respectively, by [4]

$$(V\phi)(x)=\int_{\partial S} D(x,y)\phi(y)\,ds(y) \quad \text{and} \quad (W\phi)(x)=\int_{\partial S} P(x,y)\phi(y)\,ds(y).$$

Here [5],

$$D(x,y)=L^*(\partial x)t(x,y) \tag{3}$$

is a matrix of fundamental solutions for L constructed by means of Galerkin's representation, L^* is the adjoint of L,

$$t(x,y)=\frac{k^2}{8\pi}[4k_iK_0(c_i|x-y|)+(4k_4+k_5|x-y|^2)\ln|x-y|], \tag{4}$$

K_0 is the modified Bessel function of order zero, k^2,c_i,k_j, $j=1,2,\dots,5$, are certain combinations of the elastic constants, and

$$P(x,y)=(T(\partial y)D(y,x))^{\mathrm{T}}.$$

Let $S^+ = S$ and $S^- = R^2 \setminus \overline{S}^+$. If $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ is a solution of (1), then the Somigliana formula [5] can be written in the form

$$V(Tu|_{\partial S}) - W(u|_{\partial S}) = \begin{cases} u & \text{in } S^+, \\ \frac{1}{2}u & \text{on } \partial S, \\ 0 & \text{in } S^-. \end{cases}$$

It follows that the solution of the Dirichlet problem in S^+ can be computed provided $Tu|_{\partial S}$ can be found. This procedure is known as the direct method and reduces therefore to solving uniquely the Fredholm integral equation of the first kind

$$V_0\phi = f = \tfrac{1}{2}u|_{\partial S} + W_0(u|_{\partial S}), \tag{5}$$

where $V_0\theta$ and $W_0\theta$ are the direct values of $V\theta$ and $W\theta$ (in the sense of principal value) on ∂S.

For the two-dimensional Laplace equation there are smooth curves ∂S (of logarithmic capacity one) on which the corresponding homogeneous equation (5) has nonzero solutions [2]. In [3] it is shown that a similar situation exists in the theory of bending of Mindlin plates if ∂S has a certain property (based on a generalisation of the concept of logarithmic capacity) characterised by a unique constant matrix henceforth referred to as "Constanda's matrix".

In what follows we show that in the case of system (1) there are also certain curves ∂S on which the homogeneous equation from (5) has nontrivial solutions. We identify these particular curves using a generalisation of Constanda's matrix for micropolar plates. This identification is important in determining for which boundary curves ∂S (5) may not be uniquely solvable and hence when the direct method of solution is inapplicable.

First we recall some properties of the elastic potentials V and W [4], [5].

THEOREM 1. (i) If $\phi \in C(\partial S)$ then $V\phi$ and $W\phi$ are analytic and satisfy $L(V\phi) = L(W\phi) = 0$ in $S^+ \cup S^-$.

(ii) If $\phi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, then $V_0\phi$ and $W_0\phi$ exist, the functions

$$\nu^+(\phi) = (V\phi)|_{\overline{S}^+}, \qquad \nu^-(\phi) = (V\phi)|_{\overline{S}^-}$$

are of class $C^{1,\alpha}(\overline{S}^+)$ and $C^{1,\alpha}(\overline{S}^-)$, respectively, and

$$T\nu^+(\phi) = (W_0^* + \tfrac{1}{2}\mathbf{I})\phi, \qquad T\nu^-(\phi) = (W_0^* - \tfrac{1}{2}\mathbf{I})\phi,$$

where W_0^* is the adjoint of W_0, and $\mathbf{I}$ is the identity operator.

(iii) If $\phi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, then the functions

$$\omega^+(\phi) = \begin{cases} (W\phi)|_{S^+} & \text{in } S^+, \\ (W_0 - \frac{1}{2}\mathbf{I})\phi & \text{on } \partial S, \end{cases} \qquad \omega^-(\phi) = \begin{cases} (W\phi)|_{S^-} & \text{in } S^-, \\ (W_0 + \frac{1}{2}\mathbf{I})\phi & \text{on } \partial S, \end{cases}$$

are of class $C^{1,\alpha}(\overline{S}^+)$ and $C^{1,\alpha}(\overline{S}^-)$, respectively.

(iv) $(W_0+\frac{1}{2}\mathbf{I})\phi=0$ if and only if $\phi=Fk$ where the columns $F^{(i)}$ of the matrix

$$F=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -x_1 & -x_2 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

form a basis for the space of rigid displacements and $k\in\mathcal{M}_{3\times1}$ is constant and arbitrary. (It is clear that Fk represents a vector of arbitrary rigid displacement and microrotation.) Also, $L(Fk)=0$ in $\mathbf{R}^2$ and $T(Fk)=0$ on ∂S.

(v) The solutions ϕ of the system $(W_0^*+\frac{1}{2}\mathbf{I})\phi=0$ form a subspace of $C^{1,\alpha}(\partial S)$ of dimension 5. (We represent ϕ by $\phi=G\mathfrak{l}$ where the columns $G^{(i)}$ of $G\in\mathcal{M}_{5\times3}$ are linearly independent and $\mathfrak{l}\in\mathcal{M}_{3\times1}$ is constant and arbitrary.)

(vi) $\frac{1}{4}$ is an eigenvalue of W_0^{*2}, and the corresponding eigenspace coincides with that of W_0^* for the eigenvalue $-\frac{1}{2}$.

(vii) Define the operator N_0 on $C^{1,\alpha}(\partial S)$ by $N_0\phi=T\omega^+(\phi)$. If $N_0\phi=0$, then $\phi=Fk$ where $k\in\mathcal{M}_{3\times1}$ is constant and arbitrary.

(viii) $N_0V_0=W_0^{*2}-\frac{1}{4}\mathbf{I}$ on $C^{0,\alpha}(\partial S)$.

(ix) Let $\mathfrak{A}$ be the class of functions $v\in\mathcal{M}_{5\times1}$ which, as $r=|x|\to\infty$, admit an asymptotic expansion of the form

$$\begin{aligned}
v_1(r,\theta)&=r^{-1}[a_0\sin\theta+2a_1\cos\theta-a_0\sin3\theta+(a_2-a_1)\cos3\theta]\\
&\quad+r^{-2}[(2b_1+d_1)\sin2\theta+d_2\cos2\theta-2b_1\sin4\theta+2b_2\cos4\theta]\\
&\quad+r^{-3}[2e_1\sin3\theta+2f_1\cos3\theta+3(e_2-e_1)\sin5\theta+(f_2-f_1)\cos5\theta]+O(r^{-4}),\\
v_2(r,\theta)&=r^{-1}[2a_2\sin\theta+a_0\cos\theta+(a_2-a_1)\sin3\theta+a_0\cos3\theta]\\
&\quad+r^{-2}[(2b_2+d_2)\sin2\theta-d_1\cos2\theta+2b_2\sin4\theta+2b_1\cos4\theta]\\
&\quad+r^{-3}[2f_2\sin3\theta-2e_2\cos3\theta+3(f_2-f_1)\sin5\theta+3(e_1-e_2)\cos5\theta+O(r^{-4}),\\
v_3(r,\theta)&=-(a_1+a_2)\ln r-[a_1+a_2+a_0\sin2\theta+(a_1-a_2)\cos2\theta]\\
&\quad-r^{-1}[(b_1+d_1)\sin\theta+(b_2+d_2)\cos\theta-b_1\sin3\theta+b_2\cos3\theta]\\
&\quad+r^{-2}[g_1\sin2\theta+g_2\cos2\theta+(e_2-e_1)\sin4\theta+(f_2-f_1)\cos4\theta]+O(r^{-3}),\\
v_4(r,\theta)&=-r^{-1}[2a_2\sin\theta+a_0\cos\theta+(a_2-a_1)\sin3\theta+a_0\cos3\theta]\\
&\quad-r^{-2}[(2b_2+d_2)\sin2\theta-d_1\cos2\theta+2b_2\sin4\theta+2b_1\cos4\theta]\\
&\quad-r^{-3}[(2f_2+a_3)\sin3\theta-(2e_2-a_4)\cos3\theta+3(f_2-f_1)\sin5\theta\\
&\qquad\qquad+3(e_1-e_2)\cos5\theta]+O(r^{-4}),\\
v_5(r,\theta)&=r^{-1}[a_0\sin\theta+2a_1\cos\theta-a_0\sin3\theta+(a_2-a_1)\cos3\theta]\\
&\quad+r^{-2}[(2b_1+d_1)\sin2\theta+d_2\cos2\theta-2b_1\sin4\theta+2b_2\cos4\theta]\\
&\quad+r^{-3}[(2e_1+a_4)\sin3\theta+(2f_1+a_3)\cos3\theta+3(e_2-e_1)\sin5\theta\\
&\qquad\qquad+3(f_2-f_1)\cos5\theta]+O(r^{-4}),
\end{aligned}$$

where $a_0, a_\alpha, a_{\alpha+2}, b_\alpha, d_\alpha, e_\alpha, f_\alpha$, and g_α are arbitrary constants. Also, let $\mathfrak{A}^*$ be the class of functions of the form $u = Fk + \sigma^{\mathfrak{A}}$, with $\sigma^{\mathfrak{A}} \in \mathfrak{A}$. Then $W\phi \in \mathfrak{A}$ and

$$V\phi = M^\infty(\rho\phi) + \sigma^{\mathfrak{A}}$$

where ρ is the operator defined on continuous functions $\phi \in \mathcal{M}_{5\times 1}$ on ∂S by

$$\rho\phi = \int_{\partial S} F^{\mathrm{T}}\phi\, ds,$$

and $M^\infty(r,\theta)$ is the (5×3)-matrix with columns [5]:

$$\begin{aligned}\frac{8\pi}{k^2}M^{\infty(1)}(r,\theta) = [&A_1k_5(2(\ln r+1)+\cos 2\theta), A_1k_5\sin 2\theta,\\ &-(A_1k_5r(2\ln r+1)+4(A_1k_4-k_5A_2)r^{-1})\cos\theta, -A_1k_5\sin 2\theta,\\ &A_1k_5(2(\ln r+1)+\cos 2\theta)]^{\mathrm{T}},\end{aligned}$$

$$\begin{aligned}\frac{8\pi}{k^2}M^{\infty(2)}(r,\theta) = [&A_1k_5\sin 2\theta, A_1k_5(2(\ln r+1)-\cos 2\theta),\\ &-(A_1k_5r(2\ln r+1)+4(A_1k_4-k_5A_2)r^{-1})\sin\theta,\\ &-A_1k_5(2(\ln r+1)-\cos 2\theta), A_1k_5\sin 2\theta]^{\mathrm{T}},\end{aligned}$$

$$\begin{aligned}\frac{8\pi}{k^2}M^{\infty(3)}(r,\theta) = [&A_1k_5r(2\ln r+1)\cos\theta, A_1k_5r(2\ln r+1)\sin\theta,\\ &-A_1k_5r^2\ln r-4(A_1k_4-k_5A_2)\ln r+4k_5A_2, -A_1k_5r(2\ln r+1)\sin\theta,\\ &A_1k_5r(2\ln r+1)\cos\theta]^{\mathrm{T}}.\end{aligned}$$

Here, A_α are well-defined constants expressed in terms of the elastic constants. It is easy to show that for any fixed y, $L(\partial x)M^\infty(x,y) = 0$, $x \in \mathbf{R}^2$, $x \neq y$, and $V\phi \in \mathfrak{A}$ if and only if $\rho\phi = 0$.

(x) The interior Dirichlet problem has at most one solution $u \in C^2(S^+) \cap C^1(\overline{S}^+)$.

(xi) The exterior Dirichlet problem has at most one solution $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathfrak{A}^*$. If $u|_{\partial S} \in C^{1,\alpha}(\partial S)$ and G can be chosen so that the sets $\{F^{(i)}\}$ and $\{G^{(i)}\}$ are biorthonormal, i.e., $\int_{\partial S} F^{\mathrm{T}}G = E_3$, then that solution can be expressed as the sum of a double layer potential and a specific rigid displacement Fk with $k = \int_{\partial S} G^{\mathrm{T}}u|_{\partial S}\, ds$.

We now return to the discussion of system (5). Using a generalisation of Constanda's matrix, we will characterise curves ∂S on which the homogeneous equation from (5) has nonzero solutions, thereby identifying curves ∂S on which (5) may not be uniquely solvable.

Condition for nonuniqueness.

THEOREM 2. For any closed C^2-curve ∂S and any $\alpha \in (0,1)$, there exist a unique $\Phi \in \mathcal{M}_{5\times 3} \cap C^{1,\alpha}(\partial S)$ and a unique constant $C \in \mathcal{M}_{3\times 3}$ such that the $\Phi^{(i)}$ are linearly independent,

$$V_0\Phi = FC \quad \text{and} \quad \rho\Phi = E_3. \tag{6}$$

Here, $(V_0\Phi)^{(i)} = V_0\Phi^{(i)}$.

Proof. By Theorem 1(v), (vi), (viii), we can write

$$(W_0^{*2} - \tfrac{1}{4}\mathbf{I})G = 0 = N_0(V_0G)$$

so that $V_0G = FK$ (by Theorem 1(viii)) for some constant $K \in \mathcal{M}_{3\times 3}$ (or simply note that since $VG^{(i)}$ solve homogeneous interior Neumann problems [4], $VG = FK$ in S^+ for some constant $K \in \mathcal{M}_{3\times 3}$ and, by Theorem 1(ii), $V_0G = FK$).

Suppose that $H = \rho G$, where $(\rho G)^{(i)} = \rho G^{(i)}$, is a singular matrix. Then there is a constant nonzero $h \in \mathcal{M}_{3\times 1}$ such that $Hh = 0$. Consequently (Theorem 1(i), (v), (ix)), the (5×1)-vector function $U = V(Gh) - FKh$ is a solution of the problem

$$\begin{aligned} LU &= 0 && \text{in } S^+ \cup S^-, \\ U &= (V_0G - FK)h = 0 && \text{on } \partial S, \end{aligned}$$

and

$$\begin{aligned} U &= M^\infty(\rho G)h + \sigma^{\mathfrak{A}} - FKh \\ &= M^\infty(Hh) + \sigma^{\mathfrak{A}} - FKh \\ &= -FKh + \sigma^{\mathfrak{A}}, \quad \text{as } |x| \to \infty. \end{aligned}$$

Thus, $U = V(Gh) - FKh = 0$ in $S^+ \cup S^-$ by Theorem 1(x), (xi) and since $U = 0$ on ∂S, we conclude that $FKh = 0$, which means that $V(Gh) = 0$ in $S^+ \cup S^-$. From Theorem 1(ii) we now have $Gh = 0$ which contradicts the linear independence of the $G^{(i)}$ and we conclude that H is nonsingular. Therefore, $\det H \neq 0$, and, since H is constant, we see that

$$V_0(GH^{-1}) = (V_0G)H^{-1} = FKH^{-1} \quad \text{and} \quad \rho(GH^{-1}) = (\rho G)H^{-1} = HH^{-1} = E_3.$$

Hence, $\Phi = GH^{-1}$ and $C = KH^{-1}$ is a solution pair for (6).

To prove uniqueness of this pair, we assume there are two such pairs $\{\Phi_1, C_1\}, \{\Phi_2, C_2\}$ and consider the difference $\{\Phi, C\}$ where $\Phi = \Phi_1 - \Phi_2$ and $C = C_1 - C_2$. Since $\rho\Phi = 0$ we conclude from Theorem 1(1), (iv), (ix) that

$$\begin{aligned} L(V\Phi - FC) &= 0 && \text{in } S^+ \cup S^-, \\ V_0\Phi - FC &= 0 && \text{on } \partial S, \end{aligned}$$

and

$$V\Phi - FC = \sigma^{\mathfrak{A}} \quad \text{as } |x| \to \infty.$$

As above, $V\Phi - FC = 0$ in $S^+ \cup S^-$ so that $FC = 0$. The linear independence of the $F^{(i)}$ now yields $C = 0$ and finally, using Theorem 1(ii), $\Phi = 0$, which completes the proof.

We now state the main result concerning the possible non-solubility of Eq. (5).

THEOREM 3. The homogeneous equation from (5) has nonzero solutions if and only if ∂S is such that $\det C = 0$.

Proof. If $\det C = 0$, then $Ch = 0$ for some constant nonzero $h \in \mathcal{M}_{3\times 1}$. By Theorem 2 there is a $\Phi \in \mathcal{M}_{5\times 3}$ such that the $\Phi^{(i)}$ are linearly independent and $V_0\Phi = FC$.

Therefore,

$$V_0(\Phi h) = (V_0\Phi)h = (FC)h = F(Ch) = 0,$$

with $\Phi h \neq 0$ ($\Phi^{(i)}$ are linearly independent).

Conversely, if $\det C \neq 0$, by (6), $V(\phi - \Phi\rho\phi) \in \mathfrak{A}$ and the combination

$$V(\phi - \Phi\rho\phi) + FC\rho\phi$$

is a solution of the homogeneous exterior Dirichlet problem. Hence, $V(\phi - \Phi\rho\phi) + FC\rho\phi = 0$ in S^-, which means that $FC\rho\phi = 0$. Since $\det C \neq 0$ and $F^{(i)}$ are linearly independent we have that $\rho\phi = 0$. Consequently, $V\phi \in \mathfrak{A}$ (Theorem 1(ix)) and $V\phi$ is therefore a solution of both the interior and exterior Dirichlet problems leading to the conclusion that $V\phi = 0$ in $\mathbf{R}^2$ and hence $\phi = 0$. Thus, if the homogeneous equation from (5), $V_0\phi = 0$, has nonzero solutions, we must necessarily have $\det C = 0$. This completes the proof.

Remark 1. It is clear from Theorem 3 that (5) has at most one solution if and only if $\det C \neq 0$.

From Theorem 2, the matrix C is associated with the basis G for the null space of $W_0^* + \frac{1}{2}\mathbf{I}$. In what follows, we note that the $\Phi^{(i)}$ from (6) also form a basis for the null space of $W_0^* + \frac{1}{2}\mathbf{I}$ and therefore allow a more convenient representation of C in terms of Φ.

Corollary 1. If Φ is given by (6), the matrix C of Theorem 3 can be expressed as

$$C = \int_{\partial S} \Phi^{\mathrm{T}} M^\infty \, ds. \tag{7}$$

Proof. Consider the combination $B = V\Phi - FC - M^\infty$. From (6) and Theorem 1(ix), B solves the exterior Dirichlet problem (for each $B^{(i)}$):

$$\begin{aligned} LB &= 0 \quad \text{in } S^-, \\ B &= V_0\Phi - FC - M^\infty = -M^\infty \quad \text{on } \partial S, \end{aligned}$$

and

$$\begin{aligned} B &= V\Phi - FC - M^\infty \\ &= M^\infty(\rho\Phi) + \sigma^{\mathfrak{A}} - FC - M^\infty \\ &= M^\infty E_3 + \sigma^{\mathfrak{A}} - FC - M^\infty \\ &= \sigma^{\mathfrak{A}} - FC \quad \text{as } |x| \to \infty. \end{aligned} \tag{8}$$

From Theorem 1(ix), B can be expressed as the sum of a double layer potential and a specific matrix Fk. Now, since Φ is given by (6), from the proof of Theorem 2, $\Phi = GH^{-1}$, where H is constant. Hence, the $\Phi^{(i)} = (GH^{-1})^{(i)}$ form a basis for the null space of $W_0^* + \frac{1}{2}\mathbf{I}$ and, from (6), $\{F^{(i)}\}$ and $\{\Phi^{(i)}\}$ are biorthonormal. Hence, we can write

$$k = \int_{\partial S} \Phi^{\mathrm{T}} B|_{\partial S} \, ds.$$

From (8) it follows that

$$k = -C = \int_{\partial S} \Phi^{\mathrm{T}}(-M^{\infty})\, ds,$$

i.e.,

$$C = \int_{\partial S} \Phi^{\mathrm{T}} M^{\infty}\, ds$$

as required.

Finally, we express the uniqueness condition of Theorem 3 explicitly in terms of the basis G.

COROLLARY 2. Equation (5) has at most one solution if and only if $V_0 G^{(i)}$ are linearly independent.

Proof. Let $h \in \mathcal{M}_{3\times 1}$ be constant, nonzero and arbitrary and assume that (5) has at most one solution. Then

$$\begin{aligned}(V_0 G)h = V_0(Gh) &= 0\\ &\Rightarrow Gh = 0 \text{ by assumption}\\ &\Rightarrow h = 0 \text{ (linear independence of } G^{(i)}).\end{aligned}$$

Hence $(V_0 G)^{(i)} = V_0 G^{(i)}$ are linearly independent. Suppose now that the $V_0 G^{(i)}$ are linearly independent. Then $(V_0 G)h = 0 \Leftrightarrow h = 0$. It must follow then that $\det C \neq 0$ for, if not, $\exists$ a nonzero constant $h' \in \mathcal{M}_{3\times 1}$ such that $Ch' = 0$. Hence, from the proof of Theorem 2, we can find a nonzero constant $B = (\rho G)^{-1}h' \in \mathcal{M}_{3\times 1}$ such that

$$\begin{aligned}V_0(GB) &= V_0(\Phi h')\\ &= (V_0\Phi)h'\\ &= FCh' \quad \text{(using (6))}\\ &= 0,\end{aligned}$$

i.e., $V_0(GB) = (V_0 G)B = 0$ for nonzero B, which contradicts the assumption that the $V_0 G^{(i)}$ are linearly independent. Hence, $\det C \neq 0$ and by Theorem 3, (5) has at most one solution.

EXAMPLE. It is not difficult to show [6] that another matrix of fundamental solutions for the operator L is $D^B(x,y) = D(x,y) + F(x)BF^{\mathrm{T}}(y)$, where $B \in \mathcal{M}_{5\times 5}$ is constant and symmetric. The above results remain valid when $D(x,y)$ is replaced by $D^B(x,y)$, the only difference being that C now varies with B. With this in mind, we choose ∂S to be the circle of radius R, centre the origin and replace $D(x,y)$ with $D^B(x,y)$ with B chosen such that

$$B_{33} = \frac{k^2}{8\pi}(\mu+\kappa)^2(R^{-1} + 4h^4), \qquad k^2 = [h^4(\mu+\kappa)^2\gamma(\lambda+2\mu+\kappa)(\alpha+\beta+\gamma)]^{-1}$$

and all other elements of B equal zero. We find that $V_0\phi = 0$ for every $\phi \in \mathcal{M}_{5\times 1}$ of the form $(0,0,\mathfrak{l},0,0)^{\mathrm{T}}$, $\mathfrak{l} = \text{const}$. It follows that $\Phi^{(3)} = (0,0,(2\pi R)^{-1},0,0)^{\mathrm{T}}$ and $C^{(3)} = (0,0,0)^{\mathrm{T}}$. Hence, we have found a specific curve ∂S for which the homogeneous equation from (5) has nonzero solutions and for which the matrix C is singular.

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References

[1] G. C. Hsiao and R. E. Kleinman, *On a unified characterization of capacity*, Potential Theory, Plenum, Prague, 1988, pp. 5–119

[2] E. Hille, *Analytic Function Theory*, Vol. II, Ginn and Company, Boston, New York, 1962

[3] C. Constanda, *On integral solutions of the equations of thin plates*, Proc. Royal Soc. London **A 444**, 317–323 (1994)

[4] P. Schiavone and C. Constanda, *Existence theorems in the theory of bending of micropolar plates*, Internat. J. Engrg. Sci. **27**, 463–468 (1989)

[5] P. Schiavone and C. Constanda, *Uniqueness in the elastostatic problem of bending of micropolar plates*, Arch. Mech. **41**, 5, 785–791 (1989)

[6] P. Schiavone, *The application of the boundary integral equation method to boundary value problems in the bending of thin micropolar plates*, Ph.D. Thesis, University of Strathclyde, Glasgow, Scotland, 1988