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ON A NONLOCAL DISPERSIVE EQUATION MODELING PARTICLE SUSPENSIONS

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Abstract. We study a nonlocal, scalar conservation law $u_t + ((K_a * u)u)_x = 0$, modeling sedimentation of particles in a dilute fluid suspension, where $K_a(x) = a^{-1}K(x/a)$ is a symmetric smoothing kernel, and $*$ represents convolution. We show this to be a dispersive regularization of the Hopf equation, $u_t + (u^2)_x = 0$, analogous to KdV and certain dispersive difference schemes. Using the smoothing property of convolution and the physical principle of conservation of mass, we establish the global existence of smooth solutions.

In physical applications, boundary considerations impose initial data of shock type. Up to shock formation time, we show that solutions remain close to solutions of the Hopf equation. After shock formation time, we give evidence that dispersive, hydrodynamic effects, interacting with Brownian diffusion, can generate oscillatory interface patterns in place of shocks. We discuss possible relevance to the physical phenomenon of layering.

1. Introduction. We study a nonlocal, scalar conservation law,

$$u_t + V_{ST}u_x + \alpha((K_a * u)u)_x = 0, \tag{1}$$

derived by Rubinstein, [15], as a model for sedimentation of a dilute suspension of particles in fluid, together with its diffusive regularization,

$$u_t + V_{ST}u_x + \alpha((K_a * u)u)_x = du_{xx}. \tag{2}$$

This model was derived in the context of a more general kinetic theory developed by Rubinstein and Keller [16, 17]. Particles are assumed to be identical, randomly distributed spheres settling under gravity toward an infinitely far-off horizontal wall, behaving according to Stokes flow and so sparse as to interact only through the intermediary of the encompassing fluid flow. The variable $u(x,t)$ denotes the concentration of particles at height x and time t. The background drift,

$$V_{ST} = \frac{f}{6\pi\mu a},$$

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is the Stokes velocity predicted for a single particle settling in isolation, where f is the net force (including buoyancy) on the particle, μ is the viscosity of the fluid, and a is the particle radius. The coefficient d denotes Brownian diffusivity, given by Einstein's formula as

$$d = \frac{\kappa T}{6\pi\mu a},$$

where T is temperature and κ is Boltzman's constant. Finally, $*$ denotes convolution and $K_a(x) = a^{-1}K(x/a)$ is a symmetric, Lipschitz, smoothing kernel, with $\int K(x)\,dx = 1$, $\alpha = -6.55V_{ST} > 0$. The kernel K, modeling wall-particle and particle-particle interactions, is to first approximation a truncated parabola,

$$K(x) = H(x) = \begin{cases} 2/(3(x^2/4-1)) & |x| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, we will take K_a to be supported on $[-2a, 2a]$, though much of our analysis requires only $K_{a,x} \in L^1$. Note that the "physical" states $\{u : u(x) \geq 0\}$ form an invariant region for either of (1), (2).

In coordinates $x' = x - V_{ST}t$ moving with Stokes velocity, (1) and (2) simplify to

$$u_t + \alpha((K_a * u)u)_x = 0 \tag{S}$$

and

$$u_t + \alpha((K_a * u)u)_x = du_{xx}. \tag{S-B}$$

Where convenient, we will simplify further by setting $\alpha, d = 1$ (equivalent to rescaling u and x). In the formal limit $K_a \to \delta$ as $a \to 0$, (S) and (S-B) reduce to Hopf,

$$u_t + \alpha(u^2)_x = 0, \tag{H}$$

and Burgers' equation,

$$u_t + \alpha(u^2)_x = du_{xx}, \tag{B}$$

respectively. Likewise, the original equation (1) reduces to a simple traffic model,

$$u_t + [V_{ST}(1 - 6.55u)u]_x = 0, \tag{3}$$

in which particles move more slowly in regions of higher density. The model (3) was proposed earlier, e.g. [3], on a more heuristic basis, being exactly a transport equation with velocity at (x,t) taken to be the steady-state settling speed $V = V_{ST}(1 - 6.55u)$ calculated by Batchelor, [1], for a uniform suspension of concentration $u(x,t)$. The (formal) justification of (3) from first principles was one of the main points of [15]. Accordingly, one of our goals here is to verify rigorously the convergence of (1) to (3) in the zero-radius limit, so long as the solution of (3) remains smooth (and therefore, locally, reasonably like a uniform distribution).

There is another interesting aspect to this formal limit. Typical initial data in physical applications is

$$u(x,0) = \begin{cases} u_0(x) > 0, & x < 0, \\ 0, & x \geq 0, \end{cases} \tag{4}$$

corresponding to a vat filled with fluid up to level $x = 0$. As is well known (cf. [12]) this is shock initial data for the Hopf equation (and (3)), leading to the formation of singularities in finite time. In fact, in the most basic situation, $u_0(x) \equiv \tilde{u} > 0$, of an initially homogeneous suspension, (7.5) becomes a Riemann problem, with a shock initiating from the fluid boundary $x = 0$ at time $t = 0^+$. In this situation, the traffic model cannot be expected to remain valid, and higher-order, regularizing effects become important, as for example in Burgers' equation. Yet, comparison with experiment, e.g. [19], shows that the regularization given by (B) is not valid either. For, (B) resolves the shock interface into a smooth, monotonic shock layer. Yet, experimentally observed behavior is a stable, traveling staircase pattern, appearing to the eye as several distinct bands of differing concentrations just below the primary interface between sediment and clear fluid, a phenomenon known as "layering". This is most familiar in the context of sedimentation of clay solutions, [13, 21].

The process of dynamical (nonuniform concentration) sedimentation in general, and layering in particular, remains poorly understood, despite its importance in applications. Previous analyses (e.g. [3, 19]) have focused on regularization by more complicated, non-Fickian diffusion; however, these have been carried out only at a qualitative level, and in a physically ad hoc way. Equation (S) provides an alternate, purely hydrodynamic regularization of the Hopf equation, appealingly simple and derived from first principles. Thus, it is interesting not only how the behavior of (S) conforms to that of the traffic model in the smooth regime, but more importantly how it differs in the presence of shocks. We would like to know whether (S) is indeed a regularization of (H), i.e., admits global smooth solutions, and, if so, of what nature? In particular, does the qualitative behavior of solutions of "shock" initial data, (4) have a bearing on layering?

Equation (S) has the appearance of a fundamental type, such as Burgers or KdV, that could be expected to arise quite generally; as such, it is perhaps worthy of study in its own right. As we shall show, (S) and (S-Burgers) are mathematically both interesting and extremely well-behaved. Far from being a complication, the nonlocal convolution term aids in the analysis by virtue of its smoothing properties. Further, (S) and (S-Burgers) turn out to be nonlocal analogs of KdV and KdV-Burgers, respectively, closely related to the dispersive difference schemes studied by Goodman, Hou, and Lax [5, 6]. This observation is crucial in our studies of traveling wave behavior and the zero-dispersion limit, $a \to 0$. Moreover, it suggests that, as a tractable, non-integrable dispersive equation, (S) might bear further investigation from the point of view of nonlinear waves.

Our analysis of (S), (S-B) follows two key observations. The first, and immediate, observation is that (S) is a transport equation with a smooth velocity field $V = \alpha(K_a * u)$. The smoothing in the velocity field gives (S) an essentially linear character. For example, at a discontinuity, the Rankine-Hugoniot condition

$$s[u] = [Vu] = V[u] \tag{RH}$$

shows that shock speed s is given by V, i.e., discontinuities are convected with the flow. On the other hand, differentiating the jump $[u]$ along the particle path $\frac{dx}{dt} = V$, we find

that

$$\frac{d}{dt}[u] = [-(K * u)_x u] = -(K * u)_x [u], \tag{2}$$

i.e., shocks persist in the solution (in fact, since $-(K * u)_x$ is typically positive when $[u] < 0$, shocks initially tend to grow). The conclusion is that the solution u retains whatever smoothness is given in the initial data, u_0. Likewise, the linear quality of (S) allows us to pass weak limits through the flux (vu), extending C^1 existence to L^∞ existence.

The second, and more interesting observation is that convolution is the continuous analog of a finite difference stencil. When we derive an "effective", local equation for (S), we discover that it is a close relative of certain dispersive finite difference schemes studied in [5, 6]. This connection gives the intuition behind most of our results on behavior of solutions. Further, oscillatory phenomena familiar from the study of KdV and Lax-Wendroff difference schemes, and other dispersive systems suggest a possible hydrodynamic mechanism for pattern formation in shock layers, through the interplay between dispersive and diffusive effects. This possibility directs our later analysis and discussion.

Plan of the paper. In Sec. 2, we prove global existence of solutions of (S), (S-B) in L^∞ and uniqueness in BV. The main technical point is to use conservation of mass to establish global L^∞ bounds; this requires a somewhat subtle argument incorporating the natural length-scale a in an important way. Using the smooth property of convolution, we can then obtain global existence of weak solutions in L^∞, and uniqueness in the class BV. Global existence of smooth solutions was proved in [16, 17] for more general, 3-dimensional systems, under the assumption of L^1 initial data. However, that result did not apply to the case of stratified suspensions, or more general L^∞ initial data. In the 3-dimensional case, the kernel is divergence-free, and the problem of bounding the L^∞ norm of solutions does not arise.

In Sec. 3, we derive an "effective", or approximating local equation, (S-KdV), for (S). By studying its properties, we deduce that (S) supports periodic traveling waves, but not viscous shocks. A key point is Lemma 3.2.1, which gives a rigorous sense to the observation that the symmetry of K makes (S) completely dispersive.

In Sec. 4, we verify the formal limit as $a \to 0$, by an L^2-stability argument similar to convergence proofs for difference schemes. Symmetry of K again plays a key role, this time in the proof of L^2-linearized stability.

In Sec. 5, we study the effect of artificial diffusion on the large-time behavior of (S-B). For the corresponding local equation, (S-KdV-B), we show that solutions of shock initial data converge to a stable, oscillatory traveling wave.

The final section, 6, is devoted to a general, physical discussion of sedimentation, with special attention toward the phenomenon of layering. We compare the predictions of various models proposed in the literature to the results of a classic experiment of Siano [19] demonstrating this phenomenon. In particular, we consider whether ordinary, Fickean diffusion together with dispersive hydrodynamic effects, as modeled in (S-B), (S-KdV-B), might be sufficient to explain layering, without ad hoc introduction of non-Fickean terms.

Surprisingly, we find that even without dispersive effects, the simple Burgers' model, (B), predicts the width and propagation of the layering interface rather well, through the effects of hydrodynamic convection and (Fickean) Brownian diffusion alone. This is contrary to popular wisdom that neither convection nor Brownian diffusion play much role in layering. However, the arguments for this popular view (exemplified, e.g., in [19], p. 123) are based on analyses of these effects *in isolation.* Convection alone predicts an interface of zero width, while diffusion alone predicts an interface of essentially infinite extent, i.e., spatial homogeneity; it is precisely the balance of these two competing effects that accounts for the actual interface width.

On the other hand, the same sort of scale analysis shows that, for the parameter range of the experiment, the oscillatory traveling interface of (S-KdV-B) is essentially indistinguishable from the smooth Burgers' interface. Thus, (S-KdV) does not predict the finer, banded structure of the interface. The effective equation (S-KdV-B) is expected to model (S-B) when the kernel K has a second moment; thus, the "primary" scattering effects modeled by the truncation $K(x) \sim H(x)$ *do not* account for layering structure. However, we note that, though small in L^1, the tail neglected in this truncation has infinite second moment, and might still play a role in formation of structure. The analysis of these tail effects is beyond the scope of the present paper; we point to this and to the analysis of non-Fickean effects *in the presence of convection* as important problems for future study.

2. Existence/Uniqueness. Our aim in this section is to prove global existence of classical solutions, for smooth initial data. As a consequence, we obtain global existence of weak solutions, for L^∞ data, as well.

LEMMA 2.0.1. For $f \in C^k$, $K_a * f \in C^{k+1}$.

Proof. We prove the assertion for $k = 0$. Since K_a is Lipschitz, its difference quotients $D_h K_a = (K_a(\cdot + h) - K_a(\cdot))/h$ are bounded and converge *a.e.* as $h \to 0$. Because $K_{a.x}$ has compact support and $f \in L^1_{\text{Loc}}$, the Lebesgue Dominated Convergence gives

$$D_h(K_a * f) = (D_h K_a * f) \to K_{a.x} * f.$$

Thus, $(K_a * f)_x = (K_{a.x} * f)$; this is continuous by the symmetric argument using continuity of f. For higher k, the result follows by induction. □

PROPOSITION 2.1 (Local Existence). For initial data $u_0 \in C^k$, $k \geq 1$, with $\|u_0\|_\infty$, $\|u_{0.x}\|_\infty$ bounded, there exists a unique solution $u \in C^k$ of (S), with $\|u\|_\infty, \|u_x\|_\infty$ bounded, define up to time $T > 0$ depending only on $\|u_0\|_\infty$.

Proof. We first consider the problem

$$u_t + ((K_a * f)u)_x = 0; \quad u(x,0) = u_0(x), \tag{2.1}$$

where $f \in C^0$ is given.

By our assumptions on K_a, $(K_a * f)$ is in C^1, so that (2.1) is a standard linear hyperbolic problem and can be solved by the method of characteristics.

We find that, for $T > 0$ depending only on

$$\|K_a * f\|_\infty = O(1)\|f\|_\infty, \qquad \|(K_a * f)_x\|_\infty = O(a^{-1})\|f\|_\infty,$$

and $\|u_0\|_\infty$, there exists a solution $u \in C^1$ of (2.1). Further,

$$\begin{aligned} \|u\|_\infty &= O(1)\|u_0\|_\infty, \\ \|u_x\|_\infty &= O(1)\|u_{0,x}\|_\infty, \end{aligned} \tag{2.2}$$

where $O(1)$ depends only on $\|u_0\|_\infty, \|f\|_\infty$.

Define $\mathcal{T} : L^\infty \to C^1$ by

$$\mathcal{T} f = u. \tag{2.3}$$

We will show that $\mathcal{T}$ is a contraction on $C^0(\mathbb{R} \times [0, T_*])$, for $T_* \le T$ sufficiently small.

Let $\mathcal{T} f = u$ and $\mathcal{T} g = v$. Then, $w \stackrel{\text{def}}{=} (u - v)$ satisfies

$$w_t + ((K_a * f)w)_x + (K_a * (g - f)v)_x = 0. \tag{2.4}$$

Letting $\frac{d}{dt}$ denote differentiation along the characteristic $\frac{dx}{dt} = (K_a * f)$, we have

$$\frac{d}{dt} w = -(K_{a,x} * f)w - (K_{a,x} * (g - f))v - (K_a * (g - f))v_x, \tag{2.5}$$

and thus

$$\begin{aligned} \frac{d}{dt}\|w\|_\infty &= O(a^{-1})\|f\|_\infty \|w\|_\infty \\ &\quad + O(a^{-1})\|g - f\|_\infty \|v\|_\infty \\ &\quad + O(1)\|g - f\|_\infty \|v_x\|_\infty \\ &= O(1)(\|w\|_\infty + \|f - g\|_\infty), \end{aligned} \tag{2.6}$$

by (2.2).

Since $\|w_0\|_\infty = 0$, the Gronwall inequality (2.6) gives

$$\|w\|_\infty = O(1)t\|f - g\| \tag{2.7}$$

on $[0, T]$ where, from (2.2), $O(1)$ depends only on $\|u_0\|_\infty, \|u_{0,x}\|_\infty$. Choosing $T_* \le T$ sufficiently small, we see that $\mathcal{T}$ is a contraction on $C^0(\mathbb{R} \times [0, T_*])$, where T_* depends only on $\|u_0\|_\infty, \|u_{0,x}\|_\infty$.

Thus, there exists a unique $u \in L^\infty$ satisfying $\mathcal{T} u = u$, up to time T_*. Since $\mathcal{T} : C^0 \to C^1$, u is in fact C^1, and therefore a classical solution of (2.5). The smoothing property of $(K_a * f)$ shows that, also, $\mathcal{T} : C^{j-1} \to C^j$, for $j \le k$. Therefore, the same bootstrapping argument shows that $u \in C^k$.

Finally, we remove the dependence of T on $\|u_{0,x}\|$. Letting $\frac{d}{dt}$ denote differentiation along $\frac{dx}{dt} = (K_a * u)$, (S) becomes

$$\frac{d}{dt} u = -(K_{a,x} * u)u, \tag{2.8}$$

which gives

$$\begin{aligned} \frac{d}{dt}\|u\|_\infty &= O(1)\|(K_{a,x} * u)u\|_\infty \\ &= O(a^{-1})\|u\|_\infty^2. \end{aligned} \tag{2.9}$$

This Ricatti-type equation gives a bound on $\|u\|_\infty$ up to time T depending only on $\|u_0\|_\infty$. Likewise, differentiating (S) gives

$$\frac{d}{dt}(u_x) = -(K_{a,x} * u_x)u - 2(K_{a,x} * u)u_x \tag{2.10}$$

and

$$\frac{d}{dt}\|u_x\|_\infty = O(a^{-1})\|u\|_\infty\|u_x\|_\infty. \tag{2.11}$$

This Gronwall inequality gives a bound on $\|u_x\|_\infty$ up to time T as well, T still dependent only on $\|u_0\|_\infty$. By iteration of the previous contraction argument, we obtain existence up to time T. □

To obtain global existence, we need an a priori estimate for $\|u\|_\infty$. Unfortunately, the Ricatti estimate (2.9) blows up in finite time. On the other hand, (S) possesses trivial a priori estimates in L^1. Since (S) is a transport equation with velocity given by $(K_a * u)$, it preserves both mass and L^1 weight between particle paths (characteristics),

$$\chi : \frac{d\chi}{dt} = (K_a * u)(\chi). \tag{2.13}$$

So, for example, if $\|u_0\|_1 = M$, we have $\|u\|_1 = M$ for all time, and (2.9) can be improved to

$$\begin{aligned} \frac{d\|u\|_\infty}{dt} &= O(1)\|(K_{a,x} * u)\|_\infty\|u\|_\infty \\ &= O(1)M\|K_{a,x}\|_\infty\|u\|_\infty, \end{aligned} \tag{2.14}$$

which is a Gronwall inequality. This gives an exponential bound on $\|u\|_\infty$ for all time, and we are done. Thus, the case of L^1 data is trivial.

To treat the case of L^∞ data, we must take into account the natural length scale of the problem.

LEMMA 2.2.1. Let

$$\gamma(t) = \sup_x \int_{x-2a}^{x+2a} |u(y,t)|\, dy. \tag{2.15}$$

Then,

$$\gamma(t) = O(1)e^{O(a^{-1})\|u_0\|_\infty t}.$$

Proof. Suppose, without loss of generality, that K_a is supported on $[-2a, 2a]$. Let $\chi(t)$ and $\chi_2(t)$ be the characteristics passing through $(x-2a, T)$ and $(x+2a, T)$. By conservation of mass,

$$\begin{aligned} \int_{x-2a}^{x+2a} |u(y,T)|\, ds &= \int_{\chi_1(0)}^{\chi_2(0)} |u_0(y)|\, dy \\ &\le |\chi_2(0) - \chi_1(0)|\, \|u_0\|_\infty. \end{aligned} \tag{2.17}$$

But,

$$|\chi_2(0) - \chi_1(0)| \le 4a + \int_0^T \left| \frac{d}{dt}\chi_2(s) - \frac{d}{dt}\chi_1(s) \right| ds \tag{2.18}$$

and

$$\left|\frac{d}{dt}\chi_i\right| = |(K_a * u)(\chi_i)| = O(a^{-1})\gamma(t). \tag{2.19}$$

Thus,

$$\gamma(T) \leq \left(4a + O(a^{-1})\int_0^T 2\gamma(t)\right)\|u_0\|_\infty. \tag{2.20}$$

This integral Gronwall inequality gives

$$\gamma(T) \leq 4a\|u_0\|_\infty e^{O(a^{-1})\|u_0\|_\infty T}, \tag{2.21}$$

and we are done. □

PROPOSITION 2.2. For initial data $u_0 \in C^k$, $k \geq 1$, with $\|u_0\|_\infty, \|u_{0.x}\|_\infty$ bounded, there exists a unique global solution $u \in C^k$ of (S), with $\|u\|_\infty(t) \leq M(t)$, M depending only on $\|u_0\|_\infty$.

Proof. We have

$$\begin{aligned}\|K_{a.x} * u\|_\infty &\leq \|K_{a.x}\|_\infty \gamma(t)\\ &= O(a^{-2})\gamma(t).\end{aligned} \tag{2.22}$$

Thus, (2.9) becomes

$$\begin{aligned}\frac{d}{dt}\|u\|_\infty &= O(a^{-2})\gamma(t)\|u\|_\infty\\ &= O(1)e^{O(1)t}\|u\|_\infty.\end{aligned} \tag{2.23}$$

This Gronwall inequality gives a global bound,

$$\|u\|_\infty = O(1)e^{O(1)e^{O(1)t}}\|u_0\|_\infty \tag{2.24}$$

on $\|u\|_\infty$. Together with Proposition (2.7), this proves the result. □

PROPOSITION 2.3. For initial data $u_0 \in L^\infty$, there exists a global weak solution $u \in L^\infty$.

Proof. Define $u_0^\varepsilon \in C^1$ so that $u_0^\varepsilon \rightharpoonup u_0$ as $\varepsilon \to 0$, and $\|u_0^\varepsilon\| < \|u_0\|$. By Proposition 2.2, there exist global solutions $u^\varepsilon \in C^1$ with $\|u^\varepsilon\|_\infty(t) \leq M(t)$. The uniform L^∞ bound allows us to extract a subsequence $u^{\tilde{\varepsilon}}$ converging weakly to a candidate solution $u \in L^\infty$. To verify that u is indeed a weak solution, it is sufficient to establish that weak limits pass through the nonlinear flux term $(K * u)u$ in the sense that

$$\int \varphi(K_a * u^{\tilde{\varepsilon}})u^{\tilde{\varepsilon}}_{dx\,dt} \to \int \varphi(K_a * u)u\,dx\,dt \tag{2.25}$$

as $\tilde{\varepsilon} \to 0$, for any test function $\varphi \in C_0^\infty$.

But, note that $u^{\tilde{\varepsilon}} \rightharpoonup u$ implies that $(K_a * u^{\tilde{\varepsilon}}) \xrightarrow{\text{a.e.}} (K_a * u)$. Since φ has compact support, and the $u^{\tilde{\varepsilon}}$ are uniformly bounded in L^∞, the Lebesgue dominated convergence Theorem gives

$$\varphi(K_a * u^{\tilde{\varepsilon}}) \xrightarrow{L^1} \varphi(K_a * u). \tag{2.26}$$

Now, (2.25) follows from the standard weak-strong pairing Lemma 2.3.1, below. □

LEMMA 2.3.1. Let $z^\varepsilon \rightharpoonup z$ in a Banach space X, and $w^\varepsilon \to w$ in X^*. Then,

$$\langle w^\varepsilon, z^\varepsilon\rangle_{X^*,X} \to \langle w, z\rangle_{X^*,X}. \tag{2.27}$$

Proof. Write $\langle w^\varepsilon, z^\varepsilon\rangle - \langle w, z\rangle$ as

$$\langle w, z^\varepsilon - z\rangle + \langle w^\varepsilon - w, z^\varepsilon\rangle = \mathrm{I} + \mathrm{II}. \tag{2.28}$$

Weak convergence implies that $\|z^\varepsilon\|$ is bounded, from which we find that

$$\mathrm{II} \le \|w^\varepsilon - w\|\,\|z^\varepsilon\| \to 0. \tag{2.29}$$

But, $\mathrm{I} \to 0$ by the definition of weak convergence. □

PROPOSITION 2.4. For initial data $u_0 \in \mathrm{BV}(R)$, with Total Variation$_x u_0$ bounded, there exists a unique global solution $u \in \mathrm{BV}(R^2)$ of (S), such that Total Variation$_x u(\cdot, t)$ is bounded for all $t \ge 0$.

Proof. The total variation bound follows easily for C^2 initial data via

$$\begin{aligned}\frac{d}{dt}\int \|u_x\|\,dx &= \int [-((K_a * u)|u_x|)_x - \operatorname{sgn}(u_x)(K_a * u_x)u_x - \operatorname{sgn}(u_x)(K_a * u_{xx})u]\,dx\\ &= \int [-\operatorname{sgn}(u_x)[(K_{a,x} * u)u_x - \operatorname{sgn}(u_x)(K_{a.x} * u_x)u]\,dx\\ &\le C\|u\|_\infty \|u_x\|_1 \|K_{a.x}\|_1,\end{aligned}$$

and passes to the limit to give existence. Uniqueness follows from the calculation

$$\begin{aligned}|u-v|_t &= -\operatorname{sgn}(u-v)[(K_a * u)(u-v) - (K_a * (u-v))v]_x\\ &= -[\operatorname{sgn}(u-v)(K_a * u)(u-v)]_x - \operatorname{sgn}(u-v)[(K_a * (u-v))v_x\\ &\quad + (K_{a.x} * (u-v))v],\end{aligned}$$

which gives

$$|u-v|_t + [\operatorname{sgn}(u-v)(K_a * u)(u-v)]_x = \mu,$$

where

$$\mu = -\operatorname{sgn}(u-v)[(K_a * (u-v))v_x + (K_{a.x} * (u-v))v]$$

has L^1 weight $\|\mu\|_1 \le (C_1\|v\|_\infty + C_2\|v_x\|_1)\|u-v\|_1$.

Applying the Green's Theorem for BV functions, as in [22], we find that

$$\begin{aligned}\frac{d}{dt}\|u-v\|_1 &\le \|\mu\|_1 + \sum_i \theta_i\\ &= (C_1\|v\|_\infty + C_2\|v_x\|_1)\|u-v\|_1 + \sum_i \theta_i,\end{aligned} \tag{2.29.5}$$

where

$$\begin{aligned}\theta_i &= [(K_a * u)|u-v|] - s_i[|u-v|]\\ &= (K_a * u - s_i)[|u-v|]\end{aligned}$$

is the entropy production of $\|u-v\|_t + [\operatorname{sgn}(u-v)(K_a * u)(u-v)]_x$ at the ith shock in $|u-v|$. Note that θ_i vanishes at u-shocks, where $s_i = K_a * u$.

Using the symmetric expansion,

$$\begin{aligned}\|u-v\|_t = &-[\operatorname{sgn}(u-v)(K_a * v)(u-v)]_x \\ &- \operatorname{sgn}(u-v)[(K_a * (u-v))u_x + (K_{a.x} * (u-v))u],\end{aligned}$$

we find that in fact

$$\frac{d}{dt}\|u-v\|_1 \le (C_1\|v\|_\infty + C_2\|v_x\|_1 + C_1\|u\|_\infty + C_2\|u_x\|_1)\|u-v\|_1 + \sum_i \min\{\theta_i, \tilde{\theta}_i\}, \tag{2.29.7}$$

where $\tilde{\theta}_i = (K_a * v - s_i)[|u-v|]$. The term $\sum_i \min\{\theta_i, \tilde{\theta}_i\}$ vanishes everywhere except at countably many "interaction points" that are the intersection of a u and a v-shock.

Thus, integrating (2.29.7), we find that $\|u-v\|_1(t) \le e^{Ct}\|u_0 - v_0\|_1$, where

$$C = \sup_{s\le t}(C_1\|v\|_\infty + C_2\|v_x\|_1 + C_1\|u\|_\infty + C_2\|u_x\|_1).$$

This gives uniqueness, and we are done. □

REMARK 2.4.1. Note that no entropy condition is necessary for uniqueness, supporting the claim that (S) has essentially linear regularity properties. An open problem is uniqueness in L^∞.

REMARK 2.4.2. For $u_0 \in C^1$, continuous dependence follows from calculations similar to (2.6), (2.7). However, the estimates blow up as either $\|u_{0.x}\|_\infty \to \infty$ or $a \to 0$.

REMARK 2.4.3. All of the existence theory for (S) goes over to the case of (S-B). In Proposition (2.1), we can substitute standard parabolic theory and the maximum principle for our calculations using characteristics. Likewise, Proposition 2.3 goes through without change. Proposition 2.4 actually becomes easier. Because of parabolic smoothing, the entropy production term $\sum_i \theta_i$ in (2.29.5) no longer appears; it is replaced by the simpler term

$$\int \operatorname{sgn}(u)u_{xx}\,dx = -\sum_{\{\xi: u(\xi)=0\}} 2|u_x(\xi)| < 0.$$

Proposition 2.2 requires more effort, since L^1 weight is no longer preserved between characteristics, but instead

$$\begin{aligned}\frac{d}{dt}\int_{\chi_1(t)}^{\chi_2(t)} |u(y,t)|\,dy &= \int_{\chi_1(t)}^{\chi_2(t)} \operatorname{sgn}(u)u_{xx}\,dy \\ &= \operatorname{sgn}(u)u_x(\chi_2(t)) - \operatorname{sgn}(u)u_x(\chi_1(t)) - \int_{\chi_1(t)}^{\chi_2(t)} 2\delta(u)u_x^2\,dy \\ &\le \operatorname{sgn}(u)u_x(\chi_2(t)) - \operatorname{sgn}(u)u_x(\chi_1(t)).\end{aligned} \tag{2.30}$$

However, note that

$$\frac{d^2}{dx^2}\int_{x-2a}^{x+2a} |u(y,t)|\,dy = \operatorname{sgn}(u)u_x(x+2a) - \operatorname{sgn}(u)u_x(x-2a) \le 0 \tag{2.31}$$

for x maximizing $\int_{x-2a}^{x+2a} |u(y,t)|\,dy$. Thus, in the argument of Proposition 2.2, if we follow back characteristics $\chi_1(t), \chi_2(t)$ passing through the endpoints of the maximal interval $[x-2a, x+2a]$, then $\frac{d}{dt}\chi_2(t)_{\chi_1(t)}|u(y,t)|\,dy$ will be nonpositive initially, i.e., at $t = T$,

and has second derivative bounded by $O(1)\|u_x\|_\infty$. By Proposition 2.1, we can assume without loss of generality that $\|u_x\|_\infty$ is uniformly bounded by some M on $[0,T]$. Thus, we can follow χ_1, χ_2 back in time ΔT, with change in L^1 weight of $M\Delta T^2$. At $T-\Delta T$, we choose new characteristics χ_1', χ_2' passing through the endpoints of the maximal interval of length $(\chi_2(T-\Delta T) - \chi_1(T-\Delta T))$. Continuing back in this fashion, we obtain a bound similar to (2.20), but with an error term

$$\text{error} \leq \left(\frac{T}{\Delta T}\right)(M\Delta T^2) = MT\Delta T. \tag{2.32}$$

Letting $\Delta T \to 0$, we are done.

3. The effective equation and traveling waves. Convolution can be considered as the continuous version of a finite difference stencil. Pursuing that analogy in this section, we derive the "effective", local equation for (S), by Taylor expansion, and consider the implications on qualitative behavior of solutions.

Formally, we can expand $(K_a * u)$ as

$$\begin{aligned}\int K_a(x-y)u(y)\,dy &= \sum_j \frac{D^j u(x)}{j!}\int K_a(x-y)(y-x)^j\,dy \\ &= \sum_j c_j a^j D^j u(x),\end{aligned} \tag{3.1}$$

where c_j is $\frac{(-1)^j}{j!}$ times the jth moment of K_a. Since K_a is even, all odd moments vanish, and we have

$$K_a * u = u + c_2 a^2 u_{xx} + O(a^4). \tag{3.2}$$

Substituting (3.2) into (S), we obtain the effective equation

$$u_t + (u^2)_x = -c_2 a^2 (u u_{xx})_x, \tag{S-KdV}$$

similar to KdV.

The form of (S-KdV) suggests that (S) is actually a dispersive regularization of (1.1). In fact, (S-KdV) is almost identical to the effective equation

$$u_t + (u^2)_x = -\Delta x^2 u u_{xxx} \tag{3.3}$$

for a family of dispersive difference schemes studied in [5, 6], where Δx is mesh size. In [5], the semi-discrete version of this difference scheme, with periodic boundary conditions, was shown to be completely integrable, by reduction to the Toda chain. Numerical evidence was given, suggesting that (3.3) behaves similarly to KdV in the zero-dispersion limit, in all time regimes. This statement was proved rigorously in the smooth regime of (1.1), where solutions of the difference scheme were shown to converge to the solution of (1.1) as $\Delta x \to 0$.

Unlike (3.3), (S-KdV) does not appear to be integrable. However, we can identify a qualitative similarity to KdV, by the study of traveling wave solutions, which are in some sense normal modes of the PDE. These are particularly relevant to the present discussion,

as possible asymptotic states resolving shock-type initial data. Traveling wave solutions, $u(x,t) = \varphi(x - st)$, of (S-KdV), satisfy

$$(\varphi^2 - s\varphi)' = -c_2 a^2 (\varphi\varphi'')'. \tag{3.4}$$

PROPOSITION 3.1.

(i) There exist no "viscous shock" solutions of (S-KdV), i.e., solutions $\varphi(\xi)$ of (3.4) such that

$$\lim_{\xi \to \pm\infty} \varphi(\xi) = \varphi(\pm\infty)$$

exists and $\varphi(+\infty) \neq \varphi(-\infty)$. There do exist solitary wave solutions, with $\varphi(+\infty) = \varphi(-\infty)$.

(ii) There exist both linear (sinusoidal) and nonlinear periodic solutions of (S-KdV).

Proof. Rescaling (3.4) by the coordinate change $\xi' = \frac{\xi}{a\sqrt{c_2}}$ and integrating, we obtain

$$\varphi^2 - s\varphi + \beta = -\varphi\varphi'' \tag{3.5}$$

or

$$\varphi - s + \beta/\varphi = -\varphi''. \tag{3.6}$$

Equation (3.6) is a Hamiltonian system describing a nonlinear oscillator.

For $\beta = 0$, (3.6) is a linear oscillator

$$\varphi - s = -\varphi'', \tag{3.7}$$

admitting precisely the sinusoidal solutions

$$\varphi(\xi) = s + Ae^{i\xi}.$$

To determine the behavior when $\beta \neq 0$, we multiply by φ' and integrate, obtaining

$$\frac{d}{dt}\mathcal{H}(\varphi, \varphi') = 0, \tag{3.8}$$

where

$$\mathcal{H}(\varphi, \psi) \stackrel{\text{def}}{=} \tfrac{1}{2}\varphi^2 - s\varphi + \beta \log(\varphi) + \tfrac{1}{2}\psi^2. \tag{3.8.5}$$

Denote the roots of $\varphi^2 - s\varphi + \beta = 0$ by $\varphi_- > \varphi_+$, so that

$$\begin{aligned} s &= \varphi_+ + \varphi_-, \\ \beta &= \varphi_+\varphi_-. \end{aligned} \tag{3.9}$$

First, consider the physical case that $\varphi_- > \varphi_+ \geq 0$. From

$$\frac{\partial \mathcal{H}}{\partial \varphi} = \varphi - s + \frac{\beta}{\varphi}, \tag{3.10}$$

$$\frac{\partial \mathcal{H}}{\partial \psi} = \psi,$$

$$\frac{\partial^2 \mathcal{H}}{\partial \varphi^2} = 1 - \frac{\beta}{\varphi^2} = 1 - \frac{\varphi_+ \varphi_-}{\varphi^2}, \tag{3.11}$$

$$\frac{\partial^2 \mathcal{H}}{\partial \psi^2} = 1,$$

$$\frac{\partial \mathcal{H}^2}{\partial \psi \partial \phi} = 0,$$

it follows that

CLAIM 3.1.1. $(\varphi_-, 0)$ and $(\varphi_+, 0)$ are the only stationary points of $\mathcal{H}$ with $(\varphi_-, 0)$ a local minimum and $(\varphi_+, 0)$ a saddle-point.

Since $(\varphi_-, 0)$ is a local minimum of $\mathcal{H}$, its level set is a single point, corresponding to the constant solution, $\varphi(\xi) \equiv \varphi_-$ of (3.4). In particular, there exists no viscous shock solution connecting φ_+ and φ_-, since orbits of (3.4) are contained in level sets of $\mathcal{H}$.

The other distinguished level set of $\mathcal{H}$ is the separatrix $\mathcal{S}$ that crosses at the saddle-point $(\varphi_+, 0)$. From (3.8.5), it is easily seen that $\mathcal{S}$ is bounded in φ, and is single-branched for large Ψ. Thus, it must consist of four orbits: a homoclinic orbit of (3.4) connecting φ_+ to itself, i.e., a solitary wave solution; the constant solution $\varphi(\xi) \equiv \varphi_+$; and two orbits that are unbounded at one end, $\xi = \pm\infty$, and converge to φ_+ at the other end, $\xi = \mp\infty$.

All other level sets of $\mathcal{H}$ must be simple curves corresponding to a single orbit of (3.4), either periodic or else unbounded at both ends, $\xi = \pm\infty$. In particular, all orbits within the homoclinic loop of $\mathcal{S}$ must be periodic. Closer examination of (3.8.5) shows that these are the only periodic orbits of (3.4). A typical phase portrait of (3.4) is sketched in Fig. 3.1.

The situation is similar in the nonphysical case that $\phi_\pm$ have opposite signs, except that both of $(\phi_\pm, 0)$ are local minima of $\mathcal{H}$. It follows that there can exist neither a viscous shock nor a homoclinic connection; however, periodic orbits occur near $(\phi_\pm, 0)$. In no case do orbits cross the physical barrier $\phi = 0$. □

To some extent, the conclusions of Proposition 3.1 hold for (S) as well. Traveling wave solutions $u(x,t) = \varphi(x - st)$ of (S) satisfy

$$[((K_a * \varphi) - s)\varphi]' = 0. \tag{3.12}$$

PROPOSITION 3.2.

(i) There exist no viscous shock solutions of (S).

(ii) There exist sinusoidal traveling wave solutions of (S) precisely if the Fourier transform $\widehat{K}$ has a zero.

Before proving Proposition 3.2, we make an important observation.

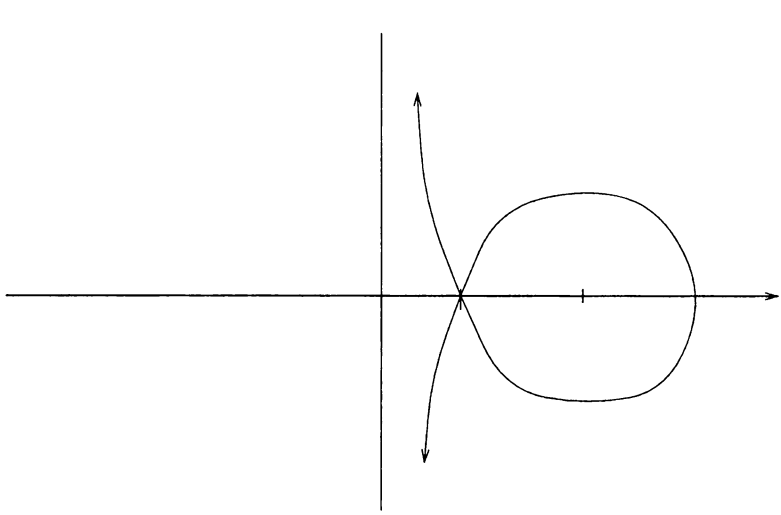

FIG. 3.1. Typical Phase Portrait

LEMMA 3.2.1. Let K_a be a symmetric kernel and $\varphi(\xi) \to \varphi\pm$ as $\xi \to \pm\infty$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} (K_a * \varphi)\varphi' \, d\xi &= \int_{-\infty}^{\infty} (K_a * \varphi')\varphi \, d\xi \\ &= \frac{1}{2}\varphi^2 \Big]_{-\infty}^{-\infty}. \end{aligned} \tag{3.13}$$

MOTIVATION. Lemma 3.2.1 follows, formally, from the expansion (3.1), since terms of the form $(\frac{\partial}{\partial x})^{2j}\varphi(\frac{\partial}{\partial x})\varphi$ are perfect derivatives.

Proof. This is a special case of the convolution identity

$$\int u(K * v) = \int v(\check{K} * u), \qquad \check{K}(x) \stackrel{\text{def}}{=} K(-x).$$

Using the change of variables $\overline{\psi} = -\psi, \overline{\xi} = \xi - \psi$, and symmetry of K_a, we have

$$\begin{aligned} \int (K_a * \varphi)\varphi' \, d\xi &= \iint K_a(\psi)\varphi(\xi - \psi)\varphi'(\psi) \, d\psi \, d\xi \\ &= \iint K_a(-\overline{\psi})\varphi(B)\varphi'(B - \overline{\psi}) \, d\overline{\xi} \, d\overline{\psi} \\ &= \iint K_a(\overline{\psi})\varphi(B)\varphi'(B - \overline{\psi}) \, d\overline{\xi} \, d\overline{\psi} \\ &= \int (K_a * \varphi')\varphi \, d\overline{\xi}, \end{aligned} \tag{3.14}$$

verifying the first line of (3.13). But, this gives

$$\begin{aligned}\int_{-\infty}^{\infty}(K_a * \varphi)\varphi'\,d\xi &= \frac{1}{2}\int_{-\infty}^{\infty}[(K_a * \varphi)\varphi]'\,d\xi \\ &= \frac{1}{2}(K_a * \varphi)\varphi\Big]_{-\infty}^{+\infty} \\ &= \frac{1}{2}\varphi^2\Big]_{-\infty}^{+\infty}. \quad \square\end{aligned} \tag{3.15}$$

Proof of Proposition 3.2. Integrating (3.12), we have

$$((K_a * \varphi) - s)\varphi + \beta = 0 \tag{3.16}$$

or

$$(K_a * \varphi) - s + \beta/\varphi = 0. \tag{3.17}$$

Suppose φ is a viscous shock connecting φ_- to φ^+. Then, letting $\xi \to \infty$ in (3.16), we find that $\varphi\pm$ are roots of $\varphi^2 - s\varphi + c$. Multiplying by φ' and integrating over $(-\infty, +\infty)$, we obtain, by Lemma 3.2.1,

$$\left[\frac{1}{2}\varphi^2 - s\varphi + \beta\log(\varphi)\right]\Big]_{-\infty}^{+\infty} = 0, \tag{3.18}$$

i.e., $\mathcal{H}(\varphi_-, 0) = \mathcal{H}(\varphi_+, 0)$, using the notation of Proposition 3.1. But, we have already shown that this is impossible.

On the other hand, for $\beta = 0$, (3.17) becomes the linear equation

$$(K_a * \varphi) = s, \tag{3.19}$$

which can be solved by Fourier transform. We find that (3.19) possesses periodic solutions

$$\varphi = s + Ae^{i\lambda\xi}, \tag{3.20}$$

for each zero λ of $\widehat{K_a}$. $\square$

REMARK 3.2.1. In Propositions 3.1, 3.2, the case $\beta = 0$, corresponding to a shock from concentration φ_- to zero, admits a simple treatment. Recall (introduction) that this case is the physically relevant one.

Further, note that for the physical kernel $K(x) = H(x)$ discussed in the introduction, assertion (ii) of Proposition 3.2 is nonvacuous. Indeed, any K that can be expressed as the product of a polynomial $p(x)$ and the characteristic function of an interval has the property that $\widehat{K}$ (and thus $\widehat{K_a}$) has infinitely many zeroes. For, by symmetry of K, $\widehat{K}$ is real for real λ. But, also

$$\begin{aligned}\widehat{K}(\lambda) &= p(i\partial/\partial\lambda)\frac{\sin c\lambda}{\lambda} \\ &= \lambda^{-n} b\,\sin(c\lambda + d) + O(\lambda^{-n-1})\end{aligned}$$

for some b, c, d, n, hence is oscillatory for large λ, and the result follows. This observation holds, too, for any $K(x) = H(x) + g(x)$, where $g(x)$ is symmetric, with $(\partial/\partial x)^{n+1}g''(x) \in L^1$. For, then $\hat{g}(\lambda) = O(\lambda^{-n-1})$, and the argument goes as before.

It is an interesting question for what class of symmetric K does $\widehat{K}$ have a zero.

REMARK 3.2.2. An interesting question is to determine whether (S) admits solitary wave solutions, or nonlinear periodic traveling waves analogous to those of (S-KdV).

4. The zero-dispersion limit. Equations (S-KdV), (2.3) suggest that the particle radius a plays a role as dispersion parameter in (S) analogous to that of the mesh size Δx in a dispersive difference scheme. In this section, we validate the formal limit

$$(\mathrm{S}) \to (\mathrm{H}) \text{ as } a \to 0,$$

within the smooth regime for (H), using an L^2-stability argument similar to those used to show convergence of finite difference schemes (cf. [5, 20]). In this limit, α, or equivalently V_{ST}, is understood to be held fixed, rather than, say, the particle density (in which case, $\alpha \to 0$, and the limit becomes trivial).

As $a \to 0$ with α held fixed, the L^∞-stability estimates of Sec. 2 blow up, and so are not helpful. However, as might be suggested by the analogy to difference schemes, (S) has better stability properties in L^2. In particular, (S) has the L^2-bounded linearized stability we require for our convergence proof. This nontrivial fact is the crucial point in our analysis. It follows from the symmetry of K, through a trick very similar to that used to prove Lemma 3.2.1.

The linearization of (S) around a given function $\overline{u}$ is

$$v_t + [(K_a * \overline{u})v + (K_a * v)\overline{u}]_x = 0, \tag{$S_{\underline{u}}$}$$

where v approximates the perturbation $(u - \overline{u})$.

LEMMA 4.0.1 (L^2-linearized stability). Let $\overline{u}(x,t)$ be a bounded, C^1 function on $\mathbb{R} \times [0,T]$, with $\overline{u}_x$ bounded as well, and let v be a solution of $(S_{\overline{u}})$. Then, for $0 \le t \le T$,

$$\|v(\cdot,t)\|_{L^2} = O(1)\|v_0\|_{L^2}, \tag{4.1}$$

where $O(1)$ depends only on $\overline{u}$, and not on v.

Proof. Multiplying $(S_{\overline{u}})$ by v and integrating gives the basic L^2-energy estimate

$$\begin{aligned} \frac{d}{dt}\|v\|_2^2 &\le \left|\int vv_x(K_a * \overline{u})\right| \\ &\quad + \left|\int v(K_a * v)\overline{u}_x\right| + \left|\int v(K_a * v_x)\overline{u}\right| \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{4.2}$$

Integrating by parts, we see that I is bounded by

$$\mathrm{I} = \left|\int v^2(K_a * \overline{u}_x)\right| = O(1)\|v\|_2^2, \tag{4.3}$$

where $O(1)$ depends on the Lipschitz constant for $\overline{u}$. Likewise, by Cauchy-Schwartz, II

is bounded by

$$\begin{aligned} \mathrm{II} &= \left| \iint v(x) K_a(y) v(x-y) \overline{u}_x(x)\, dy\, dx \right| \\ &= O(1) \|v\|_2^2 \int |K_a(y)|\, dy \\ &= O(1) \|v\|_2^2, \end{aligned} \tag{4.4}$$

where $O(1)$ again depends on $\mathrm{Lip}(\overline{u})$.

The critical term is III, which requires a trick similar to that of Lemma 3.2.1. By symmetry of K_a, $K_{a,x}$ is odd, and

$$\begin{aligned} \mathrm{III} &= \iint v(x) K_{a,x}(y) v(x-y) \overline{u}(x)\, dy\, dx \\ &= \iint_{y \geq 0} v(x) \overline{u}(x) K_{a,x}(y) [v(x-y) - v(x+y)]. \end{aligned} \tag{4.5}$$

Using the change of variables $w = x + y$, we can rewrite

$$- \iint_{y \geq 0} v(x) \overline{u}(x) K_{a,x}(y) v(x+y) \tag{4.6}$$

as

$$\iint_{y \geq 0} v(w-y) \overline{u}(w+y) K_{a,x}(y) v(w), \tag{4.7}$$

effecting the "summation by parts",

$$\mathrm{III} = \iint_{y \geq 0} v(x) v(x-y) K_{a,x}(y) [\overline{u}(x) - \overline{u}(x+y)]. \tag{4.8}$$

From (4.8), we obtain

$$\begin{aligned} \mathrm{III} &= O(1) \iint v(x) v(x-y) K_{a,x}(y) y\, dy\, dx \\ &= O(1) \|v\|_2^2 \int |K_{a,x}(y) y|\, dy \\ &= O(1) \|v\|_2^2, \end{aligned} \tag{4.9}$$

using the Lipschitz bound on $\overline{u}$, Cauchy-Schwartz, and the scaling

$$\begin{aligned} \int |K_{a,x}(y) y|\, dy &= \int |a^{-2} K_x(y/a) y|\, dy \\ &= \int |K_x(z) z|\, dz \\ &= O(1). \end{aligned} \tag{4.10}$$

Combining our estimates, we have

$$\frac{d}{dt} \|v\|_2^2 = O(1) \|v\|_2^2, \tag{4.11}$$

from which

$$\|v(\cdot, t)\|_2^2 \leq e^{O(1)t} \|v_0\|_2^2, \tag{4.12}$$

and we are done.

REMARK 4.0.1. Calculations (4.5)–(4.8) are motivated by Lemma 3.2.1. Since $v(K_a * v_x)$ is in a sense a perfect derivative, it is natural that something like integration by parts should allow us to shift the x-derivative in term III onto the Lipschitz term $\overline{u}$.

REMARK 4.0.2. The argument of Lemma 4.0.1 remains valid for $\overline{u}$, v periodic.

PROPOSITION 4.1. Let $u^0(x) \in C^4$ be either periodic, or else rapidly converging at $x = \pm\infty$, and let $\overline{u} \in C^4$ be the classical solution of (H) with initial data u_0, defined up to time $T > 0$. For each $a > 0$, define $u^a \in C^4$ to be the solution of (S) with initial data u_0. Then, for a sufficiently small,

$$\|u^a - \overline{u}\|_\infty = O(1)a^2, \quad t \le T, \tag{4.13}$$

where $O(1)$ depends only on $\overline{u}$.

Proof. We use the shorthand notation

$$\begin{aligned} S[u] &= u_t + ((K_a * u)u)_x, \\ S_{\overline{u}}[u] &= u_t + ((K_a * \overline{u})u)_x + ((K_a * u)\overline{u})_x \end{aligned} \tag{4.14}$$

($S_{\overline{u}}$ being the linearization of S around $\overline{u}$).

Setting

$$S[\overline{u}] = a^2\eta, \tag{4.15}$$

$$\eta = -a^{-2}[(K_a * \overline{u}_x - \overline{u}_x)\overline{u} + (K_a * \overline{u} - \overline{u})\overline{u}_x], \tag{4.16}$$

and Taylor expanding as in (3.1), we have

$$\eta(x) = [(C_1\overline{u}_{xxx}(\zeta)\overline{u}(x)) + C_2\overline{u}_{xx}(\zeta)\overline{u}_x(x)] \tag{4.17}$$

with

$$\zeta \in \text{ Support}_y K(x-y) = [x-a, x+a]. \tag{4.18}$$

If $u_0, \overline{u}$ are periodic, or if u_0 and (by finite propagation speed) $\overline{u}$ are rapidly converging at $x = \pm\infty$, then

$$\|\eta\|_p = O(1) \quad \forall p. \tag{4.19}$$

By similar reasoning,

$$\|\eta_x\|_p = O(1) \quad \forall p. \tag{4.19.5}$$

Now, set $v = u^a - \overline{u}$. Combining $S(u^a) = 0$ and (4.15), we have

$$S_{\overline{u}}[v] = -((K_a * v)v)_x + a^2\eta. \tag{4.20}$$

Multiplying by v and integrating in x, we obtain the standard energy estimate

$$\begin{aligned} \frac{d}{dt}\|v\|_2^2 &= \left|\int -v((K_a * \overline{u})v)_x - v((K_a * v)\overline{u})_x\right| \\ &\quad + \left|\int v((K_a * v)v)_x\right| + \left|\int va^2\eta\right| \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{4.21}$$

We have already estimated the critical term I, due to the linear (left-hand side) part of (4.20), as

$$\mathrm{I} = O(1)\|v\|_2^2, \tag{4.22}$$

in the proof of Lemma 4.0.1. Set

$$M(t) = \sup_{s \le t} \|v(\cdot, s)\|_\infty. \tag{4.23}$$

Then,

$$\begin{aligned} \mathrm{II} &= \left| \int v_x (K_a * v) v \right| \\ &= \left| \int v^2 (K_{a,x} * v) \right| \\ &= O(1) a^{-1} M \|v\|_2^2. \end{aligned} \tag{4.24}$$

Likewise, (4.19) gives

$$\mathrm{III} = O(1) M a^2. \tag{4.25}$$

Combining, we have

$$\frac{d}{dt}\|v\|_2^2 = O(1)(1 + a^{-1}M)\|v\|_2^2 + O(1)Ma^2. \tag{4.26}$$

Since $\|v_0\|_2^2 = 0$, the Gronwall inequality (4.26) implies

$$\|v\|_2^2 = O(1) M a^2 e^{O(1)(1+a^{-1}M)t}. \tag{4.27}$$

Differentiating (4.20) gives

$$S_{\overline{u}}[v_x] = -((K_a * v)v)_{xx} + a^2 \eta_x - [(K_a * \overline{u}_x)v + (K_a * v)\overline{u}_x]_x. \tag{4.28}$$

Multiplying by v_x, integrating in x, and following essentially the same steps as above, we obtain

$$\|v_x\|_2^2 = O(1) M a^2 e^{O(1)(1+a^{-1}M)t}, \tag{4.29}$$

as well.

Now, the one-dimensional Sobolev bound,

$$\|v\|_\infty^2 \le \|v_x\|_2 \|v\|_2, \tag{4.30}$$

gives

$$M^2 = O(1) M a^2 e^{O(1)(1+a^{-1}M)t}, \tag{4.31}$$

or

$$M(t) = O(1) a^2 e^{O(1)(1+a^{-1}M(t))t}. \tag{4.32}$$

Since $M(0) = 0$, continuous induction implies that

$$M = O(1)a^2, \quad 0 \le t \le T, \tag{4.33}$$

for a sufficiently small, and we are done. □

REMARK 4.1.1. Clearly, the result of Proposition 4.1 holds also for (S-B), with the scaling $d \sim a^2$ (relevant to the experimental situation we will discuss in Sec. 6).

REMARK 4.1.2. We can define a formal error expansion, $u^a = \overline{u} + \sum_{j=1}^{\infty} a^j v^j$, by expanding $S[u^a] = 0$ using (3.1). We obtain a sequence of equations

$$S_{\overline{u}}[u^j] = \phi^j, \tag{4.34}$$

where ϕ^j depends on the first j derivatives of $\overline{u}$ and the first $(j-k)$ derivatives of u^k, $1 \le k < j$. Thus, if $u_0 \in C^N$, the truncated series

$$\overline{u} + \sum_{j=1}^{N-2} a^j \overline{u}^j$$

is well-defined and C^2 for $t \le T$, iterating standard linear theory. Using

$$S\left[\overline{u} + \sum_{j=i}^{N-2} a^j \overline{u}^j\right] = O(1)a^{N-1},$$

the argument of Proposition 4.1 gives

$$\left\| u^a - \left(\overline{u} + \sum_{j=1}^{N-2} a^k \overline{u}^j\right)\right\|_\infty = O(1)a^{N-1},$$

within the smooth regime for $\overline{u}$.

An interesting open question, by analogy with KdV, is to determine the behavior of (S) after the formation of shocks in $\overline{u}$, perhaps by approximating with modulated waves.

5. Oscillatory traveling waves. We now incorporate diffusion and carry out an analysis analogous to that of Sec. 3 on the effective equation for (S-B),

$$u_t + \alpha(u^2)_x = -c_2 \alpha a^2 (u u_{xx})_x + d u_{xx}. \tag{S-KdV-B}$$

This equation is similar in form to KdV-Burgers and, likewise, admits oscillatory viscous shock solutions with the characteristic one-sided oscillation pattern of a KdV-Burgers shock.

PROPOSITION 5.1. There exists a viscous shock solution $u(x,t) = \phi(x - st) \ge 0$ of (S-KdV-B), connecting any endstates $\phi_- > \Phi_+ \ge 0$. For $\gamma = d/(\alpha\phi_- a\sqrt{c_2})$ sufficiently small, the profile $\phi(\xi)$ is oscillatory at $\xi = -\infty$, with period $\sim a\sqrt{c_2}$, number of oscillations $\sim \gamma^{-1}$, and total width $\sim c_2 a^2 \alpha \phi_- / d$. For $\tilde{\gamma} = d/(\alpha[\phi] a\sqrt{c_2})$ sufficiently large, $\phi(\xi)$ is exponentially decaying at both $\xi = \pm\infty$, with total width $\sim d/\alpha[\phi]$.

Proof. The equation for traveling wave solutions of (S-KdV-B), after integration and the rescaling $\xi' = \xi/(a\sqrt{c_2}), u' = u/\phi_-, s' = s/(\alpha\phi_-)$, becomes

$$\varphi - s + \beta/\varphi + \varphi'' = \gamma\varphi'/\varphi. \tag{5.2}$$

We first consider the case that $\beta \ne 0$, i.e., $\phi_+ > 0$. Carrying out the phase plane analysis of Sec. 3, we find that the former Hamiltonian function,

$$\mathcal{H}(\varphi, \varphi') = \tfrac{1}{2}\varphi^2 - s\varphi + \beta \log \varphi + \tfrac{1}{2}\varphi'^2, \tag{5.3}$$

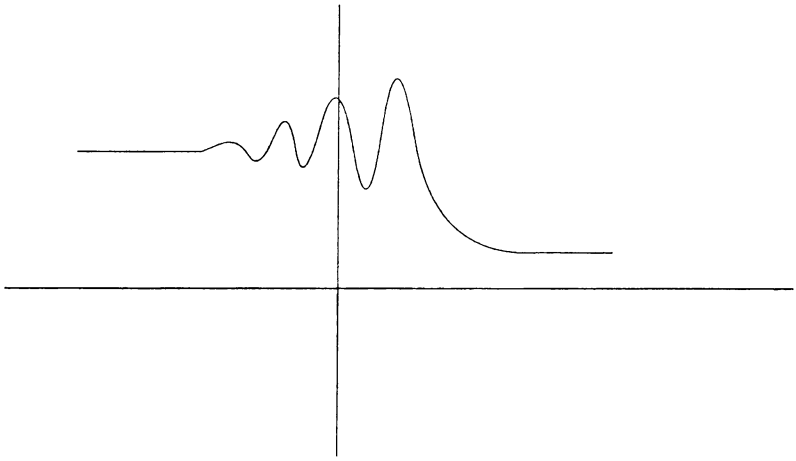

FIG. 5.1. Typical Viscous Shock Profile of S-KdV-B

becomes a Lyapunov function,

$$\frac{d\mathcal{H}}{dt} = \frac{\gamma(\varphi')^2}{\varphi} > 0. \tag{5.4}$$

As in Sec. 3, let $\varphi_- > \varphi_+$ denote the roots of $(\varphi - S + \beta/\varphi)$, so that $(\varphi_-, 0)$ and $(\varphi_+, 0)$ are the only stationary points of (5.2). From the properties of $\mathcal{H}$ derived in Sec. 3, it is easy to see that all solutions (φ, φ') of (5.2) that are bounded as $\xi \to -\infty$ must approach either $(\varphi_-, 0)$ or $(\varphi_+, 0)$ as $\xi \to -\infty$.

For, in this case, the Ω^- limit set exists, and is a connected set, contained in

$$\left\{(\varphi, \varphi') : \frac{\gamma(\varphi')^2}{\varphi} = 0\right\} = \{(\varphi, 0)\},$$

and on which $\mathcal{H}$ is constant. Since $\frac{\partial \mathcal{H}}{\partial \varphi}(\varphi, 0) = (\varphi - s\varphi + \beta/\varphi)$ vanishes only at $(\varphi^-, 0)$ and $(\varphi^+, 0)$, we find that Ω^- is an isolated point, and one of $(\varphi^-, 0), (\varphi^+, 0)$.

Now, consider (5.2) as a first-order system in (φ, φ'). Linearizing around the stationary points $(\varphi^-, 0), (\varphi^+, 0)$, we find that, similar to the Hamiltonian case, $(\varphi^-, 0)$ is a repelling spiral point and $(\varphi^+, 0)$ is a saddle point. In particular, there exists a nonconstant solution, $(\varphi(\xi), \varphi'(\xi))$ of (5.2) approaching $(\varphi^+, 0)$ as $\xi \to +\infty$, asymptotic at $\xi \to +\infty$ to the solitary wave solution of the Hamiltonian system (3.6). Since it is nonconstant, (5.4) implies that $\mathcal{H}$ increases along this orbit, so that it is trapped within the strict interior of the closed curve determined by the Hamiltonian solitary wave orbit, as $\xi \to -\infty$. In the first place, this implies that the orbit is bounded, and in the second place, that $(\varphi(\xi), \varphi'(\xi))$ cannot approach $(\varphi^+, 0)$ as $\xi \to -\infty$. As noted above, it must therefore approach the repelling spiral $(\varphi^+, 0)$. We conclude that $\varphi(\xi)$ is a viscous shock profile connecting φ_- to φ_+. Since it is a spiral-saddle connection, it is oscillatory toward $\xi = -\infty$, as shown in Fig. 5.1.

When γ is small, the behavior of (5.2) near $\xi = -\infty$ is close to that of the associated Hamiltonian system, near the critical point $(\phi_-, 0)$, i.e., is approximately that of a linear oscillator. Thus, the period in the ξ' coordinates is approximately constant, and, taking

into account the scaling $\xi' = \xi/(a\sqrt{c_2})$, the period of oscillation in x is approximately proportional to $a\sqrt{c_2}$. Further, the average of $(\phi')^2$ over a period is comparable to the average squared distance from $(\phi_-, 0)$, which is comparable to $\mathcal{H}$. Thus, $\mathcal{H}$ satisfies $d\mathcal{H}/d\xi \sim -\gamma\mathcal{H}$, and therefore the interval on which ξ is "oscillatory", i.e., with the squared amplitude of oscillations above some cutoff of detectability (relative to ϕ_-^2), is of length $\sim \gamma^{-1}$ in ξ' coordinates. Taking into account the scaling $\xi' = \xi/(a\sqrt{c_2})$, the total width is $\sim a^2 c_2 \alpha\phi_-/d$, as claimed.

The case $\beta = 0$ is very similar, the only difficulty being to establish the existence of an orbit terminating at $\xi = +\infty$ at the stationary point $(0,0)$, at which the ODE (5.2) becomes singular. This can be accomplished by a simple trick: Defining the blow-up parameter t by

$$d\xi/dt = \phi, \tag{5.5}$$

we convert (5.2) to the nonsingular system

$$\begin{aligned} d\psi/dt &= \gamma\psi + (s-\phi)\phi, \\ d\phi/dt &= \phi\psi. \end{aligned} \tag{5.6}$$

Linearizing (5.6) around $(0,0)$ gives

$$\frac{d}{dt}\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \begin{pmatrix}\gamma & s\\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix} \tag{5.6'}$$

and (5.6) therefore has an invariant center manifold,

$$\psi = \psi(\phi) \sim -s\phi/\gamma. \tag{5.7}$$

Integrating the ODE (5.7) along the center manifold, we have $\phi \sim e^{-s\xi/\gamma}$, in the ξ' coordinates, or $\phi \sim e^{-s\xi\alpha\phi_-/d}$ in the original coordinates, for ξ sufficiently large. This proves the existence of an orbit terminating at $(0,0)$; the rest of the argument proceeds as before to give the existence of a connecting orbit, and its behavior in the case that γ is small.

We remark that in neither case does Φ cross the physical boundary, $\Phi = 0$. In the $\beta \neq 0$ case, this is precluded by the logarithmic barrier $\mathcal{H} \sim \log\phi$. In the $\beta = 0$ case, the fact that the Lyapunov function $\mathcal{H} = \frac{1}{2}(\phi - \phi_-)^2 + \frac{1}{2}\psi^2$ is maximized at $(\phi(+\infty), \psi(+\infty)) = (0,0)$ implies that (ϕ, ψ) are trapped in a ball tangent to the ψ-axis.

In both cases, it remains to establish the shape of the traveling wave in the case that $\tilde{\gamma}$ is large. This can be accomplished by the alternate rescaling $\xi' = \xi\alpha[\phi]/d, u' = u/[\phi], s' = s/(\alpha[\phi])$, under which (5.2) becomes

$$\varphi - s + \beta/\varphi + \tilde{\gamma}^{-2}\varphi'' = \varphi'/\varphi.$$

As $\tilde{\gamma} \to \infty$, the behavior of this system converges to that of Burgers' equation,

$$\varphi - s + \beta/\varphi = \varphi'/\varphi,$$

for a wave of strength $[\phi'] = 1$. This convergence can be established rigorously by the center manifold argument used in the $\beta = 0$ case, with the modification $d\xi/dt = \mu\phi$, the new variable $\mu = \tilde{\gamma}^{-2}$ satisfying $d\tilde{\gamma}/dt = 0$. The intersection of the center manifold and the hyperplane $\mu = 0$ is exactly the Burgers' orbit, from which the result follows. By translation invariance (in u) of Burgers' equation, all such waves have identical profiles,

of constant width. Thus, in the original coordinates, the traveling wave profile is of width $\sim d/(a[\phi])$, as claimed. □

For $\tilde{\gamma}$ large, the traveling wave profile described above is an attracting asymptotic state.

PROPOSITION 5.2. For $\tilde{\gamma}$ sufficiently large, the viscous shock solution described in Proposition 5.1 is stable under small H^3 perturbations.

This Proposition is proved in [11], using an L^2-energy argument related to that of [10]. The proof requires $\tilde{\gamma}$ large enough that the profile is only "weakly oscillatory", in the sense that it is monotone no an interval $[\xi_0, +\infty]$ and of small total variation on the remainder, $[-\infty, \xi_0]$. We conjecture that, like KdV-Burgers profiles, the (S-KdV-B) profile is in practice stable also for small $\tilde{\gamma}$ and for large, nonsmooth perturbations, and, further, that the corresponding traveling wave solution represents a unique attracting state for (S-KdV-B) with initial data of shock type.

Numerical studies of closely related dispersive difference schemes [6] both support this conjecture and suggest that similar behavior is to be found in solutions of (S-B), an equation still closer to the difference scheme model. A very interesting open problem is to show the existence and stability of viscous shock solutions of the nonlocal equation (S-B).

6. Layering. We conclude with a discussion of the physics of sedimentation, in relation to a benchmark experiment of Siano [19] demonstrating layering behavior. The theory of sedimentation has a long and interesting history, going back to Stokes' study of the single-particle case and Einstein's study of Brownian motion. The case of an infinite, uniformly distributed system of particles already involves interesting renormalization issues. For, assuming particles to be identical spheres obeying steady Stokes flow,

$$\Delta U = \nabla P, \qquad \operatorname{div} U = 0$$

(where U is fluid velocity and P is pressure), the sum of fluid velocities U due to the effects of infinitely many particles is divergent, being roughly equal (away from particles) to $\sum_j G(x - x_j)$, where x_j is the location of the jth particle and

$$G(x) = \frac{1}{8\pi\mu}(1/|x|)(\mathrm{I} - xx^t/|x|^2)$$

is the fundamental solution of the Stokes equation on R^3.

Various devices have been introduced in order to extract a finite value for the flow, cf. [1, 2, 4, 7, 8, 9, 14, 15, 18, 23]. A particularly natural resolution of this paradox, [2, 14, 16], is to solve the flow problem first within a finite domain with solid boundary, then let the boundary of the domain recede to infinity. Though boundary and interior contributions each diverge in the limit, their sum remains finite. For a homogeneous, randomly distributed suspension, this gives the settling speed, first obtained by Batchelor, [1], of $V = V_{ST}(1 - 6.55u)$. We refer the reader to [16] for a further account of this steady-state theory, including other interesting issues involving the geometry of particle distributions.

Rubinstein and Keller extended these ideas in [17] to the case of a non-uniform random distribution, deriving a kinetic model for the temporal evolution; their model, however, does not include effects of Brownian or self-diffusion. In the special case of a stratified suspension, Rubinstein, [15], showed that this kinetic model can be reduced to an evolution equation in terms of the average concentration (expectation) u alone, namely (S).

However, the theory of dynamic (i.e., non-uniform distribution) sedimentation is as yet only partially understood, in particular the phenomenon of layering. This phenomenon is illustrated by a well-known experiment of Siano, [19], in which extremely uniform and nearly colloidal polystyrene spheres, suspended in aqueous solution, were allowed to settle in 20 cm. test tubes over periods of several days. Typical experimental parameters (in the notation of the introduction) in this study were

$$\begin{aligned} a &= 0.5 \times 10^{-4} \text{ cm}, \\ \mu &= 10^{-2} \text{ dyne sec cm}^{-2}, \\ T &= 293^\circ \text{ Kelvin}, \\ \kappa &= 1.38 \times 10^{-16} \text{ erg}^\circ\text{Kelvin}^{-1}, \\ g &= 9.8 \times 10^{1} \text{ cm sec}^{-1}, \\ u_- &= .001, \\ V_{ST} &= 0.3 \times 10^{-6}\text{cm sec}^{-1}, \\ d &= 4 \times 10^{-9}\text{cm}^2\text{sec}^{-1}, \end{aligned}$$

where u_- represents initial concentration at the lower part of the tube. Initial concentrations were taken to be either uniform or smoothly decreasing with height. Even an initially homogeneous solution was found, after some time, to produce a layering pattern: several distinct bands of differing concentrations just below the primary interface between sediment and clear fluid, of width $\sim$ 1cm per band and $\sim$ 10cm in total. These bands appeared to bifurcate from the initially sharp interface near the top of the fluid; once formed, they were fairly stable, reappearing after small perturbations such as tapping the tube. The appearance of these patterns was greatly accelerated for initial concentrations with a negative gradient, taking hours instead of days. In this case, the pattern appeared to stabilize into bands of constant width, moving with a constant velocity close to V_{ST}, i.e., a traveling wave. In either case, the layering pattern had an almost monotone, stairstep profile $u(\cdot, t)$, as pictured in Fig. 6.1. The precise width of the bands appeared to depend strongly on the concentration u_-, but more weakly on the particle radius a.

On the basis of these results, Siano suggested several conclusions about layering:

(i) It is gradient driven.

(ii) It is not driven by hydrodynamic effects.

(iii) It is not driven by standard diffusion, but rather a diffusion of the form $d(u)u_{xx} - u_{xxxx}$, where $d(u)$ is negative for some concentrations.

Conclusion (i) was supported by the observation of accelerated layering for initial data with a gradient, and by the fact that layers initiate from the primary interface, the point of largest gradient. Conclusion (ii) was drawn essentially by consideration of

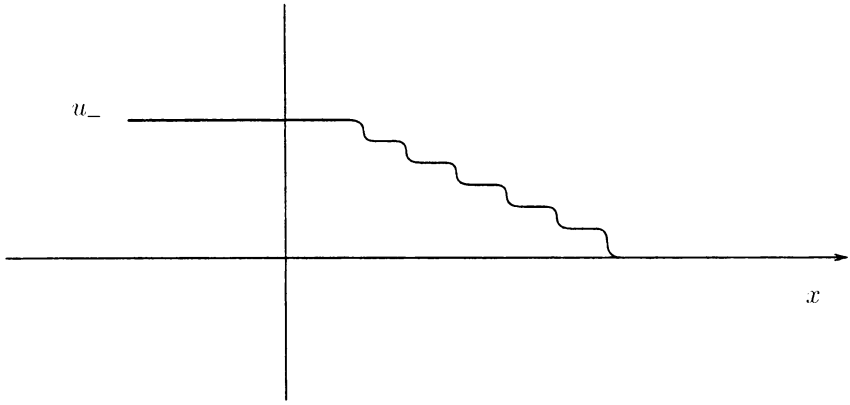

FIG. 6.1. Layered Profile

the hyperbolic traffic model, (3), presented in the introduction; as observed there, the type of initial data prescribed by Siano should give a single shock interface rather than multiple bands. Conclusion (iii) was drawn on phenomenological grounds, by analogy to the Cahn-Hilliard Theory of two-phase systems; this type of diffusion predicts the formation of sharp interfaces of width depending *only* on the concentration u. More recently, Caflisch and Papanicolaou, [3], have given evidence that terms as in (iii) can arise from interaction of Brownian and self-diffusion (though they did not verify that d can become negative); their arguments are partly by analogy as well.

Before drawing our own conclusions, it is instructive to consider layering in the context of the naive traffic model, (3). The experimental scenario can be modeled by a mixed initial boundary value problem on $(-\infty, 0]$, with the no-flux boundary condition $u \equiv 0$ at $x = 0$. Because the particle suspension immediately detaches from the fluid boundary, leaving zero-concentration fluid above (in agreement with [19], bottom p. 125), this boundary condition may be modeled more simply by extending the initial data to

$$u(x,0) = \begin{cases} u_0(x) > 0, & x < 0, \\ 0, & x \geq 0, \end{cases}$$

and solving a pure initial data problem on the whole real line. As discussed in the introduction, such initial data leads to a shock originating from the fluid boundary $x = 0$ and propagating downward, marking the boundary between positive and zero-concentration fluid. In a rough sense, this is what is observed in experiment; that is, $u(x,t)$ obeys (3) away from the primary interface, where the implicit small-gradient assumption of (3) is violated and there occur localized phenomena outside the scope of the model. From this perspective, layering would appear to be a *shock-layer phenomenon*, a conclusion that is supported by the (accelerated) case in which the layering pattern seems to form a traveling wave.

This is consistent with Siano's conclusion (i). More evidence for this point of view is given by the diffusive regularization of the traffic model, which is equivalent to Burgers' equation. For Burgers' equation, the asymptotic state for shock initial data is a traveling

"viscous shock" wave; as discussed in Sec. 5, the profile of this wave is a monotone ramp, with width $\sim d/(\alpha[\phi]) = d/(6.55|V_{ST}|u_-)$. For the experimental parameters given above, the width of the shock layer predicted by Burgers' equation is thus 2 cm, in good agreement with actual behavior.

In this light, conclusions (ii) and (iii) appear overly strong since, within their limitations, both (3) and its diffusive regularization seem to model layering rather well. Also, experiment [19] often seems to be described as demonstrating instability of the constant state. This is misleading. Taken at face value, it amounts to ignoring the boundary conditions at $x = 0$; in fact, the stability of behavior away from the slowly growing layering pattern is evidence for *stability* of the constant state u_-. What is precisely true is more circular: that instability of some constant states between 0 and u_- is consistent with (iii). What has not been and should be done is an analysis of traveling waves of the traffic model with the nonstandard diffusion of (iii), and comparison with experiment (note that the related analysis of Caflisch and Papanicolaou, [3] pp. 897–899, does not include compressive hydrodynamic effects).

Further, we point out that the dispersive hydrodynamic effects included in (S) were unknown at the writing of [3], [19]. The results of Sec. 5 point up another possible explanation for layering, based on the interaction of hydrodynamic compression and dispersion with ordinary Fickian diffusion modeled by (S-KdV-B). For, depending on the parameter $\tilde{\gamma} = \gamma = d/(\alpha u_- a\sqrt{c_2})$, which in this case is approximately $4 \times 10^1/\sqrt{c_2}$, the traveling wave ϕ resolving shock initial data can be made up of several oscillatory bands rather than the single smooth interface of a Burgers' shocklayer. This consideration leads us to an interesting observation. As described in the introduction, the numerically larger part of the kernel K is the truncated parabola $H(x)$ of the introduction, which in fact is the contribution to K of wall-particle interactions alone. For H, the second moment c_2 is of order one, so that $\tilde{\gamma}$ is quite large, and the profile predicted by (S-KdV-B) is essentially the Burgers' profile. In other words, wall-particle effects are essentially negligible for the parameter range we are interested in. On the other hand, the numerically smaller remainder of K, accounting for two-particle interactions, is $O(1+|x|^2)^{-1}$, and has infinite second moment c_2. Unexpectedly, this remainder appears to be the dominant term for behavior (whereas the wall-interactions term was critical for renormalization/derivation of the model). Unfortunately, to determine its effect is beyond the scope of the analysis in this paper and perhaps also of the model (S-B) since it would appear that the lower boundary of the test tube should play a role in the calculation, violating the assumptions of the model. However, our analyses do suggest that this hydrodynamic effect might be significant.

On this speculative note, we end. It would be very interesting to do numerical or analytic studies of (S-B) and of (Burgers) and (S) augmented with the nonstandard diffusion of (iii), to determine which if any of these regularizations gives a shock layer with the stairstep pattern observed in experiment.

REMARK 6.0.1. The Existence/Uniqueness Theory of Sec. 2 remains valid with the more general K_a discussed here, since also $K_{a,x} \in L^1$. Likewise, we still obtain convergence in the zero-dispersion limit as in Sec. 4, but at a linear rate,

$$\|u^a - \overline{u}\|_\infty = O(1)a, \quad t \le T.$$

This can be seen by breaking up K_a as $K_a^1 + K_a^2$, with

$$K_a^1 = \rho(x)K_a(x) + c(a)\tilde{\delta}(x), \qquad K_a^2 = (1-\rho(x))K_a(x) - c(a)\tilde{\delta}(x),$$

where $\rho(x)$ is a smooth cutoff function, supported on $[-2,2]$ and identically equal to one on $[-1,1]$, $\tilde{\delta} \in C^\infty$ is an approximate delta function (i.e., $\tilde{\delta} \in C^\infty$, with $\int \tilde{\delta}\, dx = 1$), and the constant $c(a)$ is chosen so that $\int K_a^1\, dx = 1$. We find that the mass of K_a^2 is of order

$$\int_{|x|\geq a^{-1}} |x|^{-2}\, dx = O(a),$$

and the moment of K_a^1 is of order

$$a^2 \int_{|x|\leq a^{-1}} dx = O(a)$$

as well. Likewise, $c(a) = O(a)$. Decompose the formal error in (4.16) as

$$\begin{aligned}
\eta &= -a^{-2}[(K_a * \overline{u}_x - \overline{u}_x)\overline{u} + (K_a * \overline{u} - \overline{u})\overline{u}_x] \\
&= -a^{-2}[(K_a^1 * \overline{u}_x - \overline{u}_x)\overline{u} + (K_a^1 * \overline{u} - \overline{u})\overline{u}_x] \\
&\quad + -a^{-2}[(K_a^2 * \overline{u}_x)\overline{u} + (K_a^2 * \overline{u})\overline{u}_x] \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

We find that $\mathrm{I} = O(a^{-1})$, by the moment argument of Sec. 4, while $\mathrm{II} = O(a^{-1})$ by the simpler bound

$$\|K_a^2 * f\|_p \leq \|K_a^2\|_1 \|f\|_p = O(a)\|f\|_p.$$

The rest of the argument proceeds as before, with the modification that error terms $O(a^2)$ become $O(a)$.

Similarly, Proposition 3.2 remains valid for this more general kernel, K_a, since (i) requires only decay at infinity and by Remark 3.2.1. Of course, the results of Secs. 3 and 5 that pertain to the local equation (S-KdV) obtained by the Taylor expansion (3.1) are not relevant when K_a does not have a second moment.

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