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KINETIC AND FLUID ASPECTS OF GAS DISCHARGES

By

A. NOURI

Laboratoire J. A. Dieudonné, Nice, France

Abstract. Gas discharges are investigated in two levels: the kinetic one, where a stationary electron distribution is determined, and the fluid one, where use of the small mobility of ions with respect to the mobility of electrons leads to a simplified model and a boundary layer analysis.

Introduction. The microscopic study of gas discharges is usually carried out using a linearised form of the Boltzmann equation. The electron distribution function $f = f(t,x,v)$ satisfies the following transport equation:

$$\partial_t f + v \cdot \nabla_x f - \frac{e}{m} E \cdot \nabla_v f = Q(f), \qquad t \geq 0, \quad x \in \Omega, \quad v \in \mathbb{R}^3. \tag{0.1}$$

The constants e and m are respectively the charge and the effective mass of an electron, $E = E(t,x)$ is the electric field, Ω is a smooth bounded domain of $\mathbb{R}^3$ describing the medium locus, for example, a tube where discharges occur. The integral operator Q, described in [10], is intended to model the interaction of the electrons with the neutral molecules and the ions. The electron densities considered here are small compared to the density of the neutral molecules, so that the ionization phenomenon alone can be taken into account. It consists of collisions of the electrons with neutral molecules, which gives rise to the creation of new electrons, and is modelled by

$$Q(f)(x,v) = 2\int_{\mathbb{R}^3} q(v,w) f(x,w)\,dw - f(x,v)\int_{\mathbb{R}^3} q(v,w)\,dw, \qquad x \in \Omega, \quad v \in \mathbb{R}^3.$$

Integrating (0.1) with respect to v over $\mathbb{R}^3$ does not lead to a conservation equation, as it occurs for the Boltzmann equation for example, because of the factor 2 in Q. It displays the ionization phenomenon, which produces new free electrons. The discharge considered here occurs in a cylindrical tube between a cathode and an anode, so that the given electric field is directed along the cylinder axis, say $x'x$, and the electron density can be written

$$f = f(t,x,v), \qquad t \in \mathbb{R}_+, \quad x \in [0,1], \quad v \in \mathbb{R}.$$

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Emission of electrons occurs at the cathode only and is assumed independent of time, which is modelled by the given input data:

$$f(t,0,v) = f_0(v), \quad t \in \mathbb{R}_+, \quad v > 0,$$
$$f(t,1,v) = 0, \qquad t \in \mathbb{R}_+, \quad v < 0.$$

If an equilibrium between the ingoing data and the inside flow is reached, stationary-in-time solutions occur. We prove in the first section that such solutions exist and are unique in two cases, either for a small electric field or for a large one. The second purpose of this paper is to study the fluid approximation of the kinetic model usually considered by physicists in the context of low $\frac{|E|}{N}$ (see [1, 2, 5, 9]) where N is the gas density. The electron density n, the ion density p, and the electric potential ϕ are solutions of the drift-diffusion type equations

$$n_t - \nabla \cdot (D_e \nabla n - \mu_e n \nabla \phi) = \nu(\nabla \phi) n, \tag{0.2}$$
$$p_t - \nabla \cdot (D_p \nabla p + \mu_p p \nabla \phi) = \nu(\nabla \phi) n, \tag{0.3}$$

together with the Poisson equation

$$\Delta \phi = -\frac{e}{\varepsilon_0}(p - n). \tag{0.4}$$

Here, ε_0 is the given permittivity, μ_e (resp. μ_p) is the electronic (resp. ionic) mobility, and D_e (resp. D_p) the electronic (resp. ionic) diffusion coefficient. They are related by Einstein's relations

$$D_e = U_T \mu_e, \qquad D_p = U_p \mu_p,$$

where the thermal voltage U_T is given by

$$U_T = \frac{k_B T}{e}.$$

k_B denotes Boltzmann's constant and T the device temperature. $\nu(\nabla \phi)$, which denotes the ionization rate, is a smooth function, typically given by

$$\nu(\nabla \phi) = A e^{-B |\nabla \phi|^{-1/2}}.$$

The mathematical analysis of the initial boundary-value problem related to (0.2)–(0.4) has been performed for analogous equations in the semiconductor frame ([3, 7, 11, 12]). Only the source term of Eqs. (0.2)–(0.3) differs, but this does not require substantial modifications of the proofs. In this paper, the smallness of the ion mobility μ_p with respect to the electron mobility μ_e is used. The limit of the solution of (0.2)–(0.4) when $\frac{\mu_p}{\mu_e}$ tends to zero is described, as well as the occurring boundary layer.

1. Stationary kinetic solution of the ionization process.

THEOREM 1.1. Let q, ν, and g_0 belong to $L^2(\mathbb{R}^2), L^\infty(\mathbb{R}_+)$, and $L^2(\mathbb{R}_+)$ respectively. Then there are bounds A^* and A_* such that the following holds. If

$$A > A^* \quad \text{and} \quad \int \frac{q^2(v,w)}{v^2}\, dv\, dw < \infty,$$

or

$$A < A_* \quad \text{and} \quad \int \frac{q^2(v,w)}{v^2}\, dv\, dw < \frac{1}{2},$$

there is a unique weak solution $f \in L^2((0,1)\times\mathbb{R})$ of

$$v\frac{\partial f}{\partial x} + A\frac{\partial f}{\partial v} = Q(f), \qquad x \in [0,1], \quad v \in \mathbb{R},$$
$$f(0,v) = g_0(v), \quad v > 0, \qquad f(1,v) = 0, \quad v < 0.$$

The proof of Theorem 1.1 is based on the following lemma.

LEMMA 1.2. The solution of

$$v\frac{\partial g}{\partial x} + A\frac{\partial g}{\partial v} + \nu g = \int q(v,w)h(x,w)\, dw,$$
$$g(0,v) = k(v), \quad v > 0, \qquad g(1,v) = 0, \quad v < 0,$$

is analytically given by

$$g(x,v) = k(\sqrt{v^2 - 2Ax})e^{\int_v^{\sqrt{v^2-2Ax}} \frac{\nu}{A}(\alpha)\, d\alpha}$$
$$+ \int_0^x \int_{\mathbb{R}} e^{\int_v^{\sqrt{v^2+2A(\sigma-x)}} \frac{\nu}{A}(\alpha)\, d\alpha} \frac{q(\sqrt{v^2+2A(\sigma-x)}, w)}{\sqrt{v^2+2A(\sigma-x)}} h(\sigma,w)\, dw\, d\sigma \quad \text{if } v \geq \sqrt{2Ax},$$

$$g(x,v) = \int_x^1 \int_{\mathbb{R}} e^{\int_v^{-\sqrt{v^2+2A(\sigma-x)}} \frac{\nu}{A}(\alpha)\, d\alpha} \frac{q(-\sqrt{v^2+2A(\sigma-x)}, w)}{\sqrt{v^2+2A(\sigma-x)}} h(\sigma,w)\, dw\, d\sigma, \quad \text{if } v < 0,$$

$$g(x,v) = \int_{x-\frac{v^2}{2A}}^1 \int_{\mathbb{R}} e^{\int_v^{-\sqrt{v^2+2A(\sigma-x)}} \frac{\nu}{A}(\alpha)\, d\alpha} \frac{q(-\sqrt{v^2+2A(\sigma-x)}, w)}{\sqrt{v^2+2A(\sigma-x)}} h(\sigma,w)\, dw\, d\sigma$$
$$+ \int_{x-\frac{v^2}{2A}}^x \int_{\mathbb{R}} e^{\int_v^{\sqrt{v^2+2A(\sigma-x)}} \frac{\nu}{A}(\alpha)\, d\alpha} \frac{q(\sqrt{v^2+2A(\sigma-x)}, w)}{\sqrt{v^2+2A(\sigma-x)}} h(\sigma,w)\, dw\, d\sigma,$$
$$\text{if } 0 < v < \sqrt{2Ax}.$$

The proof of Lemma 1.2 is based on elementary computations along the characteristics.

Proof of Theorem 1.1. The proof consists of three steps. In the first step, a sequence of approximate solutions (f_n) is constructed, using the Schauder fixed-point theorem. The second step proves that, for A small enough, the sequence (f_n) is uniformly bounded in $L^2((0,1)\times\mathbb{R}_v)$. The last step consists of passing to the limit when n tends to ∞.

Denote

$$a \wedge b = \min\{a,b\}, \qquad a \vee b = \max\{a,b\}.$$

Step 1. Our aim is to prove the existence of a solution of the boundary value problem

$$v\frac{\partial f_n}{\partial x} + A\frac{\partial f_n}{\partial v} + \nu(v)f_n$$
$$= 2\int_{w\in\mathbb{R}} q(v,w)\chi_n(v,w)\frac{f_n(x,w)}{1+\frac{f_n(x,w)}{n}}\, dw, \qquad x \in [0,1], \quad v \in \mathbb{R}, \tag{1.1}$$

$$f(0,v) = g_0(v) \wedge n, \quad v > 0, \qquad f(1,v) = 0, \quad v < 0,$$

where

$$\chi_n(v,w)=1 \quad \text{if } |v|\le n \quad \text{and} \quad |w|\le n, \qquad \chi_n(v,w)=0 \quad \text{otherwise.}$$

This can be proved by a fixed-point argument. Let K be the closed and convex set of $L^\infty((0,1)\times\mathbb{R})$ defined by

$$K=\{f\in L^\infty((0,1)\times\mathbb{R});\ \|f\|_\infty\le n(1+4\sqrt{An}\|q\|_{L_2})\}.$$

Let T be the map defined on K by

$$v\frac{\partial g}{\partial x}+A\frac{\partial g}{\partial v}+\nu g=2\int q(v,w)\chi_n(v,w)\frac{f(x,w)}{1+\frac{f(x,w)}{n}}\,dw,$$
$$g(0,v)=g_0(v)\wedge n,\quad v>0,\qquad g(1,v)=0,\quad v<0.$$

By Lemma 1.2, T maps K into K and is continuous for the weak-$*$ topology of $L^\infty((0,1)\times\mathbb{R})$. Moreover, T is compact for this topology, since K is bounded. Hence, by Schaefer's fixed-point theorem, T has a fixed point in K denoted f_n.

Step 2. It follows from Lemma 1.2 and the definition of f_n that

$$\|f_n\|^2_{L^2((0,1)\times\mathbb{R})}\le 2\|g_0\|^2_{L^2(\mathbb{R}_+)}+K(A)\|f_n\|^2_{L^2}, \tag{1.2}$$

where

$$\begin{aligned}K(A):=&\int_0^1\int_{v\ge\sqrt{2Ax}}\int_0^x\int_{\mathbb{R}}\frac{q^2(\sqrt{v^2+2A(\tau-x)},u)}{v^2+2A(\tau-x)}\,dx\,dv\,d\tau\,du\\&+\int_0^{\sqrt{2A}}\int_0^1\int_{x-\frac{v^2}{2A}}^x\int_{\mathbb{R}}\frac{q^2(\sqrt{v^2+2A(\tau-x)},u)}{v^2+2A(\tau-x)}\,dv\,dx\,d\tau\,du\\&+\int_0^1\int_{v<0}\int_x^1\int_{\mathbb{R}}\frac{q^2(-\sqrt{v^2+2A(\tau-x)},u)}{v^2+2A(\tau-x)}\,dx\,dv\,d\tau\,du\\&+\int_0^{\sqrt{2A}}\int_0^1\int_{x-\frac{v^2}{2A}}^1\int_{\mathbb{R}}\frac{q^2(-\sqrt{v^2+2A(\tau-x)},u)}{v^2+2A(\tau-x)}\,dv\,dx\,d\tau\,du.\end{aligned}$$

On one hand,

$$\lim_{A\to0^+}K(A)\le\int\frac{q^2(v,w)}{v^2}\,dv\,dw,$$

which implies that, for A small enough and $\int\frac{q^2(v,w)}{v^2}\,dv\,dw<\frac12$, (f_n) is uniformly bounded in $L^2((0,1)\times\mathbb{R})$. On the other hand, if $\int\frac{q^2(v,w)}{v^2}\,dv\,dw<\infty$, then

$$\lim_{A\to\infty}K(A)=0,$$

which implies that, for A large enough, (f_n) is uniformly bounded in $L^2((0,1)\times\mathbb{R})$.

Step 3. Recall a consequence of [4] on averaging, discussed in [8].

PROPERTY 1.3. Let $(f_n),(g_n)$, and (h_n) be bounded sequences of $L^2((0,1)\times\mathbb{R})$ that satisfy

$$v\frac{\partial f_n}{\partial x}=\frac{\partial g_n}{\partial v}+h_n$$

in the sense of distributions. Then for any Hilbert-Schmidt operator K defined on $L^2(\mathbb{R}^2)$, the sequence $(K(f_n))$ is relatively compact in $L^2((0,1)\times\mathbb{R})$.

Here,

$$v\frac{\partial f_n}{\partial x} = \frac{\partial(-Af_n)}{\partial v} + 2\int_{\mathbb{R}} q(v,w)\chi_n(v,w)\frac{f_n(x,w)}{1+\frac{f_n(x,w)}{n}}\,dw - \nu f_n,$$

so that Property 1.3 applies, with

$$g_n := -Af_n, \qquad h_n(x,v) := 2\int_{\mathbb{R}} q(v,w)\chi_n(v,w)\frac{f_n(x,w)}{1+\frac{f_n(x,w)}{n}}\,dw - \nu f_n(x,v).$$

$(f_n), (g_n)$, and (h_n) are bounded in $L^2((0,1)\times\mathbb{R})$, thanks to Step 2 and the belongness of q and ν to $L^2(\mathbb{R}^2)$ and $L^\infty(\mathbb{R})$ respectively. Hence $\int_{\mathbb{R}} q(v,w)f_n(x,w)\,dw$, and also $(2\int_{\mathbb{R}} q(v,w)\chi(v,w)\frac{f_n(x,w)}{1+\frac{f_n(x,w)}{n}})\,dw$, are relatively compact in $L^2((0,1)\times\mathbb{R})$. Finally, the previous relative compactness applied to the solution formula in Lemma 1.2 allows us to pass to the limit when $n\to\infty$ and proves the existence of a solution to the stationary problem.

Step 4. If there were two solutions f and g, then, as for (1.2),

$$\|f-g\|_{L^2} \le K(A)\|f-g\|_{L^2}.$$

It follows from $K(A)<1$ that $f=g$.

2. Approximation of the fluid model. The aim of this section is to perform an analysis of the fluid model (0.2)–(0.4) when the ratio of the ionic to the electron mobilities μ_p/μ_e tends to 0. First, n_r being a reference density, introduce the following scalings on the variables and unknowns:

$$x = \sqrt{\frac{\varepsilon_0}{e}U_T n_r}\,x', \qquad t = \frac{e_0 n_r}{e\mu_e}t',$$
$$n(x) = n_r n'(x'), \qquad p(x) = n_r p'(x'),$$
$$\nu'(\nabla\phi') = \frac{\varepsilon_0 n_r}{e\mu_e}\nu(U_T\nabla\phi),$$

so that (0.2)–(0.4) become

$$n'_{t'} - \nabla_{x'}\cdot(\nabla_{x'}n' - (\nabla_{x'}\phi')n') = \nu'(\nabla\phi')n',$$
$$p'_{t'} - \frac{\mu_p}{\mu_e}\nabla_{x'}\cdot(\nabla_{x'}p' + (\nabla_{x'}\phi')p') = \nu'(\nabla\phi')n',$$
$$\Delta_{x'}\phi' = n' - p'.$$

Denote $\varepsilon = \mu_p/\mu_e$ and drop the primes for the sake of convenience. The problem of interest is then to find a $(n^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ solution of

$$n_t^\varepsilon - \nabla_x\cdot(\nabla_x n^\varepsilon - (\nabla_x\phi^\varepsilon)n^\varepsilon) = \nu(\nabla\phi^\varepsilon)n^\varepsilon, \tag{2.1}$$
$$p_t^\varepsilon - \varepsilon\nabla_x\cdot(\nabla_x p^\varepsilon + (\nabla_x\phi^\varepsilon)p^\varepsilon) = \nu(\nabla\phi^\varepsilon)n^\varepsilon, \tag{2.2}$$
$$\Delta_x\phi^\varepsilon = n^\varepsilon - p^\varepsilon, \tag{2.3}$$

supplemented with the initial conditions

$$n^\varepsilon(0,x) = n_0(x), \qquad p^\varepsilon(0,x) = p_0(x), \quad x \in \Omega, \tag{2.4}$$

and the boundary conditions

$$n^\varepsilon(t,x) = \tilde{n}_b(t,x), \qquad p^\varepsilon(t,x) = \tilde{p}_b(t,x), \qquad \phi^\varepsilon(t,x) = \tilde{\phi}_b(t,x), \quad t>0, \quad x \in \partial\Omega. \tag{2.5}$$

Here Ω is a smooth bounded domain of $\mathbb{R}^3$.

First, recall the basic results proved in [11, 12], which are useful for the present analysis.

PROPERTY 2.1. Let T positive be given. Assume that ν is a function bounded by ν^* and that (n_0, p_0), (n_b, p_b), and ϕ_b belong to $(L^2(\Omega))^2$, $(H^1(0,T;H^{\frac{3}{2}}(\partial\Omega))^2$, and $L^2((0,T)\times\Omega)$ respectively.

For every positive ε, there is a unique solution $(n^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ of (2.1)–(2.5) belonging to $(L^2(0,T;H^1(\Omega)\cap A))^2 \times L^2(0,T;C^1(\overline{\Omega}))$, where

$$A = \{z \in L^2((0,T)\times\Omega) \text{ such that } 0 \leq z \leq Ke^{\lambda t}\},$$

K and λ being constants depending on ν, (n_0,p_0), and (n_b,p_b,ϕ_b).

As ε tends to 0, the following convergence result holds:

THEOREM 2.2. When ε tends to 0, the solution $(n^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ of (2.1)–(2.5) tends in $L^\infty(0,T;H^1(\Omega))$ weak $*$, $L^\infty(0,T;L^2(\Omega))$ weak $*$, and $L^2(0,T;H^1(\Omega))$ respectively, to $(\overline{n}, \overline{p}, \overline{\phi})$, the solution of

$$\overline{n}_t - \nabla\cdot(\nabla\overline{n} - (\nabla\overline{\phi})\overline{n}) = \nu(\nabla\overline{\phi})\overline{n}, \tag{2.6}$$

$$\overline{p}_t = \nu(\nabla\overline{\phi})\overline{n}, \tag{2.7}$$

$$\Delta\overline{\phi} = \overline{n} - \overline{p}, \tag{2.8}$$

with the initial conditions

$$\overline{n}(0,x) = n_0(x), \quad \overline{p}(0,x) = p_0(x), \quad x \in \Omega, \tag{2.9}$$

and the boundary conditions

$$\overline{n}(t,x) = \tilde{n}_b(t,x), \quad \overline{\phi}(t,x) = \phi_b(t,x), \qquad t>0, \quad x \in \partial\Omega. \tag{2.10}$$

Here, and in the sequel, ∇ and $\nabla\cdot$ respectively denote the gradient and divergence with respect to the space. Remark that there is no boundary condition on $\overline{p}$ in contrast to the p^ε case. That justifies the boundary layer analysis performed at the end of this section.

Proof of Theorem 2.2. Let n_b (resp. p_b) be an extension in $H^1(0,T;H^2(\Omega))$ of $\tilde{n}_b$ (resp. $\tilde{p}_b$) such that

$$\Delta n_b = \Delta p_b = 0, \qquad t \in (0,T), \quad x \in \Omega,$$
$$n_b(t,x) = \tilde{n}_b(t,x), \quad p_b(t,x) = \tilde{p}_b(t,x), \qquad t>0, \quad x \in \partial\Omega.$$

Then

$$\begin{aligned}(n^\varepsilon - n_b)_t - \nabla \cdot [\nabla(n^\varepsilon - n_b) - \nabla\phi^\varepsilon(n^\varepsilon - n_b)] \\ = \nu(\nabla\phi^\varepsilon)(n^\varepsilon - n_b) - \nabla \cdot ((\nabla\phi^\varepsilon)n_b) + \nu(\nabla\phi^\varepsilon)n_b - (n_b)_t,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}(p^\varepsilon - p_b)_t - \varepsilon\nabla \cdot [\nabla(p^\varepsilon - p_b) + \nabla\phi^\varepsilon(p^\varepsilon - p_b)] \\ = \nu(\nabla\phi^\varepsilon)(n^\varepsilon - n_b) + \varepsilon\nabla \cdot ((\nabla\phi^\varepsilon)p_b) + \nu(\nabla\phi^\varepsilon)n_b - (p_b)_t.\end{aligned} \tag{2.12}$$

Integrate (2.11) over Ω, so that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |n^\varepsilon - n_b|^2 + \int_\Omega |\nabla(n^\varepsilon - n_b)|^2 - \int_\Omega (n^\varepsilon - n_b)\nabla\phi^\varepsilon \cdot \nabla(n^\varepsilon - n_b) \\ = \int_\Omega \nu(\nabla\phi^\varepsilon)(n^\varepsilon - n_b)^2 + \int_\Omega n_b\nabla\phi^\varepsilon \cdot \nabla(n^\varepsilon - n_b) + \int_\Omega [\nu(\nabla\phi^\varepsilon)n_b - (p_b)_t](n^\varepsilon - n_b).\end{aligned} \tag{2.13}$$

$n^\varepsilon - p^\varepsilon$ belongs to A, and so is bounded. Hence, from (2.3), (ϕ^ε) is bounded in any $L^\infty(0,T;W^{2,p}(\Omega))$, $p \geq 1$. It follows, from Rellich-Kondrakov's imbedding theorem that $(\nabla\phi^\varepsilon)$ is bounded in $L^\infty((0,T)\times\Omega)$. From now on, c denotes any constant independent of ε. Since, for any constant α,

$$\begin{aligned}\int_\Omega (n^\varepsilon - n_b)\nabla\phi^\varepsilon \cdot \nabla(n^\varepsilon - n_b) \\ \leq \frac{c\alpha}{2}\int_\Omega |n^\varepsilon - n_b|^2 + \frac{c}{2\alpha}\int_\Omega |\nabla(n^\varepsilon - n_b)|^2,\end{aligned}$$

(2.13) leads to

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |n^\varepsilon - n_b|^2 + c\int_\Omega |\nabla(n^\varepsilon - n_b)|^2 \leq c. \tag{2.14}$$

Handling (2.12) the same way gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |p^\varepsilon - p_b|^2 + c\varepsilon\int_\Omega |\nabla(p^\varepsilon - p_b)|^2 \leq c. \tag{2.15}$$

This implies that $(\sqrt{\varepsilon}\nabla p^\varepsilon)$ is bounded in $L^2((0,T)\times\Omega)$, so that

$$\Delta(\phi_t^\varepsilon) = n_t^\varepsilon - p_t^\varepsilon = \nabla \cdot [\nabla n^\varepsilon - (\nabla\phi^\varepsilon)n^\varepsilon - \sqrt{\varepsilon}(\sqrt{\varepsilon}\nabla p^\varepsilon)] \tag{2.16}$$

is bounded in $L^\infty(0,T;H^{-1}(\Omega))$. It follows from (2.14)–(2.16) and (2.3) that

(n^ε) is bounded in $L^\infty(0,T;H^1(\Omega))$,

(p^ε) and $(\sqrt{\varepsilon}\nabla p^\varepsilon)$ are bounded in $L^\infty(0,T;L^2(\Omega))$,

$(\nabla\phi^\varepsilon)$ is bounded in $L^\infty(0,T;H^1(\Omega))$,

$((\nabla\phi^\varepsilon)_t)$ is bounded in $L^\infty(0,T;L^2(\Omega))$.

So there exists $(\overline{n}, \overline{p}, \overline{q}, \overline{\phi})$ such that, up to subsequences,

$$\begin{aligned} &n^\varepsilon \rightharpoonup \overline{n} \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak } *,\\ &p^\varepsilon \rightharpoonup \overline{p} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak } *,\\ &\sqrt{\varepsilon}(\nabla p^\varepsilon) \rightharpoonup \overline{q} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak } *,\\ &\nabla\phi^\varepsilon \to \nabla\overline{\phi} \text{ in } L^2((0,T)\times\Omega) \text{ strong,} \end{aligned}$$

thanks to Aubin's Lemma ([6]). Thus it is possible to pass to the limit in the weak formulation of (2.1) and in the weak formulation of (2.2), written in the following way:

$$p_t^\varepsilon - \sqrt{\varepsilon}(\sqrt{\varepsilon}\nabla p^\varepsilon + \sqrt{\varepsilon}\nabla\phi^\varepsilon p^\varepsilon) = \nu(\nabla\phi^\varepsilon)n^\varepsilon.$$

Then, from (2.3), it is straightforward to obtain

$$\Delta\overline{\phi} = \overline{n} - \overline{p},$$

which ends the proof of Theorem 2.2.

Finally, since there is no other boundary condition on $\overline{p}$, some boundary layer study is necessary. Let Ω_δ denote

$$\Omega_\delta = \{x \in \Omega \text{ such that } \operatorname{dist}(x,\Omega) \leq \delta\}.$$

Since $\overline{\Omega}$ is smooth and compact, the variables $\xi \in \partial\Omega$ and $y \in \mathbb{R}_+$ provide a set of coordinates in Ω_δ for δ small enough, such that for any $x \in \Omega_\delta$ there is a unique $(\xi, y) \in \partial\Omega \times \mathbb{R}_+$ that satisfies

$$x = \xi - y\nu(\xi),$$

where $\nu(\xi)$ denotes the outward normal to $\partial\Omega$ at ξ. Let $(\hat{n}, \hat{p}, \hat{\phi})$ be defined by

$$\begin{aligned} \hat{n}(t,\xi,y) &= n^\varepsilon(t, \xi - \sqrt{\varepsilon}y\nu(\xi)),\\ \hat{p}(t,\xi,y) &= p^\varepsilon(t, \xi - \sqrt{\varepsilon}y\nu(\xi)),\\ \hat{\phi}(t,\xi,y) &= \phi^\varepsilon(t, \xi - \sqrt{\varepsilon}y\nu(\xi)). \end{aligned}$$

The boundary layer analysis states the following theorem.

THEOREM 2.3. $(\hat{n}, \hat{\phi})$ are given by

$$\begin{aligned} \hat{n}(t,\xi,y) &= n_b(t,\xi),\\ \hat{\phi}(t,\xi,y) &= \phi_b(t,\xi), \end{aligned}$$

and $\hat{p}$ is the solution of

$$\begin{aligned} &\hat{p}_t - c(\xi,y)\partial_{yy}^2\hat{p} = 0,\\ &\hat{p}(0,\xi,y) = p_0(\xi,y), \quad \xi \in \partial\Omega, \quad y \in \mathbb{R}_+,\\ &\hat{p}(t,\xi,0) = p_b(t,\xi), \quad t > 0, \quad \xi \in \partial\Omega, \end{aligned}$$

where $c(\xi, y)$ is a given positive function.

Proof of Theorem 2.3. Express $(n^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ with respect to $(\hat{n}, \hat{p}, \hat{\phi})$ in the initial system (2.1)–(2.3). Keeping the terms appearing with the smallest power of ε leads to

$$\partial^2_{yy}\hat{n} - \partial_y(\hat{n}\partial_y\hat{\phi}) = 0, \tag{2.17}$$

$$\hat{p}_t - c(\xi, y)[\partial^2_{yy}\hat{p} + \partial_y(\hat{p}\partial_y\hat{\phi})] = \nu(\nabla\hat{\phi})\hat{n}, \tag{2.18}$$

where

$$\varepsilon^{-1}c(\xi, y) \text{ is the } \varepsilon^{-1}\text{-order term appearing in } |\nabla_x y|^2, \text{ and } \partial^2_{yy}\hat{\phi} = 0, \tag{2.19}$$

supplemented with the initial conditions

$$\hat{n}(0,\xi,y) = n_0(\xi,y), \quad \hat{p}(0,\xi,y) = p_0(\xi,y), \qquad \xi \in \partial\Omega, \quad y \in \mathbb{R}_+, \tag{2.20}$$

and the boundary conditions

$$\hat{n}(t,\xi,0) = n_b(t,\xi), \qquad \hat{p}(t,\xi,0) = p_b(t,\xi), \qquad \hat{\phi}(t,\xi,0) = \phi_b(t,\xi), \quad t > 0, \quad \xi \in \partial\Omega. \tag{2.21}$$

Since $\hat{\phi}$ is a bounded function, (2.19) and (2.21) imply that

$$\hat{\phi}(t,\xi,y) = \phi_b(t,\xi), \qquad t > 0, \quad \xi \in \partial\Omega, \quad y \in \mathbb{R}_+.$$

Then, from (2.17) and (2.21), $\hat{n}$ is given by

$$\hat{n}(t,\xi,y) = n_b(t,\xi), \qquad t > 0, \quad \xi \in \partial\Omega, \quad y \in \mathbb{R}_+.$$

Finally, $\hat{p}(\cdot,\xi,\cdot)$ is obtained as the unique solution in $C([0,+\infty); L^2(\mathbb{R}_+)) \cap C((0,+\infty); H^2(\mathbb{R}_+))$ of the parabolic system (2.18)–(2.21). This ends the proof of Theorem 2.3.

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