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ASYMPTOTIC STABILITY AND GLOBAL EXISTENCE IN THERMOELASTICITY WITH SYMMETRY

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Abstract. First we prove an exponential decay result for solutions of the equations of linear, homogeneous, isotropic thermoelasticity in bounded regions in two or three space dimensions if the rotation of the displacement vanishes. As a consequence, we describe the decay in radially symmetrical situations, and in a cylinder in $\mathbb{R}^3$. Then we establish the global existence of solutions to the corresponding nonlinear equations for small smooth initial data and a certain class of nonlinearities.

1. Introduction. In this article we will first prove the exponential decay of solutions to the equations of linear thermoelasticity for a homogeneous and isotropic medium in a bounded region in $\mathbb{R}^n$, $n = 2, 3$, if the rotation of the displacement vanishes identically. As a consequence, we obtain the exponential decay in radially symmetrical situations (and in a cylinder in $\mathbb{R}^3$). Then we establish the global existence of smooth small solutions to the corresponding nonlinear equations for a certain class of nonlinearities.

The asymptotic behavior as $t \to \infty$ of solutions to the equations of linear thermoelasticity in a bounded domain has been studied by many authors. In one dimension, it is well known that solutions decay (to zero) exponentially for all the classical boundary conditions (see, for example, [2, 8, 10, 11, 13, 16, 17, 25]), while in more than one dimension the situation becomes more delicate. Dafermos [7] (also cf. [15, 20]) investigated the linear equations of n-dimensional thermoelasticity and showed that, e.g., if the displacement u and the temperature difference θ satisfy Dirichlet boundary conditions, then θ tends to zero and u tends to a function $\tilde{u}$ as time goes to infinity. Whether the

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function $\tilde{u}$ is zero depends on the geometry of the domain, e.g., $\tilde{u} = 0$ for a rectangle but $\tilde{u} \neq 0$ for the unit ball in $\mathbb{R}^2$. However, no decay rate was given in [7] and [20]. Henry, Perissinitto, and Lopes [10, 9] proved that in more than one dimension there is no uniform decay rate of solutions for spatially periodic boundary conditions or for the domain containing a finite cylinder the ends of which are in the boundary. Racke [24] studied some special boundary conditions and proved the exponential decay of θ and the curl-free part of u. Recently Jiang [12] showed that solutions with radial symmetry decay exponentially in annular domains with appropriately large diameter. We also mention the works of Carvalho Pereira and Prela Menzala [4, 5], Racke [24] who showed that if an additional damping term u_t is added to the equations, then solutions converge to zero exponentially.

The purpose of the present work is to show the exponential decay for the case when the rotation of the displacement vanishes identically, in particular, for radially symmetrical solutions to the linear equations of homogeneous and isotropic thermoelasticity, and to prove the global existence of solutions of the corresponding nonlinear equations for small smooth initial data and a certain class of nonlinearities. The main results are Theorems 2.1, 3.4, and 4.2 in Secs. 2, 3, and 4.

The paper is organized as follows: In Sec. 2 we prove the exponential decay of solutions with vanishing rotation for the linear case, in Sec. 3 radial symmetry is considered, and in Sec. 4 we establish the global existence of solutions to the nonlinear system for small smooth initial data.

We now introduce the notation used throughout this paper. ^{T} denotes transposition. Let G be a domain in $\mathbb{R}^n$. By $W^{m,p}(G)$ ($m \in \mathbb{N}_0$, $1 \leq p \leq \infty$) we denote the usual Sobolev space defined over G with norm $\|\cdot\|_{W^{m,p}}$ (see, e.g., [1]); $W^{m,2}(G) \equiv H^m(G)$ with norm $\|\cdot\|_{H^m}$, $W^{0,p}(G) \equiv L^p(G)$ with norm $\|\cdot\|_{L^p}$, $H_0^1 \equiv H_0^1(G)$: completion of the test functions $C_0^\infty(G)$ in H^1. For simplicity we also use the abbreviations $L^p \equiv L^p(G)$ and $H^m \equiv H^m(G)$. The norm and inner product in $L^2(G)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively. $C^L(I,B)$ (resp. $L^2(I,B)$) denotes the space of B-valued functions that are L-times continuously differentiable (resp. square integrable) in I, $I \subset \mathbb{R}$ an interval, B a Banach space, L a nonnegative integer. We denote by $O(n)$ the set of orthogonal $n \times n$ real matrices and by $SO(n)$ the set of matrices in $O(n)$ that have determinant 1. $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ is the inner product in $\mathbb{R}^n$. For a vector-valued function $h = (h_1, \ldots, h_m)^{\mathrm{T}}$ and a normed space X with the norm $|||\cdot|||$, $h \in X$ means that each component of h is in X; we put $|||h||| := |||h_1||| + \cdots + |||h_m|||$.

C or C_1 will denote various positive constants which, in particular, do not depend on t and the initial data.

2. Exponential stability in linear thermoelasticity with vanishing rotation. We consider the equations of linear thermoelasticity for a homogeneous, isotropic medium with bounded reference configuration $G \subset \mathbb{R}^n$, $n = 2$ or 3, having a smooth boundary ∂G:

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\theta = 0, \tag{2.1}$$

$$c\theta_t - \kappa\Delta\theta + \beta \operatorname{div} u_t = 0, \tag{2.2}$$

where $u(x,t) = (u_1, \dots, u_n)$, $x \in G$, $t \geq 0$, is the displacement, $\theta = \theta(x,t)$ is the temperature difference, μ and λ are the Lamé moduli satisfying $\mu > 0$ and $2\mu + n\lambda > 0$, and $c, \kappa > 0$ and $\beta \neq 0$ are given constants depending on the material properties.

We shall investigate the large-time behavior of solutions to the initial-boundary value problem (2.1), (2.2) with initial conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad \theta(x,0) = \theta^0(x), \quad x \in G, \tag{2.3}$$

and Dirichlet boundary conditions

$$u(\cdot,t)|_{\partial G} = 0, \quad \theta(\cdot,t)|_{\partial G} = 0, \quad t \geq 0. \tag{2.4}$$

Defining the rotation of a vector field $u = (u_1, u_2)$ in $\mathbb{R}^2$ to be the scalar

$$\operatorname{rot} u := \partial_1 u_2 - \partial_2 u_1$$

we can formulate the main theorem for two and three space dimensions as follows:

THEOREM 2.1. Let (u, θ) be the solution to (2.1)–(2.4) and assume

$$\operatorname{rot} u = 0 \quad \text{in } G \times [0, \infty). \tag{2.5}$$

Then there are constants $\Gamma \geq 1$ and $\gamma > 0$ independent of the initial data and of t such that

$$E(t) + \int_0^t e^{\gamma s} \|\nabla \theta_t(s)\|^2 \, ds \leq \Gamma E(0), \tag{2.6}$$

where

$$E(t) := e^{\gamma t} \left\{ \sum_{k=0}^{2} \|\partial_t^k u(t)\|_{H^{2-k}}^2 + \|\theta_t(t)\|^2 + \|\theta(t)\|_{H^2}^2 \right\}. \tag{2.7}$$

REMARK 2.2. The existence of solutions to (2.1)–(2.4) for data such that the right-hand side of (2.6) is finite, is standard; see, for example, [15], [20]. If we define the rotation of a scalar field f in $\mathbb{R}^2$ to be

$$\operatorname{rot} f := (\partial_2 f, -\partial_1 f)^{\mathrm{T}},$$

the classical formula for the vector Laplace operator

$$\Delta = \nabla \operatorname{div} - \operatorname{rot} \operatorname{rot}$$

is valid in two and in three space dimensions.

Proof of Theorem 2.1. Since u has rotation zero we have

$$\nabla \operatorname{div} u = \Delta u \quad \text{and} \quad \|\nabla u\| = \|\operatorname{div} u\|. \tag{2.8}$$

Let $\tau := 2\mu + \lambda$ and

$$\begin{aligned}
F_1(t) &:= \tfrac{1}{2}(\|u_t\|^2 + \tau\|\nabla u\|^2 + c\|\theta\|^2)(t), \\
F_2(t) &:= \tfrac{1}{2}(\|u_{tt}\|^2 + \tau\|\nabla u_t\|^2 + c\|\theta_t\|^2)(t), \\
F_3(t) &:= \tfrac{1}{2}(\|\nabla u_t\|^2 + \tau\|\nabla \operatorname{div} u\|^2 + c\|\nabla\theta\|^2)(t), \\
F(t) &:= \sum_{j=1}^{3} F_j(t).
\end{aligned}$$

Multiplying (2.1) by u_t and (2.2) by θ in $L^2(G)$, integrating by parts, we obtain (dropping t in most places)

$$\frac{d}{dt}F_1(t) = -\|\nabla\theta\|^2. \tag{2.9}$$

Analogously, after differentiating (2.1), (2.2) with respect to t:

$$\frac{d}{dt}F_2(t) = -\|\nabla\theta_t\|^2. \tag{2.10}$$

Multiplying (2.1) by $-\Delta u_t$ and (2.2) by $-\Delta\theta$ in $L^2(G)$ respectively, integrating by parts with respect to x, we get

$$\frac{d}{dt}F_3(t) = -\kappa\|\Delta\theta\|^2 + \beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\operatorname{div} u_t, \tag{2.11}$$

where $\vec{n} = (n_1, \dots, n_n)$ denotes the outer normal on ∂G.

$$\left|\beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\operatorname{div} u_t\right| \le \varepsilon\tau\int_{\partial G}|\operatorname{div} u_t|^2 + \frac{C}{\varepsilon}\int_{\partial G}\left|\frac{\partial\theta}{\partial\vec{n}}\right|^2 \equiv I_1(t) + I_2(t), \tag{2.12}$$

where $0 < \varepsilon < 1$ will be chosen later.

In order to estimate the boundary terms I_1 and I_2, we shall use the following lemma.

LEMMA 2.3. a) Let $v = (v_1, v_2, v_3)$ be a solution to the equations of elasticity:

$$\begin{aligned} v_{tt} - \mu\Delta v - (\mu+\lambda)\nabla\operatorname{div} v &= h_1 \quad \text{in } G\times[0,\infty),\\ v|_{\partial G} &= 0 \quad \text{in } [0,\infty). \end{aligned}$$

Then

$$\begin{aligned} \mu\int_{\partial G}\left|\frac{\partial v}{\partial\vec{n}}\right|^2 + (\mu+\lambda)\int_{\partial G}|\operatorname{div} v|^2 &= 2\frac{d}{dt}\int_G v_t\sigma_k\partial_k v\\ &+ \int_G \operatorname{div}\sigma|v_t|^2 + 2\mu\int_G \partial_j v^i\partial_j\sigma_k\partial_k v^i - \mu\int_G\operatorname{div}\sigma|\nabla v|^2\\ &+ 2(\mu+\lambda)\int_G \operatorname{div} v\nabla\sigma_k\partial_k v - (\mu+\lambda)\int_G\operatorname{div}\sigma|\operatorname{div} v|^2 - 2\int_G h_1\sigma_k\partial_k v, \end{aligned} \tag{2.13}$$

where $\sigma \in (C^1(\overline{G}))^3$ such that $\sigma_i = n_i$ on ∂G, $i = 1, 2, 3$, and the Einstein summation convention is used.

b) Let θ be a solution to the heat equation

$$\begin{aligned} c\theta_t - \kappa\Delta\theta &= h_2 \quad \text{in } G\times[0,\infty),\\ \theta|_{\partial G} &= 0 \quad \text{in } [0,\infty). \end{aligned}$$

Then

$$\kappa\int_{\partial G}\left|\frac{\partial\theta}{\partial\vec{n}}\right|^2 = 2c\int_G\theta_t\sigma\nabla\theta + 2\kappa\int_G\nabla\theta\nabla\sigma_k\partial_k\theta - \kappa\int_G\operatorname{div}\sigma|\nabla\theta|^2 - \int_G h_2\sigma\nabla\theta. \tag{2.14}$$

REMARK 2.4. The easy proof of Lemma 2.3 is presented for the sake of completeness at the end of this section; cf. [14], [18].

Now we are able to estimate $I_1(t), I_2(t)$ from (2.12) as follows, using (2.1)–(2.2), Cauchy-Schwarz's and Poincaré's inequalities:

$$I_1 \le \varepsilon \frac{2\tau}{\mu+\lambda}\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C\varepsilon(\|u_{tt}\|^2 + \|\nabla u_t\|^2 + \|\nabla\theta_t\|^2) - \frac{\varepsilon\mu\tau}{\mu+\lambda}\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2, \tag{2.15}$$

$$I_2 \le \varepsilon\|\operatorname{div} u_t\|^2 + \frac{C}{\varepsilon^2}(\|\theta_t\|^2 + \|\nabla\theta\|^2) \le \varepsilon\|\operatorname{div} u_t\|^2 + \frac{C}{\varepsilon^2}(\|\nabla\theta_t\|^2 + \|\nabla\theta\|^2). \tag{2.16}$$

The relations (2.11), (2.12), (2.15), (2.16) imply

$$\begin{aligned}\frac{d}{dt}F_3(t) \le -\kappa\|\Delta\theta\|^2 - \frac{\varepsilon\mu\tau}{\mu+\lambda}\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2 + \frac{2\varepsilon\tau}{\mu+\lambda}\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t \\ + C\varepsilon(\|u_{tt}\|^2 + \|\nabla u_t\|^2) + \frac{C}{\varepsilon^2}(\|\nabla\theta_t\|^2 + \|\nabla\theta\|^2).\end{aligned} \tag{2.17}$$

Choosing $\eta \ge \frac{C}{\varepsilon^2} + 1$ and using Poincaré's inequality, we conclude from (2.9), (2.10), (2.17) that

$$\begin{aligned}\frac{d}{dt}(\eta F_1 + \eta F_2 + F_3) \le -C_1(\|\theta\|^2 + \|\nabla\theta\|^2 + \|\theta_t\|^2 + \|\nabla\theta_t\|^2 + \|\Delta\theta\|^2) \\ - \frac{\varepsilon\mu\tau}{\mu+\lambda}\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2 \\ + \frac{2\tau\varepsilon}{\mu+\lambda}\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C\varepsilon(\|u_{tt}\|^2 + \|\nabla u_t\|^2).\end{aligned}$$

From the differential equation (2.2) we know

$$\|\operatorname{div} u_t\|^2 \le C(\|\theta_t\|^2 + \|\Delta\theta\|^2);$$

hence, using Poincaré's inequality and (2.8), we obtain

$$\begin{aligned}\frac{d}{dt}(\eta F_1 + \eta F_2 + F_3) \le -C_1\|(\theta, \nabla\theta, \theta_t, \nabla\theta_t, \Delta\theta, u_t, \nabla u_t)\|^2 - \frac{\varepsilon\mu\tau}{\mu+\lambda}\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2 \\ + \frac{2\tau\varepsilon}{\mu+\lambda}\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C\varepsilon\|(u_{tt}, \nabla u_t)\|^2.\end{aligned} \tag{2.18}$$

Using the differential equation (2.1) and integrating by parts we have

$$\frac{d}{dt}\int_G u_t u = \tau\int_G u\nabla\operatorname{div} u - \beta\int_G u\nabla\theta + \|u_t\|^2 \le -\frac{\tau}{2}\|\nabla u\|^2 + C\|(\theta, u_t)\|^2, \tag{2.19}$$

$$\begin{aligned}\frac{d}{dt}\int_G \operatorname{div} u \operatorname{div} u_t &= \|\operatorname{div} u_t\|^2 - \int_G \nabla\operatorname{div} u\, u_{tt} \\ &= \|\operatorname{div} u_t\|^2 - \tau\|\nabla\operatorname{div} u\|^2 + \beta\int_G \nabla\operatorname{div} u\nabla\theta \\ &\le -\frac{\tau}{2}\|\nabla\operatorname{div} u\|^2 + C\|(\nabla u_t, \nabla\theta)\|^2.\end{aligned} \tag{2.20}$$

Since by (2.1) and (2.8), $\|u_{tt}\|^2 \le 2\tau^2\|\nabla\operatorname{div} u\|^2 + 2\beta^2\|\nabla\theta\|^2$, whence

$$-\frac{\tau}{2}\|\nabla\operatorname{div} u\|^2 \le -\frac{\tau}{4}\|\nabla\operatorname{div} u\|^2 - \frac{1}{8\tau}\|u_{tt}\|^2 + \frac{\beta^2}{4\tau}\|\nabla\theta\|^2, \tag{2.21}$$

we conclude from (2.18)–(2.21), (2.8), and Poincaré's inequality

$$\begin{aligned}
&\frac{d}{dt}\underbrace{\left\{\eta F_1+\eta F_2+F_3-\frac{2\varepsilon\tau}{\mu+\lambda}\int_G u_{tt}\sigma_k\partial_k u_t+\sqrt{\varepsilon}\int_G u_t u+\sqrt{\varepsilon}\int_G \operatorname{div} u \operatorname{div} u_t\right\}}_{=:\,H(t)}\\
&\quad\le -C_1\|(\theta,\nabla\theta,\theta_t,\nabla\theta_t,\Delta\theta,u_t,\nabla u_t)\|^2-C_1\sqrt{\varepsilon}\|(u,\nabla u,\Delta u,u_{tt})\|^2\\
&\qquad\qquad -\frac{\varepsilon\mu\tau}{\mu+\lambda}\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2+C(\varepsilon\|u_{tt}\|^2+\sqrt{\varepsilon}\|(\theta,u_t,\nabla\theta,\nabla u_t)\|^2).
\end{aligned}$$

Choosing ε such that $C\sqrt{\varepsilon}=C_1/2$ we get

$$\forall t\ge 0:\quad \frac{d}{dt}H(t)+C\left(\|\nabla\theta_t(t)\|^2+\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2 dx+F(t)\right)\le 0. \tag{2.22}$$

Observing (choosing η large enough)

$$\exists\, k_1,k_2>0\quad \forall t\ge 0:\quad k_1F(t)\le H(t)\le k_2F(t),$$

hence we conclude from (2.22) with $\gamma:=C/k_2$

$$e^{\gamma t}F(t)+\int_0^t e^{\gamma s}\|\nabla\theta_t(s)\|^2\,ds+\int_0^t e^{\gamma s}\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2 dx\,ds\le CF(0)\le CE(0)\quad\forall t\ge 0. \tag{2.23}$$

If we apply the elliptic regularity property

$$\|h\|_{H^2}^2\le\widehat{\Gamma}\|\Delta h\|^2,\quad h\in H_0^1,\ \Delta h\in L^2, \tag{2.24}$$

with $\widehat{\Gamma}\ge 1$ being a constant, and (2.2), we get $\|\theta(t)\|_{H^2}\le C\|\Delta\theta(t)\|^2\le C(\|\theta_t\|^2+\|\operatorname{div} u_t\|^2)$, which together with (2.23) yields the assertion (2.6). □

REMARK 2.5. As the proof shows, a Lipschitz boundary ∂G (used for the H^2-regularity in (2.24)) would have been sufficient.

Proof of Lemma 2.3. a) Multiplying the differential equations for v by $\sigma_k\partial_k v$ and integrating, we obtain

$$\int v_{tt}\sigma_k\partial_k v-\mu\int\Delta v_i\sigma_k\partial_k v_i-(\mu+\lambda)\int\nabla\operatorname{div} v\sigma_k\partial_k v=\int h_1\sigma_k\partial_k v, \tag{2.25}$$

where $\int := \int_G$. By virtue of $v|_{\partial\Omega} = 0$, $\nabla v|_{\partial\Omega} = (\nabla v \cdot \vec{n})\vec{n}|_{\partial\Omega}$. So integrating by parts, we get

$$\int v_{tt}\sigma_k\partial_k v = \frac{d}{dt}\int v_t\sigma_k\partial_k v + \frac{1}{2}\int \operatorname{div}\sigma|v_t|^2, \tag{2.26}$$

$$\begin{aligned}-\mu\int \Delta v_i\sigma_k\partial_k v_i = &-\frac{\mu}{2}\int_{\partial G}\left|\frac{\partial v}{\partial\vec{n}}\right|^2 + \mu\int \partial_j v_i\partial_j\sigma_k\partial_k v_i \\ &-\frac{\mu}{2}\int \operatorname{div}\sigma|\nabla v|^2,\end{aligned} \tag{2.27}$$

$$\begin{aligned}-(\mu+\lambda)\int \nabla\operatorname{div} v\,\sigma_k\partial_k v = &-\frac{\mu+\lambda}{2}\int_{\partial G}|\operatorname{div} v|^2 + (\mu+\lambda)\int \operatorname{div} v\nabla\sigma_k\partial_k v \\ &-\frac{\mu+\lambda}{2}\int \operatorname{div}\sigma|\operatorname{div} v|^2.\end{aligned} \tag{2.28}$$

From (2.25)–(2.28) we conclude (2.13).

b) Multiplying the differential equation for θ by $\sigma_k\partial_k\theta$ and integrating, we obtain

$$c\int \theta_t\sigma_k\partial_k\theta - \kappa\int \Delta\theta\sigma_k\partial_k\theta = \int h_2\sigma_k\partial_k\theta. \tag{2.29}$$

Integrating by parts, similarly to (2.27) we obtain

$$-\kappa\int \Delta\theta\sigma_k\partial_k\theta = -\frac{\kappa}{2}\int_{\partial G}\left|\frac{\partial\theta}{\partial\vec{n}}\right|^2 + \kappa\int \nabla\theta\nabla\sigma_k\partial_k\theta - \frac{\kappa}{2}\int \operatorname{div}\sigma|\nabla\theta|^2. \tag{2.30}$$

From (2.29), (2.30) we conclude (2.14). □

3. Radial symmetry—the linear case. We consider the equations of linear thermoelasticity for a homogeneous and isotropic medium with unit reference density in a smoothly bounded, radially symmetrical domain G in $\mathbb{R}^n$, $n = 2, 3$; i.e.,

$$x \in G \Rightarrow \forall\Omega \in O(2)(\text{if } n = 2) \text{ resp. } SO(3)(\text{if } n = 3): \ \Omega x \in G.$$

That is, G can be obtained by rotation of an $(n-1)$-dimensional domain around the nth axis; typical examples are balls and annular domains, the latter having been considered in [12].

We recall the definition of radially symmetrical vector fields and functions, respectively:

DEFINITION 3.1. A vector field $u : G \to \mathbb{R}^n$ [a function $\theta : G \to \mathbb{R}$] is called radially symmetrical, if

$$\forall\Omega \in O(2)(\text{if } n = 2) \text{ or } SO(3)(\text{if } n = 3)\forall x \in G : u(\Omega x) = \Omega u(x) \ [\theta(\Omega x) = \theta(x)].$$

Radially symmetrical functions are characterized by the following (folklore) lemma.

LEMMA 3.2. i) θ is a radially symmetrical function $\Leftrightarrow$ there exists a function $\psi : \mathbb{R}_0^+ \to \mathbb{R}$ with $\theta(x) = \psi(r)$, $r = |x|$, $x \in G$.

ii) u is a radially symmetrical vector field $\Leftrightarrow$ there exists a function $\phi : \mathbb{R}_0^+ \to \mathbb{R}$ with $u(x) = x\phi(r)$.

iii) In ii) one has $0 \in G \Rightarrow u(0) = 0$.

iv) A radially symmetrical vector field has vanishing rotation.

Proof. iii) and the if-part "$\Leftarrow$" in i) and ii) are obvious, also iv), if ii) is proved. The only-if-part of i) is also clear choosing

$$\psi(r) := \theta(re_1),$$

where e_1 denotes the first unit vector in $\mathbb{R}^n$.

Finally, let u be a radially symmetrical vector field.

Case I. $n = 2$. Let $x = (x_1, x_2)^{\mathrm{T}} \in G$, $x \neq 0$, be arbitrary but fixed and let

$$\Omega := \begin{pmatrix} x_1/r & x_2/r \\ -x_2/r & x_1/r \end{pmatrix} \in O(2).$$

From the assumption it follows that

$$u(x,t) = \begin{pmatrix} x_1/r & -x_2/r \\ x_2/r & x_1/r \end{pmatrix} \begin{pmatrix} u_1(re_1,t) \\ u_2(re_1,t) \end{pmatrix}. \tag{3.1}$$

Taking $\Omega := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in O(2)$, we obtain that $u_2(re_1,t) = 0$, which together with (3.1) gives $u(x,t) = x\phi(r,t)$ with $\phi(r,t) := u_1(re_1,t)/r$.

Case II. $n = 3$. For $x \in G$, let $\widetilde{SO}_x(n) := \{\Omega \in SO(n) \mid \Omega x = x\}$ denote the set of all rotations about the x-direction. From the assumption we have

$$u(x,t) = u(\widetilde{\Omega}x,t) = \widetilde{\Omega}u(x,t) \quad \text{for any } \widetilde{\Omega} \in \widetilde{SO}_x(3). \tag{3.2}$$

By (3.2) we conclude that there is a $\zeta(x) \in \mathbb{R}$ such that $u(x,t) = \zeta(x)x$. Obviously, $\zeta(x) = \langle u(x), x\rangle_{\mathbb{R}^3}/|x|^2$ for $x \neq 0$. It follows from the assumption that for any $\Omega \in SO(3)$

$$\zeta(\Omega x) = \frac{\langle u(\Omega x), \Omega x\rangle_{\mathbb{R}^3}}{|\Omega x|^2} = \frac{\langle \Omega u(x), \Omega x\rangle_{\mathbb{R}^n}}{|x|^2} = \frac{\langle u(x), x\rangle_{\mathbb{R}^3}}{|x|^2} = \zeta(x),$$

which implies (cf. i)) $\zeta(x) = \phi(|x|) := \zeta(|x|e_1)$. Therefore, $u(x,t) = x\phi(|x|,t)$ for $x \in G$, $x \neq 0$. □

Radially symmetrical data produce radially symmetrical solutions to the equations of thermoelasticity, since we have

LEMMA 3.3. If the data u^0, u^1, θ^0 in (2.3) are radially symmetrical, then the solution u, θ to (2.1)–(2.4) is radially symmetrical and the rotation of u vanishes for all times.

Proof. Let $\Omega \in O(2)$ for $n = 2$ resp. $\in SO(3)$ for $n = 3$ be arbitrary but fixed, and denote $v(x,t) := \Omega^{\mathrm{T}} u(\Omega x, t), \Theta(x,t) := \theta(\Omega x, t)$. After a straightforward calculation we get

$$\begin{gathered} v_{tt}(x,t) = \Omega^{\mathrm{T}} u_{tt}(\Omega x,t), \qquad \Delta v(x,t) = \Omega^{\mathrm{T}}(\Delta u)(\Omega x,t), \\ \operatorname{div} v_t(x,t) = (\operatorname{div} u_t)(\Omega x,t), \qquad \nabla \operatorname{div} v = \Omega^{\mathrm{T}}(\nabla \operatorname{div} u)(\Omega x,t), \\ \Theta_t(x,t) = \theta_t(\Omega x,t), \quad \nabla\Theta(x,t) = \Omega^{\mathrm{T}}(\nabla\Theta)(\Omega x,t), \quad \Delta\Theta(x,t) = (\Delta\theta)(\Omega x,t). \end{gathered} \tag{3.3}$$

Hence we see that v and Θ satisfy Eqs. (2.1)–(2.2). From the uniqueness of solutions to (2.1)–(2.4) we get $u \equiv v, \theta \equiv \Theta$, which proves the radial symmetry of $v(\cdot,t), \Theta(\cdot,t)$, and the vanishing of the rotation of $u(\cdot,t)$ by Lemma 3.2. □

As a consequence, we can apply Theorem 2.1, and we obtain the exponential decay for radially symmetrical situations:

THEOREM 3.4. Assume that the domain G, having a smooth boundary, is radially symmetrical, and that the initial data u^0, u^1, θ^0 are radially symmetrical. Let (u, θ) be the solution to (2.1)–(2.4). Then there are constants $\Gamma \geq 1$ and $\gamma > 0$ independent of the initial data and of t such that

$$E(t) + \int_0^t e^{\gamma s} \|\nabla \theta_t(s)\|^2 \, ds \leq \Gamma E(0), \tag{3.4}$$

where

$$E(t) = e^{\gamma t} \left\{ \sum_{k=0}^{2} \|\partial_t^k u(t)\|_{H^{2-k}}^2 + \|\theta_t(t)\|^2 + \|\theta(t)\|_{H^2}^2 \right\}. \tag{3.5}$$

REMARK 3.5. As mentioned before, a Lipschitz boundary ∂G is sufficiently smooth.

As the end of this section, we consider the system (2.1)–(2.2) in a cylinder $G := \{x = (x', x_3)^{\mathrm{T}} \in \mathbb{R}^3 \mid |x'| < 1, x' = (x_1, x_2)^{\mathrm{T}};\ 0 < x_3 < l\}$ for some $l > 0$ with the initial conditions (2.3) and the following boundary conditions: Dirichlet on the lateral surface of the cylinder and periodicity on the top and on the bottom,

$$\begin{gathered} u(x', x_3) = 0, \quad \theta(x', x_3) = 0, \quad \text{if } |x'| = 1, \\ u(x', x_3 + l, t) = u(x', x_3, t), \qquad \theta(x', x_3 + l, t) = \theta(x', x_3, t) \\ \text{for all } x = (x', x_3)^{\mathrm{T}} \in G,\ t \geq 0. \end{gathered} \tag{3.6}$$

Assume that u^0, u^1, θ^0 satisfy

$$u^0(x) = (w^0(x'), 0)^{\mathrm{T}}, \quad u^1(x) = (w^1(x'), 0)^{\mathrm{T}}, \quad \theta^0(x) = \xi^0(x')$$

for any $x = (x', x_3)^{\mathrm{T}} \in G$, for some w^0, w^1, ξ^0, and that

$$w^0, w^1, \xi^0 \text{ are radially symmetrical in } \{x' \in \mathbb{R}^2 \mid |x'| < 1\}.$$

Following the same arguments as used in the proof of Theorem 3.4 and Theorem 2.1, respectively, we can also obtain the exponential decay of the solution to the problem (2.1)–(2.3), (3.6). Here we do not want to go into the details.

4. A global existence theorem for nonlinear thermoelasticity. Without loss of generality, we restrict ourselves to three space dimensions. Then the equations of nonlinear thermoelasticity for a homogeneous medium with unit reference density in $G \subset \mathbb{R}^3$ read (for a derivation see [3, 21, 25])

$$\frac{\partial^2 u_i}{\partial t^2} = C_{i\alpha j\beta}(\nabla u, \theta) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} + \widetilde{C}_{i\alpha}(\nabla u, \theta) \frac{\partial \theta}{\partial x_a}, \quad i = 1, 2, 3, \tag{4.1}$$

$$a(\nabla u, \theta)\theta_t = \frac{1}{b(\theta)} \operatorname{div} q(\nabla u, \theta, \nabla \theta) + \widetilde{C}_{i\alpha}(\nabla u, \theta) \frac{\partial^2 u_i}{\partial x_\alpha \partial t}. \tag{4.2}$$

Here (and throughout this section) the Einstein summation convention is used; $u = (u_1, u_2, u_3)^{\mathrm{T}}$, $q = (q_1, q_2, q_3)^{\mathrm{T}}$, $a \geq a_0 > 0$, b is a C^∞-function such that $b(\theta) = \theta + T_0$ for $|\theta| \leq T_0/2$ and $0 < b_1 \leq b(\theta) \leq b_2 < \infty$, b_1, b_2 constants, $-\infty < \theta < \infty$, $T_0 > 0$ the reference temperature. (4.1)–(4.2) are derived for small values of $|\theta|$, i.e., for $|\theta| \leq T_0/2$, which is a posteriori justified by the smallness of the solutions obtained later.

We consider (2.3)–(2.4) as the initial and boundary conditions for (4.1)–(4.2). We assume that the medium is *initially isotropic*

$$\begin{gathered} C_{i\alpha j\beta}(0,0) = \lambda\delta_{i\alpha}\delta_{j\beta} + \mu(\delta_{ij}\delta_{\alpha\beta} + \delta_{\alpha j}\delta_{i\beta}), \qquad \widetilde{C}_{i\alpha}(0,0) = -\beta\delta_{i\alpha}, \\ \frac{\partial q_i(0,0,0)}{\partial(\partial\theta/\partial x_j)} = k\delta_{ij}, \qquad a(0,0) = c, \end{gathered} \tag{4.3}$$

and that

$$\begin{gathered} C_{i\alpha j\beta}(\nabla u,\theta) = C_{j\beta i\alpha}(\nabla u,\theta), \qquad \frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial\theta/\partial x_j)} = \frac{\partial q_j(\nabla u,\theta,\nabla\theta)}{\partial(\partial\theta/\partial x_i)}, \\ \frac{\partial q_i(0,0,0)}{\partial\theta} = 0, \qquad \frac{\partial q_i(0,0,0)}{\partial(\partial u_\alpha/\partial x_\beta)} = 0, \qquad 1 \le i,j,\alpha,\beta \le 3. \end{gathered} \tag{4.4}$$

Using (4.3), we can write (4.1)–(4.2) in the form:

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\theta = f(\nabla u,\theta,\nabla^2 u,\nabla\theta), \tag{4.5}$$

$$c\theta_t - \kappa\Delta\theta + \beta \operatorname{div} u_t = g(\nabla u,\theta,\nabla\theta,\nabla^2 u,\nabla^2\theta,\nabla u_t), \tag{4.6}$$

where $\kappa := k/T_0 > 0$, $f = (f_1, f_2, f_3)^{\mathrm{T}}$, and

$$f_i := (C_{i\alpha j\beta}(\nabla u,\theta) - C_{i\alpha j\beta}(0,0))\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} + (\widetilde{C}_{i\alpha}(\nabla u,\theta) - \widetilde{C}_{i\alpha}(0,0))\frac{\partial\theta}{\partial x_\alpha}, \quad i = 1,2,3, \tag{4.7}$$

$$\begin{aligned} g := c&\left(\frac{1}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial\theta/\partial x_j)} - \frac{1}{a(0,0)b(0)}\frac{\partial q_i(0,0,0)}{\partial(\partial\theta/\partial x_j)}\right)\frac{\partial^2\theta}{\partial x_i\partial x_j} \\ &+ c\left(\frac{\widetilde{C}_{i\alpha}(\nabla u,\theta)}{a(\nabla u,\theta)} - \frac{\widetilde{C}_{i\alpha}(0,0)}{a(0,0)}\right)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + \frac{c}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial\theta}\frac{\partial\theta}{\partial x_i} \\ &+ \frac{c}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial u_\alpha/\partial x_\beta)}\frac{\partial^2 u_\alpha}{\partial x_i\partial x_\beta}. \end{aligned} \tag{4.8}$$

Concerning the global existence for (4.5)–(4.6), only the Cauchy problem has been investigated. Racke [21], Ponce and Racke [19] essentially proved that small smooth solutions exist globally in time if one excludes quadratic nonlinearities in the displacement. On the other hand, one has to expect a blow-up in the general "genuinely nonlinear" case (cf. [22]).

In this section, by combining Theorem 2.1, the local existence and uniform a priori estimates, we establish the global existence of smooth small solutions to the system (4.5), (4.6), (2.3), (2.4) in the case of $\operatorname{rot} u = 0$. As an application, the global existence of radially symmetrical solutions is shown at the end of the section.

Let u^j $(j = 2,3,4)$, θ^j $(j = 1,2,3)$ be defined through (4.5)–(4.6) by

$$u^j := \partial_t^j u|_{t=0}, \quad j = 2,3,4; \qquad \theta^j := \partial_t^j \theta|_{t=0}, \quad j = 1,2,3. \tag{4.9}$$

In fact, u^j, θ^j are obtained successively from u^0, u^1, and θ^0 by differentiating (4.5)–(4.6) with respect to t at $t = 0$.

Assuming ∂G to be smooth (cf. Secs. 2 and 3), we now state a local existence theorem, which can be established by a standard contraction mapping argument; we omit the details of the proof here (see [6, 23] and the references therein).

LEMMA 4.1. Assume that $u^j \in H^{4-j}$ $(j = 0, \dots, 4)$, $\theta^j \in H^{4-j}$ $(j = 0, 1, 2)$, $\theta^3 \in L^2$, and the initial data are compatible with the boundary conditions (2.4). Then there is a positive constant $K_0 \le \min\{1, T_0/2\}$ such that if

$$|\nabla u^0(x)|, |\theta^0(x)|, |\nabla \theta^0(x)| < K_0 \quad \text{for all } x \in \overline{G}, \tag{4.10}$$

then there exists a unique solution (u, θ) of (4.5), (4.6), (2.3), (2.4) defined on a maximal interval of existence $[0, T)$, $T \le \infty$ such that for any $\hat{t} \in [0, T)$

$$\begin{gathered} u \in \bigcap_{j=0}^{4} C^j([0, \hat{t}], H^{4-j}), \qquad \theta \in \bigcap_{j=0}^{2} C^j([0, \hat{t}], H^{4-j}), \\ \theta_{ttt} \in C^0([0, \hat{t}], L^2) \cap L^2([0, \hat{t}], H^1), \end{gathered} \tag{4.11}$$

$$\forall (x, t) \in \overline{G} \times [0, T): \quad |\nabla u(x, t)|, |\theta(x, t)|, |\nabla \theta(x, t)| < K_0. \tag{4.12}$$

Furthermore, if

$$\sup_{t \in [0,T)} \left(\sum_{j=0}^{4} \|\partial_t^j u\|_{H^{4-j}}^2 + \sum_{j=0}^{2} \|\partial_t^j \theta\|_{H^{4-j}}^2 + \|\theta_{ttt}\|^2 \right)(t) + \int_0^T \|\nabla \theta_{ttt}(s)\|^2 \, ds < \infty \tag{4.13}$$

and

$$\sup_{x \in \overline{G}, t \in [0,T)} \{|\nabla u(x, t)|, |\theta(x, t)|, |\nabla \theta(x, t)|\} < K_0, \tag{4.14}$$

then $T = \infty$.

To get the global existence we require in addition that for $(\tilde{u}, \tilde{\theta})$ with $\operatorname{rot} \tilde{u} = 0$

$$C_{i\alpha j\beta}(\nabla \tilde{u}, \tilde{\theta}) \frac{\partial^2 \tilde{u}_j}{\partial x_\alpha \partial x_\beta} \equiv A_{ij}(\nabla \tilde{u}, \tilde{\theta}) \Delta \tilde{u}_j, \quad i = 1, 2, 3. \tag{4.15}$$

Then the main result in this section reads

THEOREM 4.2. Let (u, θ) be the solution of (4.5), (4.6), (2.3), (2.4) established in Lemma 4.1. Assume that $\operatorname{rot} u = 0$ for $(x, t) \in G \times (0, T)$. Then there is a constant $\varepsilon > 0$ such that if

$$\sum_{j=0}^{4} \|u^j\|_{H^{4-j}}^2 + \sum_{j=0}^{2} \|\theta^j\|_{H^{4-j}}^2 + \|\theta^3\|^2 \le \varepsilon^2, \tag{4.16}$$

then $T = \infty$. Moreover, $\|u(t)\|_{H^4}$, $\|\theta(t)\|_{H^4}$ decay to zero exponentially as $t \to \infty$.

Proof. By virtue of (4.16) and continuity, there is a $t_0 \in (0,T]$ such that

$$\begin{aligned} M(t) := e^{\gamma t} &\left\{ \sum_{j=0}^{4} \|\partial_t^j u\|_{H^{4-j}}^2 + \sum_{j=0}^{2} \|\partial_t^j \theta\|_{H^{4-j}}^2 + \|\theta_{ttt}\|^2 \right\}(t) \\ &+ \int_0^t e^{\gamma s} \|\nabla \theta_{ttt}(s)\|^2 \, ds \le \Lambda \varepsilon^2 \text{ for all } t \in [0, t_0), \end{aligned} \tag{4.17}$$

where the constant Λ is defined by

$$\Lambda := 258[(\beta^2 + c^2 + 1)\widetilde{\Gamma}]^3 \Gamma(1 + \kappa^{-2}) \sum_{j=0}^{2} \tau^{-2j} > 1. \tag{4.18}$$

Here $\widetilde{\Gamma}$ is defined through the elliptic regularity property (cf. (2.24))

$$\|h\|_{H^{j+2}}^2 \le \widetilde{\Gamma} \|\Delta h\|_{H^j}^2, \quad h \in H_0^1, \quad \Delta h \in L^2, \quad j = 0, 1, 2. \tag{4.19}$$

Denote

$$0 < T^* := \sup\{t_1 > 0 \mid M(t) \le \Lambda \varepsilon^2 \text{ in } [0, t_1)\} \le T. \tag{4.20}$$

Then we have either $T^* = T$ or $T^* < T$. The former case implies that the solution is bounded and small for all $t \in [0,T)$ by Sobolev's imbedding theorem provided ε is small enough; therefore (by virtue of Lemma 4.1) $T = \infty$. It remains only to consider the latter case. We will show by contradiction that the latter case does not happen if ε is sufficiently small.

From (4.20) and Sobolev's imbedding theorem ($H^2(G) \hookrightarrow L^\infty(G)$) we have

$$\|(u,\theta)(t)\|_{W^{2,\infty}}, \|(u_t, \theta_t)(t)\|_{W^{1,\infty}}, \|(u_{tt}, \theta_{tt})\|_{L^\infty} \le C\varepsilon e^{-\gamma t/2} \quad \forall t \in [0, T^*). \tag{4.21}$$

Denote

$$\mathcal{N}(t; u, \theta) := e^{\gamma t} \left(\sum_{j=0}^{2} \|\partial_t^j u\|_{H^{2-j}}^2 + \|\theta_t\|^2 + \|\theta\|_{H^2}^2 \right)(t) + \int_0^t e^{\gamma s} \|\nabla \theta_t(s)\|^2 \, ds. \tag{4.22}$$

Recall that $\operatorname{rot} u = 0$ in $G \times (0,T)$. Repeating the same procedure as in the derivation of (2.23) and (2.6) in Sec. 2, we obtain for Eqs. (4.5)–(4.6) (and also apply Cauchy-Schwarz's inequality) that

$$\begin{aligned} &\mathcal{N}(t; u, \theta) + \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial u_t}{\partial \vec{n}} \right|^2 dx \, ds \\ &\quad \le \Gamma \mathcal{N}(0; u, \theta) + C e^{\gamma t} \|g(t)\|^2 + C \int_0^t e^{\gamma s} \{ \|f\| (\|u\| + \|u_t\| + \|\Delta u\| + \|f\|) \\ &\qquad + \|g\| (\|\theta\| + \|\nabla \theta\| + \|\Delta \theta\| + \|g\|) \} \, ds + C \left\{ \left| \int_0^t e^{\gamma s} (f_t, u_{tt}) \, ds \right| \right. \\ &\qquad \left. + \left| \int_0^t e^{\gamma s} (f, \Delta u_t) \, ds \right| + \left| \int_0^t e^{\gamma s} (g_t, \theta_t) \, ds \right| + \left| \int_0^t e^{\gamma s} (f_t, \sigma_k \partial_k u_t) \, ds \right| \right\} \\ &\quad \equiv \Gamma \mathcal{N}(0; u, \theta) + \mathcal{P}(t; u, \theta, f, g), \quad t \in [0, T^*). \end{aligned} \tag{4.23}$$

Here Γ is the same as in Theorem 2.1.

If we take ∂_t^l $(l = 1, 2)$ on both sides of (4.5)–(4.6), we see that $\partial_t^l u$ and $\partial_t^l \theta$ $(l = 1, 2)$ solve (4.5), (4.6), (2.3), (2.4) with the right-hand sides f and g replaced by $\partial_t^l f$ and $\partial_t^l g$ respectively. Thus by (4.23) we have for $t \in [0, T^*)$, $l = 1, 2$,

$$\begin{aligned} &\mathcal{N}(t; \partial_t^l u, \partial_t^l \theta) + \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial}{\partial \vec{n}} \partial_t^{l+1} u \right|^2 dx\, ds \\ &\quad \le \Gamma \mathcal{N}(0; \partial_t^l u, \partial_t^l \theta) + \mathcal{P}(t; \partial_t^l u, \partial_t^l \theta, \partial_t^l f, \partial_t^l g). \end{aligned} \tag{4.24}$$

In the sequel we estimate $\mathcal{P}(t; \partial_t^l u, \partial_t^l \theta, \partial_t^l f, \partial_t^l g)$ for $l = 0, 1, 2$. It follows from Leibniz's formula, (4.20)–(4.21), (4.4), and the mean value theorem that

$$\begin{aligned} e^{\gamma t} \|\partial_t^l g(t)\|^2 + \int_0^t e^{\gamma s} \{ &\|\partial_t^l f\| (\|\partial_t^l u\| + \|\partial_t^{l+1} u\| + \|\partial_t^l \Delta u\| + \|\partial_t^l f\|) \\ &+ \|\partial_t^l g\| (\|\partial_t^l \theta\| + \|\partial_t^l \nabla \theta\| + \|\partial_t^l \Delta \theta\| + \|\partial_t^l g\|) \}(s)\, ds \le C\varepsilon^3 \end{aligned} \tag{4.25}$$

for $t \in [0, T^*)$, $l = 0, 1, 2$. Since $u = 0$ on ∂G, $|\nabla u|^2 = |(\nabla u, \vec{n})\vec{n}|^2 = |\frac{\partial u}{\partial \vec{n}}|^2$ on ∂G. So if we apply Leibniz's formula, use (4.20)–(4.21), (4.4), the mean value theorem, (4.15), and integrate by parts, we obtain that for $l = 0, 1, 2$

$$\begin{aligned} &\left| \int_0^t e^{\gamma s} (\partial_t^l f, \partial_t^{l+1} \Delta u)\, ds \right| \\ &\quad \le C\varepsilon^3 + \left| \int_0^t e^{\gamma s} (f_{tt}, \Delta u_{ttt})\, ds \right| \\ &\quad \le C\varepsilon^3 + \left| \int_0^t e^{\gamma s} (f_{ttt}, \Delta u_{tt})\, ds \right| \\ &\quad \le C\varepsilon^3 + \left| \int_0^t e^{\gamma s} ([A_{ij}(\nabla u, \theta) - A_{ij}(0,0)] \partial_t^3 \Delta u_j, \partial_t^2 \Delta u_i)\, ds \right| \\ &\quad \le C\varepsilon^3 + \frac{1}{2} \left| \int_0^t e^{\gamma s} \frac{d}{dt} ([A_{ij}(\nabla u, \theta) - A_{ij}(0,0)] \partial_t^2 \Delta u_j, \partial_t^2 \Delta u_i)\, ds \right| \\ &\quad \le C\varepsilon^3 + \frac{1}{2} e^{\gamma t} |([A_{ij}(\nabla u, \theta) - A_{ij}(0,0)] \partial_t^2 \Delta u_j, \partial_t^2 \Delta u_i)|(t) \\ &\qquad + \frac{\gamma}{2} \left| \int_0^t e^{\gamma s} ([A_{ij}(\nabla u, \theta) - A_{ij}(0,0)] \partial_t^2 \Delta u_j, \partial_t^2 \Delta u_i)\, ds \right| \\ &\quad \le C\varepsilon^3, \quad t \in [0, T^*), \end{aligned} \tag{4.26}$$

and

$$
\begin{aligned}
&\left|\int_0^t e^{\gamma s}(\partial_t^{l+1} f, \sigma_k \partial_k \partial_t^{l+1} u)\, ds\right| \\
&\quad \le C\varepsilon^3 + \left|\int_0^t e^{\gamma s}([C_{i\alpha j\beta}(\nabla u, \theta) - C_{i\alpha j\beta}(0,0))\partial_t^3 \partial_\alpha \partial_\beta u_j, \sigma_k \partial_k \partial_t^3 u_i)\, ds\right| \\
&\quad \le C\varepsilon^3 + C\varepsilon \int_0^t e^{\gamma s} \int_{\partial G} |\partial_t^3 \nabla u|\, dx\, ds \\
&\qquad + \frac{1}{2}\left|\int_0^t e^{\gamma s} \int_G [C_{i\alpha j\beta}(\nabla u, \theta) - C_{i\alpha j\beta}(0,0)]\sigma_k \partial_k [\partial_\beta \partial_t^3 u_j \partial_\alpha \partial_t^3 u_i]\, dx\, ds\right| \\
&\quad \le C\varepsilon^3 + C\varepsilon \int_0^t e^{\gamma s} \int_{\partial G} \left|\frac{\partial}{\partial \overline{n}} \partial_t^3 u\right|^2 dx\, ds, \quad t \in [0, T^*).
\end{aligned}
\tag{4.27}
$$

In the same manner, we can show that the terms

$$
\left|\int_0^t e^{\gamma s}(\partial_t^{l+1} f, \partial_t^{l+2} u)\, ds\right|, \qquad \left|\int_0^t e^{\gamma s}(\partial_t^{l+1} g, \partial_t^{l+1} \theta)\, ds\right|, \qquad l = 0, 1, 2
$$

are bounded from above by $C\varepsilon^3$ for $t \in [0, T^*)$. Hence, recalling the definition of $\mathcal{P}(t; \partial_t^l u, \partial_t^l \theta, \partial_t^l f, \partial_t^l g)$, and making use of (4.25)–(4.27), we find that

$$
\mathcal{P}(t; \partial_t^l u, \partial_t^l \theta, \partial_t^l f, \partial_t^l g) \le C\varepsilon^3 + C\varepsilon \int_0^t e^{\gamma s} \int_{\partial G} \left|\frac{\partial}{\partial \overline{n}} \partial_t^3 u\right|^2 dx\, ds, \quad t \in [0, T^*),\ l = 0, 1, 2. \tag{4.28}
$$

Combining (4.23), (4.24), and (4.28), utilizing (4.16), and letting $0 < \varepsilon \le \min\{1, C^{-1}\}$, we arrive at

$$
\sum_{l=0}^{2} \mathcal{N}(t; \partial_t^l u, \partial_t^l \theta) \le 3\Gamma\varepsilon^2 + C\varepsilon^3 \quad \forall t \in [0, T^*). \tag{4.29}
$$

Recalling (2.8), the proof of (4.25) and the definition of Λ, we apply (4.19), (4.5), and (4.29) to deduce

$$
\begin{aligned}
\|u_t(t)\|_{H^3}^2 &\le \widetilde{\Gamma} \|\Delta u_t(t)\|_{H^1}^2 \\
&\le \frac{3(\beta^2+1)\widetilde{\Gamma}}{\tau^2}(\|u_{ttt}\|_{H^1}^2 + \|\nabla \theta_t\|_{H^1}^2 + \|f_t\|_{H^1}^2)(t) \\
&\le \frac{9(\beta^2+1)\Gamma\widetilde{\Gamma}}{\tau^2}\varepsilon^2 + C\varepsilon^3 \le \frac{3}{86}\Lambda\varepsilon^2 + C\varepsilon^3 \quad \forall t \in [0, T^*).
\end{aligned}
\tag{4.30}
$$

Similarly, using Eq. (4.6) and (4.30) one gets

$$
\begin{aligned}
\|\theta(t)\|_{H^4}^2 + \|\theta_t(t)\|_{H^3}^2 &\le \widetilde{\Gamma}(\|\Delta\theta\|_{H^2}^2 + \|\Delta\theta_t\|_{H^1}^2) \\
&\le \frac{3\widetilde{\Gamma}(c^2+\beta^2+1)}{\kappa^2}(\|u_t\|_{H^3}^2 + 3\Gamma\varepsilon^2 + C\varepsilon^3) \\
&\le \frac{27[(\beta^2+c^2+1)\widetilde{\Gamma}]^2\Gamma}{\kappa^2}(1+\tau^{-2})\varepsilon^2 + C\varepsilon^3 \\
&\le \frac{9}{86}\Lambda\varepsilon^2 + C\varepsilon^3
\end{aligned}
\tag{4.31}
$$

for $t \in [0, T^*)$, whence (also by (2.4))

$$\|u(t)\|_{H^4}^2 \le \widetilde{\Gamma}\|\Delta u(t)\|_{H^2}^2 \le \frac{3(\beta^2+1)\widetilde{\Gamma}}{\tau^2}(\|\theta(t)\|_{H^3}^2 + 3\Gamma\varepsilon^2 + C\varepsilon^3) \le \frac{30}{86}\Lambda\varepsilon^2 + C\varepsilon^3 \quad (4.32)$$

for $t \in [0, T^*)$. Adding (4.30)–(4.32) to (4.29), recalling the definition of $M(t)$, we conclude

$$M(t) \le \Lambda\varepsilon^2/2 + C\varepsilon^3 \quad \forall t \in [0, T^*). \tag{4.33}$$

Now choosing ε in (4.33) so small that $C\varepsilon \le \Lambda/3$, one obtains $M(t) \le 5\Lambda\varepsilon^2/6$ for any $t \in [0, T^*)$. Letting $t \uparrow T^*$, we get $M(T^*) \le 5\Lambda\varepsilon^2/6 < \Lambda\varepsilon^2$. This is a contradiction to the maximality of T^* (cf. (4.20)). So the solution is globally defined and (4.17) holds for all $t \ge 0$. From (4.17) the exponential decay of the solution follows. This completes the proof. □

Next we present an application of Theorem 4.2. We consider $G \subset \mathbb{R}^3$ to be radially symmetrical. We assume furthermore that the nonlinearities f and g (defined by (4.7)–(4.8)) satisfy

$$f(\nabla v, \Theta, \nabla^2 v, \nabla\Theta)(x,t) = \Omega^{\mathrm{T}} f(\nabla u, \theta, \nabla^2 u, \nabla\theta)(\Omega x, t), \tag{4.34}$$

$$\begin{aligned} g(\nabla v, \Theta, \nabla\Theta, \nabla^2 v, \nabla^2\Theta, \nabla v_t)(x,t) &= g(\nabla u, \theta, \nabla\theta, \nabla^2 u, \nabla^2\theta, \nabla u_t)(\Omega x, t) \\ &\qquad \text{for } x \in \overline{G},\ t \ge 0,\ \Omega \in SO(3), \end{aligned} \tag{4.35}$$

where $\{u, \theta\}$ is a solution of (4.5), (4.6), (2.3), (2.4), and $v(x,t) := \Omega^{\mathrm{T}} u(\Omega x, t), \Theta(x,t) := \theta(\Omega x, t)$.

REMARK 4.3. We give an example satisfying our assumptions. Let

$$\begin{aligned} C_{i\alpha j\beta}(\nabla u, \theta) &:= \tilde{\lambda}(\operatorname{div} u, \theta)\delta_{i\alpha}\delta_{j\beta} + (\delta_{ij}\delta_{\alpha\beta} + \delta_{\alpha j}\delta_{i\beta})\tilde{\mu}, \\ \widetilde{C}_{i\alpha}(\nabla u, \theta) &:= -\tilde{\beta}(\operatorname{div} u, \theta)\delta_{i\alpha}, \\ q(\nabla u, \theta, \nabla\theta) &:= \tilde{\kappa}(\theta)\nabla\theta, \quad a(\nabla u, \theta) := \tilde{a}(\operatorname{div} u, \theta), \end{aligned} \tag{4.36}$$

where $\tilde{\lambda}, \tilde{\beta}, \tilde{\kappa}$, and $\tilde{a}$ are smooth functions of $\operatorname{div} u$ and $\theta, \tilde{\mu}$ is a positive constant. Let $\tilde{\lambda}, \tilde{\mu}, \tilde{\beta}, \tilde{\kappa}, \tilde{a}$ satisfy

$$\begin{aligned} &3\tilde{\lambda}(0,0) + 2\tilde{\mu} > 0, \quad \tilde{\beta}(0,0) \ne 0, \quad \tilde{\kappa}(0) > 0, \quad \tilde{a}(0,0) > 0, \\ &-\tilde{\beta}_w(w,\theta) = \tilde{\lambda}_\theta(w,\theta), \quad -\tilde{\beta}_\theta(w,\theta) = \tilde{a}_w(w,\theta), \quad w, \theta \in \mathbb{R}. \end{aligned}$$

Then it is easy to see that the nonlinearities defined by (4.36) satisfy (4.3)–(4.4), (4.15), (4.34)–(4.35).

These conditions also guarantee the existence of a Helmholtz potential $\Psi = \Psi(\nabla u, \theta)$ (cf. [23]) satisfying

$$C_{i\alpha j\beta} = \frac{\partial^2\Psi}{\partial(\partial u_i/\partial x_\alpha)\partial(\partial u_j/\partial x_\beta)}, \quad \widetilde{C}_{i\alpha} = \frac{\partial^2\Psi}{\partial(\partial u_i/\partial x_\alpha)\partial\theta}, \quad \tilde{a} = -\frac{\partial^2\Psi}{\partial\theta^2}.$$

For the problem (4.5), (4.6), (2.3), (2.4) in the radially symmetrical domain G we have

THEOREM 4.4. Assume that $u^j \in H^{4-j}$ $(j = 0, \dots, 4)$, $\theta^j \in H^{4-j}$ $(j = 0, 1, 2)$, $\theta^3 \in L^2$, and u^0, u^1, θ^0 are radially symmetrical. Also, assume that the initial data are compatible

with the boundary conditions. Then there is a constant $\varepsilon > 0$ such that if (4.16) holds, then there exists a unique solution $\{u, \theta\}$ of (4.5), (4.6), (2.3), (2.4) on $[0, \infty)$ such that

$$u \in \bigcap_{j=0}^{4} C^j([0,\infty), H^{4-j}), \qquad \theta \in \bigcap_{j=0}^{2} C^j([0,\infty), H^{4-j}),$$
$$\theta_{ttt} \in C^0([0,\infty), L^2) \cap L^2([0,\infty), H^1).$$

Moreover, $\|u(t)\|_{H^4}$, $\|\theta(t)\|_{H^4}$ decay to zero exponentially as $t \to \infty$.

Proof. By virtue of Lemma 4.1 and Theorem 4.2, it suffices to show that u satisfies $\operatorname{rot} u = 0$ in $G \times (0, T)$. Let $v(x,t) := \Omega^{\mathrm{T}} u(\Omega x, t), \Theta(x,t) := \Theta(\Omega x, t)$. Then using (3.3) and the conditions (4.34)–(4.35), following the same procedure as used in the proof of Lemma 3.3, we see that $v = u$ and $\Theta = \theta$, in particular, that u is radially symmetrical and hence $\operatorname{rot} u = 0$ holds. The proof is complete. □

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975

[2] J. A. Burns, Z. Liu, and S. Zheng, *On the energy decay of a linear thermoelastic bar*, J. Math. Anal. Appl. **179**, 574–591 (1993)

[3] D. E. Carlson, *Linear thermoelasticity*, Handbuch der Physik VIa/2, Springer-Verlag, Berlin, 1972, pp. 297–346

[4] D. Carvalho Pereira and G. Perla Menzala, *Exponential decay of solutions to a coupled system of equations of linear thermoelasticity*, Mat. Applic. Comp. **8**, 193–240 (1989)

[5] D. Carvalho Pereira and G. Perla Menzala, *Exponential stability in linear thermoelasticity*: the inhomogeneous case, Applic. Anal. **44**, 21–36 (1992)

[6] A. Chrzęszczyk, *Some existence results in dynamical thermoelasticity, Part* I. *Nonlinear case*, Arch. Mech. **39**, 605–617 (1987)

[7] C. M. Dafermos, *On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity*, Arch. Rat. Mech. Anal. **29**, 241–271 (1968)

[8] S. W. Hansen, *Exponential energy decay in a linear thermoelastic rod*, J. Math. Anal. Appl. **167**, 429–442 (1992)

[9] D. B. Henry, *Topics in Analysis*, Publ. Secc. Mat., Univ. Autòn. Barc. **31**, 29–84 (1987)

[10] D. B. Henry, A. Perissinitto Jr., and O. Lopes, *On the essential spectrum of a semigroup of thermoelasticity*, Nonlinear Anal. **21**, 65–75 (1993)

[11] S. Jiang, *Global solutions of the Neumann problem in one-dimensional nonlinear thermoelasticity*, Nonlinear Anal. **19**, 107–121 (1992)

[12] S. Jiang, *Exponential decay and global existence of spherically symmetric solutions in thermoelasticity*, SFB 256 Preprint, No. 351, Univ. Bonn, 1994

[13] J. U. Kim, *On the energy decay of a linear thermoelastic bar and plate*, SIAM J. Math. Anal. **23**, 889–899 (1992)

[14] I. Lasiecka, J.-L. Lions, and R. Triggiani, *Non homogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures et Appl. **65**, 149–192 (1986)

[15] R. Leis, *Initial boundary value problems in mathematical physics*, B. G. Teubner Verlag, Stuttgart, John Wiley and Sons, Chichester, 1986

[16] Z. Liu and S. Zheng, *Exponential stability of semigroup associated with thermoelastic system*, Quart. Appl. Math. **51**, 535–545 (1993)

[17] J. E. Muñoz Rivera, *Energy decay rates in linear thermoelasticity*, Funkcial. Ekvac. **35**, 19–30 (1992)

[18] J. E. Muñoz Rivera, *Asymptotic behavior of the energy in thermo-visco-elasticity*, Mat. Applic. Comp. **11**, 45–71 (1992)

[19] J. Ponce and R. Racke, *Global existence of small solutions to the initial value problem for nonlinear thermoelasticity*, J. Diff. Equations **87**, 70–83 (1990)

[20] R. Racke, *On the time-asymptotic behaviour of solutions in thermoelasticity*, Proc. Roy. Soc. Edinburgh **107A**, 289–298 (1987)

[21] R. Racke, *On the Cauchy problem in nonlinear 3-d-thermoelasticity*, Math. Z. **203**, 649–682 (1990)

[22] R. Racke, *Blow-up in nonlinear three-dimensional thermoelasticity*, Math. Meth. Appl. Sci. **12**, 267–273 (1990)

[23] R. Racke, *Lectures on Nonlinear Evolution Equations*, Aspects of Math., Vol. 19, Vieweg-Verlag, Braunschweig, 1992.

[24] R. Racke, *Exponential decay for a class of initial boundary value problems in thermoelasticity*, Mat. Applic. Comp. **12**, 67–80 (1993)

[25] M. Slemrod, *Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional nonlinear thermoelasticity*, Arch. Rat. Mech. Anal. **76**, 97–133 (1981)