QUARTERLY OF APPLIED MATHEMATICS
VOLUME LVII, NUMBER 3
SEPTEMBER 1999, PAGES 561-572

ON THE PROPAGATION OF THE BULK OF A MASS SUBJECT TO PERIODIC CONVECTION AND DIFFUSION

By

W. A. DAY (*Hertford College, Oxford, England*)

AND

G. SACCOMANDI (*MEMOMAT, La Sapienza, 00161, Roma, Italy*)

1. Introduction. This paper is concerned with classical solutions of the initial-value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a(x)u) = \frac{\partial}{\partial x}\left(b(x)\frac{\partial u}{\partial x}\right) && (-\infty < x < \infty, t > 0), \\ &u(x,0) = f(x) && (-\infty < x < \infty) \end{aligned} \tag{1}$$

for the Fokker-Planck equation.

It is supposed that the following hypotheses are in force:

(i) *$a(x)$ and $b(x)$ are periodic, with period p, $a(x) \in C^1$, $b(x) \in C^2$ and $b(x)$ is positive*;

(ii) *$f(x)$ is nonnegative and continuous and has compact support and, to exclude the trivial case, $f(x)$ is not identically zero*;

(iii) *$u(x,t)$ is nonnegative, the moments*

$$\int_{-\infty}^{\infty} x^n u\,dx \quad (n = 0, 1, 2) \tag{2}$$

exist, and $u \to 0$ and $\partial u/\partial x \to 0$ as $|x| \to \infty$ (at rates sufficiently rapid to justify certain integrations by parts).

Here $u(x,t)$ is the density, and $f(x)$ is the initial density, of a mass distribution that is convected with velocity $a(x)$ (the drift coefficient) and is subject to diffusion, the positive coefficient $b(x)$ being the diffusion coefficient. Our purpose is to study the manner in which the bulk of the mass is dispersed.

The well posedness of the problem is ensured by the standard theory of C_0-semigroups, which also guarantees that the positivity of the initial density is preserved [1].

Equation (1) arises in many scientific and technical applications; a recent review of the Fokker-Planck equation has been given by Miyazawa [2], where the relationship of this equation with the Schroedinger equation is discussed and an entire section is

Received September 24, 1997.
1991 *Mathematics Subject Classification.* Primary 35P99, 35B40.

dedicated to periodic coefficients. Moreover, it is well known that (1) arises in the study of quasistatic brownian motion in a periodic potential, a problem of interest in physics, chemical physics and communication theory ([3], chapter 11). In the mechanical sciences the Fokker-Planck equation with periodic coefficients arises in the study of diffusion in inhomogeneous media (for example in composite materials, [4]), in flow enhanced diffusion in porous media, [5], and in simple caricature models of turbulence [6]. A referee has suggested that it may be possible to weaken the hypothesis of periodicity on the drift and diffusion coefficients; we hope to report on this possibility in a later paper.

In the context of the heat equation, for which $a(x)$ is identically zero and $b(x)$ is identically constant, many authors have observed that $u(x,t) > 0$ for every x and every $t > 0$. This observation has led some to conclude that the speed of propagation is infinite (e.g. Racke [7], pp. 27, 141). Such a conclusion is often deemed to be nonphysical or paradoxical and has lead to the construction of alternative theories yielding finite speeds of propagation (e.g. Cattaneo [8], Gurtin and Pipkin [9], Joseph and Preziosi [10]). More recently Fichera [11] and Day [12, 13] have defended the classical theory of diffusion against the charge that it is paradoxical and, following Maxwell [14], have drawn attention to the fact that while an infinitesimally small fraction of the distribution does propagate at an arbitrarily large rate, the propagation of the bulk is in fact a very slow process and the time taken is proportional to the square of the distance.

In the context of the initial-value problem (1) we prove results that shed light on the propagation of the bulk of the mass. It does not seem possible to extract these conclusions in any straightforward way from solutions of (1) obtained by classical methods. By a change of variables we can replace (1) by an equation in which the drift coefficient is constant, but such a transformation seems not to lead to any significant shortening of the proofs and the results are more difficult to interpret. We prefer to work directly with (1) and to state our conclusions in terms of the physical coefficients $a(x)$ and $b(x)$.

The first result concerns the total mass and the centre of mass, which is the point

$$\overline{x}(t) = \frac{\int_{-\infty}^{\infty} xu\,dx}{\int_{-\infty}^{\infty} u\,dx}.$$

THEOREM 1. The total mass is conserved, i.e.,

$$\int_{-\infty}^{\infty} u\,dx = \int_{-\infty}^{\infty} f\,dx. \tag{3}$$

Furthermore, there is a constant α, depending only upon $a(x)$ and $b(x)$, such that

$$\overline{x}(t) = \alpha t + O(1). \tag{4}$$

Thus, if $\alpha \neq 0$, then $\overline{x}(t) \sim \alpha t$ as $t \to \infty$, and the speed of the centre of mass is asymptotically equal to α; but if $\alpha = 0$, then the centre of mass is confined to a bounded interval.

The second result is concerned with the moment of inertia with respect to the centre of mass.

THEOREM 2. There is a positive constant β, depending only upon $a(x)$ and $b(x)$, such that

$$\int_{-\infty}^{\infty} (x-\overline{x})^2 u\,dx \sim \beta t \int_{-\infty}^{\infty} f\,dx \quad \text{as } t \to \infty. \tag{5}$$

Our arguments yield explicit formulae for α and β; when the drift and diffusion coefficients are constant, it turns out that $\alpha = a$ and $\beta = 2b$, but, in general, each of α and β depends upon both coefficients.

Now let ϵ be any number in $0 < \epsilon < 1$, and let the distance $d(t;\epsilon)$ be defined by the requirement that the fraction of mass lying within distance d of the centre of mass is equal to ϵ, i.e.,

$$\int_{|x-\overline{x}|\le d} u\,dx = \epsilon \int_{-\infty}^{\infty} f\,dx. \tag{6}$$

It will be shown that $d(t;\epsilon)$ grows no faster than a constant multiple of $\sqrt{\beta t}$.

COROLLARY 3. There is a constant A, depending only upon ϵ, such that

$$\limsup_{t\to\infty} \frac{d(t;\epsilon)}{\sqrt{\beta t}} \le A. \tag{7}$$

Lastly, let

$$\|u\|_\infty(t) = \sup\{u(x,t) : -\infty < x < \infty\}$$

be the maximum density at time t, let $0 < \epsilon < 1$, and let $\Sigma(t;\epsilon)$ be the set

$$\{x \colon u(x,t) \ge \epsilon \|u\|_\infty(t)\},$$

which may be regarded as the densest part of the mass distribution. We shall establish that $m(\Sigma)$, the Lebesgue measure of the set Σ, grows no faster than a constant multiple of $\sqrt{\beta t}$.

COROLLARY 4. There is a constant B, depending only upon ϵ, such that

$$\limsup_{t\to\infty} \frac{m(\Sigma(t;\epsilon))}{\sqrt{\beta t}} \le B. \tag{8}$$

Our proofs yield the inequalities (7) and (8) with

$$A = \frac{1}{\sqrt{1-\epsilon}}, \qquad B = \frac{\sqrt{27}}{\epsilon},$$

but almost certainly these values are not the best possible.

It is a standard technique for the treatment of integrodifferential equations to take moments of the equations concerned and consider suitably *truncated* sets of the resulting equations. For example, tensor moments of the hydrodynamical equations yield the so-called virial equations, [15], and for the Fokker-Planck equations a study of this type has been carried out in [16]. In the context of a solute in a fluid flowing through a tube the method of moments has been used in a paper by Aris [17], and the search for finite dimensional approximations of this problem is still a current topic of research [18]. To the best of our knowledge there are no studies in which qualitative exact behaviour of the moments for equations with space-dependent coefficients has been determined. In a recent paper on the heat equation with variable coefficients, Toscani [19] has found bounds on the time evolution of the second moment of the solution, but only for special classes of initial data. For related arguments and a bibliography of qualitative results for partial differential equations the reader is referred to [20].

2. The total mass and the centre of mass. The proofs of Theorems 1 and 2 depend upon studying the evolution of the moments (2).

Let $\phi(x,t)$ be any function that is suitably differentiable but, for the present, is otherwise arbitrary. On multiplying the Fokker-Planck equation through by ϕ and rearranging the resulting equation, we arrive at the identity

$$\frac{\partial}{\partial t}(\phi u) = \left[\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial \phi}{\partial x}\right)\right] u + \frac{\partial}{\partial x}\left[-a\phi u + b\left(\phi\frac{\partial u}{\partial x} - \frac{\partial \phi}{\partial x}u\right)\right].$$

Next, we integrate with respect to x and suppose u and $\partial u/\partial x$ to be such that, for each $t > 0$,

$$-a\phi u + b\left(\phi\frac{\partial u}{\partial x} - \frac{\partial \phi}{\partial x}u\right) \to 0 \quad \text{as } |x| \to \infty,$$

and in this way we arrive at the identity

$$\frac{d}{dt}\int_{-\infty}^{\infty} \phi u \, dx = \int_{-\infty}^{\infty}\left[\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial \phi}{\partial x}\right)\right] u \, dx.$$

Thus the integral

$$\int_{-\infty}^{\infty} \phi u \, dx$$

is a conserved quantity whenever ϕ is a solution of the adjoint equation

$$\frac{\partial \phi}{\partial t} + a(x)\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\left(b(x)\frac{\partial \phi}{\partial x}\right) = 0. \tag{9}$$

It should be noted that, by means of a closely related argument, Steinberg and Wolf [21] have determined the evolution with time of all the moments for the heat equation and for the particular Fokker-Planck equation with $a(x) = -x$ and $b(x) = 1$. Pucci and Saccomandi [22] have determined all the conservation laws of the first degree for the Fokker-Planck equation by a direct method.

The choice $\phi = 1$ in the adjoint equation (9) leads immediately to the conclusion that the total mass is conserved, as Theorem 1 asserts.

In order to study the behaviour of the first moment and, hence, the motion of the centre of mass, we ask if the adjoint equation has a solution of the form

$$\phi = x - \alpha t - \theta(x), \tag{10}$$

where α is a constant and $\theta(x)$ has period p. On substituting from (10) into (9) we arrive at the differential equation

$$a(x)\theta' + (b(x)\theta')' = a(x) + b'(x) - \alpha \tag{11}$$

and thus we are led to pose the problem: *determine the constant* α, *if any, such that the differential equation* (11) *has a solution* $\theta(x)$ *with period* p.

It will be shown presently that this problem has exactly one solution α; $\theta(x)$ is determined only to within an arbitrary additive constant, but it can be rendered unique by imposing the additional constraint

$$\int_0^p \theta(x)\,dx = 0.$$

If, temporarily, we take the existence of α and $\theta(x)$ for granted, then, since (10) is a solution of the adjoint equation

$$\int_{-\infty}^{\infty} (x - \alpha t - \theta) u\,dx = \int_{-\infty}^{\infty} (x - \theta) f\,dx,$$

and, hence,

$$\int_{-\infty}^{\infty} u\,dx = \alpha t \int_{-\infty}^{\infty} f\,dx + \int_{-\infty}^{\infty} x f\,dx - \int_{-\infty}^{\infty} \theta f\,dx + \int_{-\infty}^{\infty} \theta u\,dx.$$

Thus the centre of mass satisfies

$$\overline{x}(t) = \alpha t + \overline{x}(0) + \xi(t), \tag{12}$$

where

$$\xi(t) = \frac{[-\int_{-\infty}^{\infty} \theta f\,dx + \int_{-\infty}^{\infty} \theta u\,dx]}{\int_{-\infty}^{\infty} f\,dx}.$$

Since u and f are nonnegative, we have

$$|\xi(t)| \le M \frac{[\int_{-\infty}^{\infty} f\,dx + \int_{-\infty}^{\infty} u\,dx]}{\int_{-\infty}^{\infty} f\,dx} = 2M,$$

where $M = \max_{0 \le x \le p} |\theta(x)|$, and we conclude from (12) that the order relation (4) is correct.

The proof of the existence of α and $\theta(x)$ involves an appeal to the following lemma.

LEMMA 5. *If $a(x)$ and $b(x)$ satisfy hypothesis (i), and if $c(x)$ is continuous and has period p, then there is a unique constant λ such that the differential equation*

$$a(x)y + (b(x)y)' = c(x) - \lambda \tag{13}$$

has a solution $y(x)$ which has period p and satisfies the constraint

$$\int_0^p y\,dx = 0. \tag{14}$$

The constant λ is equal to the quotient

$$\frac{\int_0^p \int_x^p \frac{c(\sigma)k(\sigma)}{k(x)b(x)} d\sigma\,dx + k(p)\int_0^p \int_0^x \frac{c(\sigma)k(\sigma)}{k(x)b(x)} d\sigma\,dx}{\int_0^p \int_x^p \frac{k(\sigma)}{k(x)b(x)} d\sigma\,dx + k(p)\int_0^p \int_0^x \frac{k(\sigma)}{k(x)b(x)} d\sigma\,dx}, \tag{15}$$

where

$$k(x) = \exp\left[\int_0^x \frac{a(\sigma)}{b(\sigma)} d\sigma\right]. \tag{16}$$

In order to verify this we note that $k(x+p) = k(x)k(p)$, and that $k(x)$ is an integrating factor for Eq. (13). Thus

$$y(x) = \frac{1}{k(x)b(x)} \int_0^x [c(\sigma) - \lambda]k(\sigma)d\sigma + \frac{\mu}{k(x)b(x)}, \tag{17}$$

where μ is a constant. There are unique choices of λ and μ which ensure that $y(x)$ has period p and (14) holds. To see this we argue that

$$\begin{aligned} \int_0^{x+p} [c(\sigma) - \lambda]k(\sigma)d\sigma &= \int_0^p [c(\sigma) - \lambda]k(\sigma)d\sigma + \int_p^{x+p} [c(\sigma) - \lambda]k(\sigma)d\sigma \\ &= \int_0^p [c(\sigma) - \lambda]k(\sigma)d\sigma + \int_0^x [c(\sigma+p) - \lambda]k(\sigma+p)d\sigma \\ &= \int_0^p [c(\sigma) - \lambda]k(\sigma)d\sigma + k(p)\int_0^x [c(\sigma) - \lambda]k(\sigma)d\sigma, \end{aligned}$$

and hence that

$$y(x+p) - y(x) = \frac{1}{k(x)k(p)b(x)} \left[\int_0^p [c(\sigma) - \lambda]k(\sigma)d\sigma + \mu(1 - k(p)) \right].$$

Thus $y(x)$ has period p if and only if the sum enclosed within the square brackets vanishes, that is to say, if and only if

$$\lambda \int_0^p k(x)\,dx + \mu(k(p) - 1) = \int_0^p c(x)k(x)\,dx. \tag{18}$$

Moreover, it follows from (17) that the integral constraint (14) is satisfied if and only if

$$\lambda \int_0^p \int_0^x \frac{k(\sigma)}{k(x)b(x)}\,d\sigma\,dx - \mu \int_0^p \frac{dx}{k(x)b(x)} = \int_0^p \int_0^x \frac{c(\sigma)k(\sigma)}{k(x)b(x)}\,d\sigma\,dx. \tag{19}$$

By virtue of (18) and (19)

$$\lambda\Delta = \int_0^p \frac{dx}{k(x)b(x)} \int_0^p c(x)k(x)\,dx + (k(p) - 1)\int_0^p \int_0^x \frac{c(\sigma)k(\sigma)}{k(x)b(x)}\,d\sigma\,dx, \tag{20}$$

and

$$\mu\Delta = \int_0^p \int_0^x \frac{k(\sigma)}{k(x)b(x)}\,d\sigma\,dx \int_0^p c(x)k(x)\,dx - \int_0^p k(x)\,dx \int_0^p \int_0^x \frac{c(\sigma)k(\sigma)}{k(x)b(x)}\,d\sigma\,dx, \tag{21}$$

where the determinant Δ equals

$$\int_0^p \frac{dx}{k(x)b(x)} \int_0^p k(x)\,dx + (k(p) - 1)\int_0^p \int_0^x \frac{k(\sigma)}{k(x)b(x)}\,d\sigma\,dx \tag{22}$$

$$= \int_0^p \int_x^p \frac{k(\sigma)}{k(x)b(x)}\,d\sigma\,dx + k(p)\int_0^p \int_0^x \frac{k(\sigma)}{k(x)b(x)}\,d\sigma\,dx. \tag{23}$$

Hence $\Delta > 0$ and Eqs. (20) and (21) determine λ and μ uniquely, while $y(x)$ is uniquely determined by Eq. (17).

Finally, a simple rearrangement reveals that the right-hand side of Eq. (20) agrees with the numerator of the quotient (15). Thus λ is equal to the quotient, and the proof of the lemma is complete.

In order to solve the problem associated with the differential equation (11) we appeal to the lemma, with $c(x) = a(x) + b'(x)$ and $\alpha = \lambda$, and define $\theta(x)$ uniquely by the conditions

$$\theta'(x) = y(x) \quad \text{and} \quad \int_0^p \theta(x)\,dx = 0.$$

Then

$$[\theta(x+p) - \theta(x)]' = \theta'(x+p) - \theta'(x) = y(x+p) - y(x) = 0,$$

and so

$$\theta(x+p) - \theta(x) = \theta(p) - \theta(0) = \int_0^p y(\sigma)\,d\sigma = 0.$$

Thus $\theta(x)$ has period p, as required, and we have proved Theorem 1.

The proofs yield an explicit formula for the constant α, viz.

$$\alpha\Delta = \int_0^p \int_x^p \frac{(a(\sigma) + b'(\sigma))k(\sigma)}{k(x)b(x)}\,d\sigma\,dx + k(p)\int_0^p \int_0^x \frac{(a(\sigma) + b'(\sigma))k(\sigma)}{k(x)b(x)}\,d\sigma\,dx, \tag{24}$$

where Δ is given by (23).

In the important case in which the diffusion coefficient $b(x)$ is constant ($= b_0$, say),

$$b'(x) = 0, \qquad a(x)k(x) = b_0 k'(x),$$

and the right-hand side of (24) reduces to

$$\begin{aligned} &\int_0^p \int_x^p \frac{k'(\sigma)}{k(x)}\,d\sigma\,dx + k(p)\int_0^p \int_0^x \frac{k'(\sigma)}{k(x)}\,d\sigma\,dx \\ &\quad = \int_0^p \frac{k(p) - k(x)}{k(x)}\,dx + k(p)\int_0^p \frac{k(x) - 1}{k(x)}\,dx \\ &\quad = p(k(p) - 1) \\ &\quad = p\left[\exp\left(\frac{1}{b_0}\int_0^p a(x)\,dx\right) - 1\right]. \end{aligned}$$

Hence we have

COROLLARY 6. *If the diffusion coefficient is identically constant, then α is positive, zero, or negative according to whether the mean drift coefficient*

$$\frac{1}{p}\int_0^p a(x)\,dx$$

is positive, zero, or negative.

It might be expected that α would coincide with the mean drift coefficient, but this is not generally so. Thus if $a(x) = 1 + \delta\cos x$, $b(x) = 1$, $p = 2\pi$, then the mean drift coefficient

$$\frac{1}{2\pi}\int_0^{2\pi} a(x)\,dx = 1.$$

However, calculation on the basis of (24) reveals that, as $\delta \to 0$,

$$\alpha = 1 - \tfrac{1}{2}\pi\delta^2 + O(\delta^2).$$

3. The moment of inertia with respect to the centre of mass. We turn to the proof of Theorem 2. Let α and $\theta(x)$ be as in the preceding section, and let us put

$$\phi^* = x - \alpha t - \theta(x) \tag{25}$$

for the solution, already constructed, of the adjoint equation (9). We now ask whether the adjoint equation also has a solution of the form

$$\phi = (\phi^*)^2 - \beta t - \psi(x), \tag{26}$$

where the constant β and the periodic function $\psi(x)$ are to be determined. Then,

$$\begin{aligned}
&\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial \phi}{\partial x}\right) \\
&\quad = 2\phi^*\left[\frac{\partial \phi^*}{\partial t} + a\frac{\partial \phi^*}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial \phi^*}{\partial x}\right)\right] \\
&\qquad + 2b\left(\frac{\partial \phi^*}{\partial x}\right)^2 - \beta - a\psi' - (b\psi')' \\
&\quad = 2b(1-\theta')^2 - \beta - a\psi' - (b\psi')'.
\end{aligned}$$

Thus, ϕ is a solution of the adjoint equation if and only if ψ is a solution of the differential equation

$$a(x)\psi' + (b(x)\psi')' = 2b(x)(1-\theta'(x))^2 - \beta, \tag{27}$$

and, as before, we are led to pose the problem: *determine the constant* β, *if any, such that the differential equation* (27) *has a solution* $\psi(x)$ *that is periodic with period* p.

The lemma, with

$$c(x) = 2b(x)(1-\theta'(x))^2, \tag{28}$$

tells us immediately that the problem has exactly one solution β; $\psi(x)$ is determined only to within an arbitrary additive constant, but it can be rendered unique by imposing the additional condition

$$\int_0^p \psi\, dx = 0.$$

The value of β can be read off from Eq. (15) on substituting for $c(x)$ from (28), and it is clear that $\beta > 0$, as asserted.

In order to derive the asymptotic relation (5) we argue that, since ϕ is a solution of the adjoint equation,

$$\int_{-\infty}^{\infty} [(x - \alpha t - \theta(x))^2 - \beta t - \psi(x)]u\, dx = \int_{-\infty}^{\infty} [(x-\theta(x))^2 - \psi(x)]f\, dx.$$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} (x - \alpha t - \theta(x))^2 u\, dx = \beta t \int_{-\infty}^{\infty} f\, dx + \int_{-\infty}^{\infty} \psi u\, dx \\
+ \int_{-\infty}^{\infty} [(x-\theta(x))^2 - \psi(x)]f\, dx.
\end{aligned}$$

On setting $M = \max_{0\le x\le p} |\psi(x)|$, and remembering that u is nonnegative, we see that

$$\left|\int_{-\infty}^{\infty} \psi u\, dx\right| \le M \int_{-\infty}^{\infty} u\, dx = M \int_{-\infty}^{\infty} f\, dx,$$

and it follows that

$$\int_{-\infty}^{\infty} (x - \alpha t - \theta(x))^2 u\, dx = \beta t \int_{-\infty}^{\infty} f\, dx + O(1). \tag{29}$$

This is not quite what is required, but, according to Eq. (12),

$$x - \alpha t - \theta(x) = x - \overline{x}(t) + \omega(x,t),$$

where $\omega = \overline{x}(0) + \xi(t) - \theta(x)$ and $\omega(x,t)$ is bounded for $-\infty < x < \infty$ and $t \ge 0$. Thus, if we choose a constant γ such that $|\omega(x,t)| \le \gamma$ and put

$$I(t) = \int_{-\infty}^{\infty} (x - \overline{x}(t))^2 u\, dx,$$

for the moment of inertia, we have

$$\begin{aligned}\int_{-\infty}^{\infty} (x - \alpha t - \theta)^2 u\, dx &= \int_{-\infty}^{\infty} (x - \overline{x} + \omega)^2 u\, dx \\ &= I + 2\int_{-\infty}^{\infty} (x - \overline{x})\omega u\, dx + \int_{-\infty}^{\infty} \omega^2 u\, dx.\end{aligned} \tag{30}$$

Now let δ be any number in $0 < \delta < 1$. In view of the elementary inequalities

$$2|(x - \overline{x})\omega u| \le 2\gamma|x - \overline{x}|u \le \delta(x - \overline{x})^2 u + \frac{\gamma^2}{\delta} u$$

we have

$$2\left|\int_{-\infty}^{\infty} (x - \overline{x})^2 \omega u\, dx\right| \le \delta I + \frac{\gamma^2}{\delta} \int_{-\infty}^{\infty} f\, dx, \tag{31}$$

and it follows from (30) and (31) that

$$\int_{-\infty}^{\infty} (x - \alpha t - \theta)^2 u\, dx \le (1+\delta) I + \gamma^2 \left(\frac{1}{\delta} + 1\right) \int_{-\infty}^{\infty} f\, dx$$

and that

$$\int_{-\infty}^{\infty} (x - \alpha t - \theta)^2 u\, dx \ge (1-\delta) I - \frac{\gamma^2}{\delta} \int_{-\infty}^{\infty} f\, dx.$$

These last two inequalities and the order relation (29) imply that

$$\frac{1}{1+\delta}\int_{-\infty}^{\infty} f\, dx \le \liminf_{t\to\infty} \frac{I}{\beta t} \le \limsup_{t\to\infty} \frac{I}{\beta t} \le \frac{1}{1-\delta}\int_{-\infty}^{\infty} f\, dx$$

for every δ in $0 < \delta < 1$ and, finally, that

$$\lim_{t\to\infty} \frac{I}{\beta t} = \int_{-\infty}^{\infty} f\, dx,$$

which is (5), and, hence, the proof of Theorem 2 is complete.

4. A comparison with the heat equation. The constant β depends upon the coefficients $a(x)$ and $b(x)$ in a complicated way. There is an important case, however, in which β is given by a comparatively simple formula (32) from which we can deduce the following corollary.

COROLLARY 7. Suppose that, in addition to hypothesis (i),

$$\int_0^p a(x)\,dx = 0,$$

that $a(x)$ is not identically zero, and that $b(x) = b_0$ for all x, where b_0 is a positive constant. Then $0 < \beta < 2b_0$.

The hypotheses ensure, as we know, that $k(p) = 1$ and $\alpha = 0$. Moreover, calculation reveals that

$$\theta'(x) = 1 - \frac{p}{\int_0^p \frac{d\sigma}{k(\sigma)}} \frac{1}{k(x)},$$

$$\Delta = \frac{1}{b} \int_0^p \frac{dx}{k(x)} \int_0^p k(x)\,dx,$$

and that

$$\beta = \frac{2b_0 p^2}{\int_0^p \frac{dx}{k(x)} \int_0^p k(x)\,dx}, \tag{32}$$

where

$$k(x) = \exp\left(\frac{1}{b_0}\int_0^x a(\sigma)\,d\sigma\right).$$

The desired conclusion now follows from (32) and the Cauchy-Schwarz inequality.

Here we are, in effect, comparing the Fokker-Planck equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a(x)u) = b_0 \frac{\partial^2 u}{\partial x^2}$$

with the heat equation

$$\frac{\partial u}{\partial t} = b_0 \frac{\partial^2 u}{\partial x^2},$$

for which $\beta = 2b_0$. Thus, by comparison with the heat equation, the effect of a periodic drift coefficient, with mean value 0, is to reduce the rate of growth of the moment of inertia with respect to the centre of mass.

5. Corollaries 3 and 4. In order to prove Corollary 3 we return to Eq. (3), and to Eq. (6), which defines the distance $d(t;\epsilon)$, and argue that

$$\int_{-\infty}^{\infty} (x-\overline{x})^2 u\,dx \geq \int_{|x-\overline{x}|>d} (x-\overline{x})^2 u\,dx \geq d^2 \int_{|x-\overline{x}|>d} u\,dx = d^2(1-\epsilon)\int_{-\infty}^{\infty} f\,dx.$$

Hence

$$\frac{d(t;\epsilon)}{\sqrt{\beta t}} \leq \frac{1}{\sqrt{1-\epsilon}} \left(\frac{\int_{-\infty}^{\infty}(x-\overline{x})^2 u\,dx}{\beta t \int_{-\infty}^{\infty} f\,dx}\right)^{1/2};$$

and on letting $t \to \infty$ and appealing to Theorem 2, we see that the inequality (7) holds, with $A = 1/\sqrt{1-\epsilon}$, and, therefore, we have established Corollary 3.

To prove Corollary 4 let r be any positive number to be chosen presently. Then

$$\int_{|x-\overline{x}|\le r} u\,dx \le 2\|u\|_\infty r,$$

and

$$\int_{|x-\overline{x}|>r} u\,dx \le \frac{1}{r^2}\int_{|x-\overline{x}|>r} (x-\overline{x})^2 u\,dx \le \frac{1}{r^2}\int_{-\infty}^{\infty} (x-\overline{x})^2 u\,dx = \frac{1}{r^2} I,$$

and, hence,

$$\int_{-\infty}^{\infty} u\,dx \le 2\|u\|_\infty r + \frac{1}{r^2} I.$$

The choice $r = (I/\|u\|_\infty)^{1/3}$ minimizes the right-hand side and leads to the inequality

$$\int_{-\infty}^{\infty} u\,dx \le 3(\|u\|_\infty)^{2/3} I^{1/3}. \tag{33}$$

On the other hand, in view of (3) we can argue that

$$\begin{aligned}\int_{-\infty}^{\infty} u\,dx &= \left(\int_{-\infty}^{\infty} u\,dx\right)^{2/3}\left(\int_{-\infty}^{\infty} f\,dx\right)^{1/3}\\ &\ge \left(\int_{\Sigma} u\,dx\right)^{2/3}\left(\int_{-\infty}^{\infty} f\,dx\right)^{1/3}\\ &\ge (\epsilon\|u\|_\infty m(\Sigma))^{2/3}\left(\int_{-\infty}^{\infty} f\,dx\right)^{1/3},\end{aligned} \tag{34}$$

and, on combining (33) with (34), we see that

$$(m(\Sigma))^2 \le \frac{27}{\epsilon^2}\frac{I}{\int_{-\infty}^{\infty} f\,dx}.$$

Thus the inequality (8) holds, with $B = \sqrt{27}/\epsilon$, and the proof of Corollary 4 is complete.

Acknowledgment. The support of the Gruppo Nazionale per la Fisica Matematica of the Italian INDAM is gratefully acknowledged.

References

[1] M. E. Taylor, *Partial Differential Equations II*, Applied Math. Sci. **116**, Springer-Verlag, Berlin, 1996

[2] T. Miyazawa, *Theory of one-variable Fokker-Planck equation*, Physical Review A **39**, 1447–1468 (1989)

[3] H. Risken, *The Fokker-Planck Equation*, Springer-Verlag, Berlin, 1984

[4] C. C. Mei, J.-L. Auriault, and Chiu-on Ng, *Some applications of homogenization theory*, Advances in Applied Mechanics **32**, 278–345 (1996)

[5] G. de Marsily, *Quantitative Hydrology*, Academic Press, New York, 1986

[6] A. J. Majda, *Vorticity, turbulence, and acoustics in fluid flow*, SIAM Review **33**, 349–388 (1991)

[7] R. Racke, *Lectures on Nonlinear Evolution Equations: Initial Value Problems*, Vieweg u. Sohn, Braunschweig u. Wiesbaden, 1992

[8] C. Cattaneo, *Sulla conduzione del calore*, Atti del Seminario Matematico Univ. di Modena **3**, 83–101 (1948–49)

[9] M. E. Gurtin and A. C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31**, 113–126 (1968)

[10] D. D. Joseph and L. Preziosi, *Heat waves*, Rev. Modern Physics **61**, 41–73 (1989)

[11] G. Fichera, *Is the Fourier theory of heat propagation paradoxical?*, Rend. del Circolo Matematico di Palermo **41**, 5–28 (1992)

[12] W. A. Day, *On rates of propagation of heat according to Fourier's theory*, Quart. Appl. Math. **55**, 127–138 (1997)

[13] W. A. Day, *A note on the propagation of temperature disturbances*, Quart. Appl. Math. **55**, 565–572 (1997)

[14] J. C. Maxwell, *Theory of Heat*, 2nd edition, Longmans, Green and Co., London, 1872

[15] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*, Yale University Press, New Haven, 1969

[16] R. M. Wilcox and R. Bellman, *Truncation and preservation of moment properties for Fokker-Planck moment equations*, J. Math. Anal. Appl. **32**, 532–542 (1970)

[17] R. Aris, *On the dispersion of a solute in a fluid flowing through a tube*, Proc. Roy. Soc. London Ser. A **235**, 67–77 (1956)

[18] S. D. Watt and A. J. Roberts, *The accurate dynamic modelling of contaminant dispersion in channels*, SIAM J. Appl. Math. **55**, 1016–1038 (1995)

[19] G. Toscani, *Kinetic approach to the asymptotic behaviour of the solution to diffusion equations*, Rend. di Matematica e Applicazioni Serie VII **16**, 329–346 (1996)

[20] J. Flavin and S. Rionero, *Qualitative Estimates for Partial Differential Equations*, CRC Press, Boca Raton, FL, 1995

[21] S. Steinberg and K. B. Wolf, *Symmetry, conserved quantities and moments in diffusive equations*, J. Math. Anal. and Applications **80**, 36–45 (1981)

[22] E. Pucci and G. Saccomandi, *Potential symmetries of Fokker-Planck equations* in Modern Group Analysis, N. H. Ibragiminov, M. Torrisi and A. Valenti (Eds.), Kluwer Academic Publisher, Dordrecht, 1993