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MATCHED ASYMPTOTIC EXPANSION CALCULATION OF THE EQUILIBRIUM SHAPE OF A HOLE IN A THIN LIQUID FILM

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Abstract. We consider the occurrence of small axisymmetric pinholes in an otherwise uniform infinite thin liquid film. Corresponding to any particular undisturbed film thickness there exists precisely one (unstable) equilibrium solution reflecting a balance between surface tension and gravity effects. If a pinhole is smaller than this critical size the pinhole tends to close over and "heal". If a pinhole is larger it tends to open out. So determination of this critical hole size is crucial. We examine this problem in the case of a "small" pinhole where the fundamental length-scale in the film is much smaller than the capillary length. Solutions are obtained using matched asymptotic expansions for which several different scalings are necessary.

1. Introduction. In many industrial processes, liquid coatings are applied to different types of substrate. Such coatings are usually thin, typical examples being coatings on TV screens [6]. The generic film coating process consists essentially of an initially dry substrate that is wetted with the coating liquid. The liquid is then partially removed from the substrate (e.g., by drainage under gravity or by spinning the substrate) and when the optimal coating thickness has been reached, the coating is dried off and the process is complete. A standard criterion for such coatings is that they should (usually) be uniform. For example, consider the case of the application of anti-reflective coating to a television screen. If the coating thickness is not uniform everywhere, light incident on the screen is differentially absorbed/reflected and the picture may appear distorted. This non-uniformity in the coating film thickness must be avoided at all costs. Even in the case when the coating has been applied successfully (and uniformly) it is possible for defects to spontaneously evolve, for example in the case of pinholing.

Basically a very thin film (typically of the order of microns) is very sensitive to outside disturbances and holes may form in the film. There are many ways in which this can occur; an instructive example occurs when the film has an alcohol/water base. Small

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fluctuations in the (local) alcohol concentration may occur (due to differential evaporation) and this can give rise to surface tension gradients and Marangoni-type effects. If these give rise to an outgoing radial type flow, a hole may start to open (and this will be further exacerbated by the effects of disjoining pressure (van der Waals forces) which ultimately may lead to complete film break-up and hole formation).

On the assumption then that holes will form, the question of the stability (or ephemerality) of these holes is critical. If all holes spontaneously close there is no real problem and the issue is of no importance. However, experience shows that this is not the case and an analysis of the stability of pinholes is necessary [12]. In this seminal paper, it was shown that for the case of an infinite expanse of liquid, corresponding to each undisturbed film thickness there exists one critical static equilibrium solution that is unstable and some numerical solutions are calculated. If a hole occurs in the film that is larger than that corresponding to the critical solution it tends to open out; if a hole opens that is smaller than the critical size, it tends to close over the "heal" (obviously desirable from the coating point of view). The present paper calculates an asymptotic expansion for this unstable equilibrium solution. (The problem has already been solved numerically in [12].

The dynamics of the situation has recently been examined by [8] but only in the case of small contact angle/slope when the lubrication approximation was employed. Some of the results suggest that static criteria are not always completely successful in predicting the stability of the films but nevertheless give a good idea of the basic trends and thresholds. The obvious next step in the modelling of these processes is to carry out a full Stokes flow analysis for the case of arbitrary contact angles and surface slopes. A suitable algorithm for such flows has recently been developed and it is hoped to use it to initiate such a study [11].

We will consider (partly from the mathematical point of view) the following problem: given an infinite expanse of liquid in which a hole of "small" radius has opened up (assumed axisymmetric for simplicity), what is the size of such a hole and, in particular, what is the mathematical structure of the solution describing such a hole? As we shall see the basic equation describing the shape of such holes is the well-known Laplace capillary equation and problems of this type were first considered by Laplace [7]. The equation is, in general, not amenable to analytic solution. Thus our approach will be asymptotic based on the presence of a suitable (realistic) small parameter and we will demonstrate the existence and structure of several different balances. Mathematically what follows is apparently the first attempt to obtain a solution to this problem using formal matched asymptotics and is a natural extension of the problem of finding the meniscus shape near a small cylinder [4]. In the present paper we will concentrate on the problem of determining the asymptotic structure of the solutions and we will demonstrate the success of van Dyke matching.

2. Problem formulation. The basic problem is represented schematically in Fig. 1 and shows an infinite film configuration after a pinhole has opened up. The essential force balance is between gravity and surface tension and gives rise to the well-known

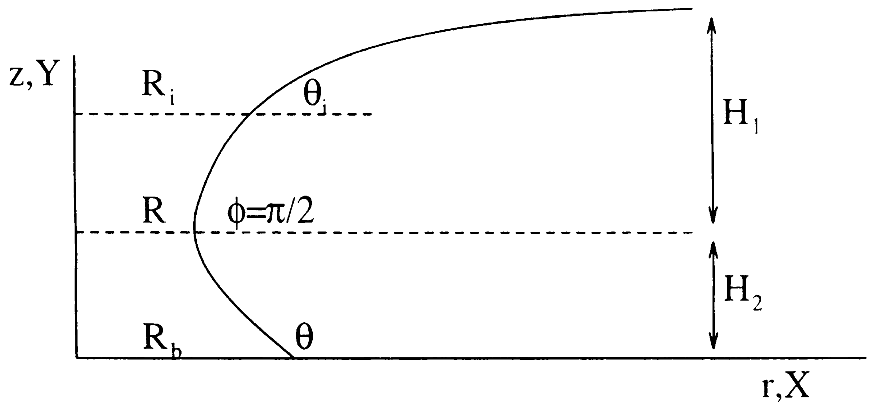

FIG. 1. Schematic of a pinhole and coordinate system used

equation (in cylindrical polars):

$$\sigma\left(\frac{z''}{(1+z'^2)^{3/2}}+\frac{z'}{r(1+z'^2)^{1/2}}\right)\pm\rho g z=0 \tag{1}$$

where use of the + or − depends on whether the point under consideration is below or above the position where the drop profile becomes vertical.

Following previous work on capillary problems ([2], [4], [9], [10]), we will first non-dimensionalise (2) using the capillary length a as reference length ($a=\sigma/(\rho g)^{1/2}$) (which for water is about 2.7mm), where σ, ρ, g are the surface tension, liquid density, and gravitational acceleration, respectively. Thus, writing

$$z=aY, \qquad r=aX, \tag{2}$$

(1) becomes

$$\frac{Y''}{(1+Y'^2)^{3/2}}+\frac{Y'}{X(1+Y'^2)^{1/2}}\pm Y=0. \tag{3}$$

Equation (3) is a second under ODE and requires two boundary conditions. We pose these in the following way:

$$\text{as } X\to\infty, Y, Y', Y''\to 0; \qquad \text{as } X\to\varepsilon, Y'\to\pm\infty \tag{4}$$

where the $\pm$ sign depends on whether we are considering the upper or lower branch and $\varepsilon=R/a$ is the dimensionless half-width of the profile at its narrowest point (and also a Bond number). The parameter ε is assumed to be small; this physically corresponds to the radius R in Fig. 1 being less than the capillary length. In [13] the problem was formulated in terms of the film thickness, R being unknown.

Equation (3) should be considered as an arbitrary non-dimensionalisation to get the problem into a simpler form, but does not reflect, at this stage, any particular balances in the equation. In the course of the paper, various new scalings will occur, but it should be borne in mind that the fundamental governing equation is (3) and the corresponding fundamental variables are (X, Y).

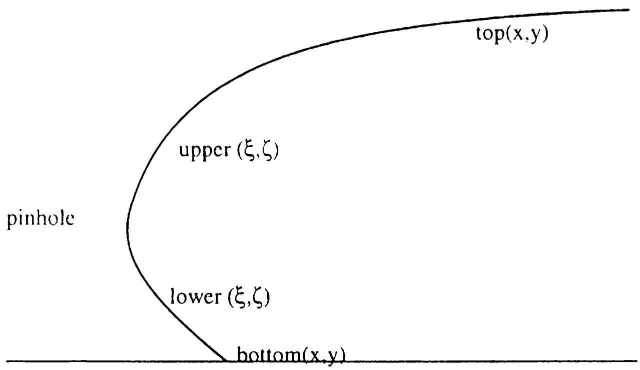

FIG. 2. Constituent asymptotic solutions and corresponding variables

We are interested in developing the general case where the contact angle θ is anywhere in the interval $0 \leq \theta \leq \pi$. In fact, we can consider the particular situation where $\theta = \pi$. If the actual contact angle is less than π, the profile can be truncated when the inclination reaches the appropriate value. We note that (3) implicitly contains the condition that $Y(X = \infty) = 0$ and so this fixes the origin of our coordinate system. Our solution technique will thus consist of starting the integration on the upper branch, integrating in towards $X = \varepsilon$ and then back out along the lower branch. We will actually obtain a solution for the top part of the profile first before proceeding to the lower half.

We will consider the problem for the realistic situation when $\varepsilon \ll 1$ corresponding in a water film to a hole of radius less than 2.7mm. We will seek a parameter perturbation solution based on ε. Before proceeding we will sketch the basic asymptotic structure of the solution: we will find that the solution consists of an outer solution (along the top branch) running into an inner solution near $X = \varepsilon$. On continuing the solution beyond this point we find another inner solution that runs into yet another outer solution as the contact angle approaches π and the inclination again approaches the horizontal. The four basic constituent solutions will be referred to as the top (small slope capillary-gravity), upper (zero mean curvature), lower (zero mean curvature) and bottom (small slope capillary-gravity) solutions (as given in Fig. 2), this structure being reminiscent in many ways of the multi-layer structure occurring in the profiles of small liquid drops as in [9] and [10]. Finally, we comment that the upper and lower solutions are merely mirror images (at least to leading order) of one another while the top and bottom solutions are quite different in character. This latter fact is consistent with the physical expectation that (for the case of $\theta = \pi$) the profile should become horizontal for some finite value of X.

3. Zero-order solutions.

3.1. *The top and upper solutions.* The top solution is obtained by making the (well-known) assumption of small slope and rescaling as follows:

$$x = X, \quad y = \frac{1}{\varepsilon}Y. \tag{5}$$

The scaled "outer" equation is then

$$\frac{y_0''}{(1+\varepsilon^2 y'^2)^{3/2}} + \frac{y_0'}{x(1+\varepsilon^2 y'^2)^{1/2}} - y = 0. \tag{6}$$

To leading order we obtain

$$y_0'' + \frac{y_0'}{x} - y_0 = 0 \tag{7}$$

with solution $y_0 = AK_0(x) + BI_0(x)$ where the constant A will be determined from matching with the upper (inner) solution and $B = 0$ to satisfy boundary condition (4).

The upper solution is obtained by rescaling as follows:

$$X = \varepsilon\xi, \qquad Y = \varepsilon\zeta \tag{8}$$

(which essentially brings the curvature terms into a balance in a thin boundary layer region while neglecting gravity effects to leading order). The upper equation is thus

$$\zeta'' + \frac{\zeta'}{\xi}(1+\zeta'^2) \pm \varepsilon^2(1+\zeta'^2)^{3/2} = 0. \tag{9}$$

Writing

$$\zeta = \zeta_0 + \varepsilon^2\zeta_1 \tag{10}$$

gives the following "upper" equation to leading order:

$$\zeta_0'' + \frac{\zeta_0'}{\xi}(1+\zeta_0'^2) = 0 \tag{11}$$

with first integral $d\zeta_0/d\xi = \pm C/\sqrt{\xi^2 - C^2}$ which integrates to

$$\zeta_0 = \ln(\xi + \sqrt{\xi^2 - 1}) + B \tag{12}$$

where the boundary condition at $\xi = 1$ has been satisfied by choosing the positive root and the constant B will be obtained from matching with the top (outer) solution.

3.2. *Matching of top and upper solutions.* We will use Van Dyke matching which in the present instance can be paraphrased as "The to $O(\varepsilon)$ inner expansion of the to $O(\varepsilon)$ outer expansion equals the to $O(\varepsilon)$ outer expansion of the to $O(\varepsilon)$ inner expansion". (See [1].) Recalling that the outer variable is X and the inner variable is $\xi = \frac{X}{\varepsilon}$ we find that the relevant expansion of the outer solution in terms of the dimensionless variables (X, Y) is

$$\varepsilon \ln X - \varepsilon \ln 2 + \varepsilon\gamma \tag{13}$$

where γ_c is Euler's constant. The relevant expansion of the inner solution is

$$\varepsilon B + \varepsilon \ln 2 + \varepsilon \ln X - \varepsilon \ln \varepsilon. \tag{14}$$

In (14) we notice the occurrence of terms of the form $\varepsilon \ln \varepsilon$ that have no counterpart in (13), indicating that the matching is not working. This is a well-known phenomenon indicating that there are terms missing called "switchbacks" (see [3]). Essentially the form of the upper series is incorrect and instead of looking for a solution in the form

$\zeta = \zeta_0 + \varepsilon^2\zeta_1$, we should seek $\zeta = \ln\varepsilon\zeta_{sw1} + \zeta_0 + \varepsilon^2\zeta_1$. Substitution of this revised series into the inner equation yields the result

$$\frac{d\zeta_{sw1}}{d\xi} = 0 \rightarrow \zeta_{sw1} = \text{constant} \tag{15}$$

and so we are free to add a term of the form constant $\varepsilon\ln\varepsilon$ to facilitate the matching. This yields the following (fully-determined) upper and top solutions:

$$\zeta = \ln(\xi + \sqrt{\xi^2 - 1}) - 2\ln 2 + \gamma + \ln\varepsilon, \tag{16}$$

$$y = -K_0(X). \tag{17}$$

We note that in the case where the contact angle $\theta < 90°$, (16) and (17) take the form:

$$\zeta = \sin\theta(\ln(\xi + \sqrt{\xi^2 - \sin\theta^2}) - 2\ln 2 + \gamma + \ln\varepsilon), \tag{18}$$

$$y = -\cos\theta K_0(X). \tag{19}$$

3.3. *The bottom and lower solution.* The development for the bottom and lower solutions is quite similar to that for the top and upper solutions in that the basic balances (and hence scales) are the same. However, referring to Fig. 1 we note that the governing equation for the lower part of the meniscus must incorporate the + sign of Eq. (3) rather than the − sign once the profile has passed through the vertical (singular point). To see this we note that the curvatures for the axisymmetric profile in (3) are in fact given by

$$\kappa_1 = \left|\frac{Y''}{(1+Y'^2)^{3/2}}\right| \quad \text{and} \quad \kappa_2 = \left|\frac{Y'}{X(1+Y'^2)^{1/2}}\right|$$

where the sign is chosen depending on whether the profile is concave or convex. One of the simplest rules for determining the sign is to choose a positive curvature if the centre curvature is located in the liquid and a negative curvature otherwise. Obviously κ_1 should have the same sign for all points on the profile but y'' changes sign at the (singular) point when the profile becomes vertical. A similar argument holds for the second curvature κ_2. The governing equation for this part of the meniscus is thus

$$\frac{Y''}{(1+Y'^2)^{\frac{3}{2}}} + \frac{Y'}{X(1+Y'^2)^{\frac{1}{2}}} + Y = 0. \tag{20}$$

To avoid cumbersome notation we will use the same notation for the lower and bottom solution as was used for the top/upper solutions, i.e., (ζ, ξ) and (y, x), respectively, and we will demand continuity at the crossover point where $\xi = 1$.

Using the rescaling (8), and seeking ζ in the form (10) the lower equation is easily seen to be

$$\frac{\zeta''}{(1+\zeta'^2)^{\frac{3}{2}}} + \frac{\zeta'}{\xi(1+\zeta'^2)^{\frac{1}{2}}} + O(\varepsilon^2) = 0 \tag{21}$$

with solution to leading order

$$\zeta_0 = -\ln(\xi + \sqrt{\xi^2 - 1}) + D. \tag{22}$$

Since $\zeta_0' = -\frac{1}{\sqrt{\xi^2-1}}$ this satisfies the condition that $\zeta'(\xi = 1) \rightarrow -\infty$ as it should do. The integration constant in (22) will depend only on the continuity of the lower and upper solutions (the upper and lower solutions should agree at $\xi = 1$). We note that following our experience with the upper solution we predict the re-occurrence of switchbacks in the lower solution. This will be borne out (and resolved) during the matching.

The bottom equation with the rescaling $x \equiv X$, $y = Y/\varepsilon$ is easily found to be:

$$\frac{y''}{(1+\varepsilon^2 y'^2)^{\frac{3}{2}}} + \frac{y'}{x(1+\varepsilon^2 y'^2)^{\frac{1}{2}}} + y = 0. \tag{23}$$

Seeking a solution in the form $y = y_0 + \varepsilon^2 y_1$ we find that to leading order this reduces to the ordinary Bessel equation of zero order, viz $y_0'' + \frac{y_0'}{x} + y_0 = 0$ with solution

$$y_0 = EJ_0(x) + FY_0(x). \tag{24}$$

In this instance we have no boundary conditions for E and F and their values must be determined by asymptotic matching. As we previously observed, the bottom solution can be expected to intersect the substrate at some finite value of x and so we expect that the profile should become horizontal for some finite x. This is a direct consequence of the change in sign in (20) leading to oscillating solutions (24) rather than the exponential modified Bessel functions of (17).

3.4. *Matching of lower and bottom solutions.* The lower part of the solution is the mirror image of the upper solution (at least to leading order). We clearly require continuity in the profile at the switchover point where $\xi = 1$. At this point (16) shows that the upper branch has the value

$$\zeta_0(\xi = 1) = \ln\varepsilon - 2\ln 2 + \gamma, \tag{25}$$

while the lower branch, according to (22), attains the value

$$\zeta_0(\xi = 1) = D. \tag{26}$$

So for continuity we obtain a contradiction (because of the presence of the $\ln\varepsilon$ term). However, inserting an extra switchback term in the lower solution ($\ln\varepsilon\zeta_{\mathrm{sw}1}$) we find that the equation governing $\zeta_{\mathrm{sw}1}$ reduces to $d\zeta_{\mathrm{sw}1}/d\xi = 0$ and so we are free to add $\ln\varepsilon$ to the lower solution. Continuity now demands that

$$D = -2\ln 2 + \gamma \tag{27}$$

and at this point the lower solution (13) is fully determined and takes the form

$$\zeta = -\ln(\xi + \sqrt{\xi^2 - 1}) - 2\ln 2 + \gamma + \ln\varepsilon. \tag{28}$$

In the case where the contact angle θ is greater than $\pi/2$ but significantly smaller than π, the lower solutions can be written as

$$\zeta = \sin\theta(-\ln\xi + \sqrt{\xi^2 - \sin\theta^2} - 2\ln 2 + \gamma + \ln\varepsilon). \tag{29}$$

We note from (28) that $d\zeta/d\xi = -1/\sqrt{\xi^2 - 1}$ and this can only become zero (i.e., the profile becomes horizontal) when $\xi \to \infty$. Since we expect physically that the profile will become horizontal at some finite value of x, this result emphasizes the need for the introduction of an extra outer solution whose general form was already given in (24).

We now proceed to match the lower and bottom solutions. The lower solutions expanded according to Van Dyke's rule yield

$$-3\ln 2 - \ln x + 2\ln\varepsilon + \gamma \tag{30}$$

while the bottom solution gives

$$E + \frac{2}{\pi} F(\ln x - \ln 2 + \gamma). \tag{31}$$

Clearly the $\ln \varepsilon$ terms in (30) have no counterpart in (31) and the need for switchbacks arises once more. We seek a modified bottom solution in the form

$$y_0 = \ln \varepsilon Y_{\mathrm{sw}}(x) + E J_0(x) + F Y_0(x) \tag{32}$$

and to leading order we find that $Y_{\mathrm{sw}}(x)$ satisfies the Bessel equation of zero order. Hence we seek a bottom solution in the form

$$y_0 = \ln \varepsilon (G J_0(x) + H Y_0(x)) + E I_0(x) + F Y_0(x). \tag{33}$$

Using Van Dyke's rule this has the following inner (lower) expansion:

$$\varepsilon G \ln \varepsilon + \varepsilon \frac{2}{\pi} H \ln \varepsilon (\ln x - \ln 2 + \gamma) + \varepsilon E + \varepsilon F \frac{2}{\pi} (\ln x - \ln 2 + \gamma) \tag{34}$$

and on matching this with (30) we find that

$$H = 0, \tag{35}$$

$$G = 2, \tag{36}$$

$$F = -\frac{\pi}{2}, \tag{37}$$

$$E = -4 \ln 2 + 2\gamma, \tag{38}$$

and we obtain the following final expression for the leading-order bottom solution:

$$y_0 = 2 \ln \varepsilon J_0(x) + J_0(x)(2\gamma - 4 \ln 2) - \frac{\pi}{2} Y_0(x). \tag{39}$$

3.5. *Composite solutions.* It is possible to find composite solutions using the rule: composite = inner + outer − common part as shown in [1]. In the present instance, because the solution branches at the crossover point we obtain two distinct solutions: one for the top/upper branch, the other for the bottom/lower branch.

For the top/upper branch, we combine (16) and (17) and obtain the result (in terms of the fundamental variables (2)):

$$Y_{\mathrm{tu}} = -\varepsilon K_0(X) + \varepsilon \ln \left(\frac{X}{\varepsilon} + \sqrt{\frac{X^2}{\varepsilon^2} - 1} \right) + \varepsilon(-\ln x - \ln 2 + \ln \varepsilon) \tag{40}$$

while for the lower/bottom solution we use (22), (27), (39) and we obtain

$$\begin{aligned} Y_{\mathrm{lb}} = {} & \varepsilon \ln \varepsilon (-1 + 2 J_0(X)) - \varepsilon \ln \left(\frac{X}{\varepsilon} + \sqrt{\frac{X^2}{\varepsilon^2} - 1} \right) \\ & + \varepsilon(\ln 2 + \ln X) + \varepsilon J_0(X)(-4 \ln 2 + 2\gamma) - \varepsilon \frac{\pi}{2} Y_0(X). \end{aligned} \tag{41}$$

3.6. *Discussion of zero-order solutions.* We note first that given R we are immediately in a position to estimate the distance R_b (see Fig. 1) for any value of the contact angle. If $\theta \leq 90°$ we can do this by using the top/upper solutions effectively truncating the graph at the point where the profile reaches the required inclination. However, we have tacitly assumed that $\theta > 90°$ by using R (Fig. 1) as a length scale (since this is an experimentally measurable quantity). This is not necessary and the preceding analysis may also be used for the case where $\theta \leq 90°$ though this will entail the solution of one extra algebraic equation. Suppose, for example, that the actual profile is of the form shown in Fig. 1 cut off at the point where $\theta = \theta_i$. The distance R_i and the corresponding height Z_i can now be measured, the contact angle θ_i is assumed known, and using (40) we obtain the algebraic equation

$$-\varepsilon K_0\left(\frac{R_i}{a}\right) + \varepsilon \ln\left(\frac{R_i}{a\varepsilon} + \sqrt{\frac{R_i^2}{a^2\varepsilon^2} - 1}\right) + \varepsilon\left(-\ln 2 - \ln\left(\frac{R_i}{a}\right) + \ln\varepsilon\right) = \frac{Z_i}{a}, \qquad (42)$$

which can be solved for the value of ε ($\equiv R/a$) corresponding to this particular case whereupon the theory in the present paper is directly applicable.

In the case where $\theta > 90°$, R_b can be estimated as follows. The slope of the lower/bottom curve, obtained by differentiating (41), is

$$\frac{dY_{\mathrm{lb}}}{dX} = -\varepsilon\frac{1}{\sqrt{X^2-1}} + \frac{\varepsilon}{X} - \varepsilon J_1(X)(-4\ln 2 + 2\gamma) + \varepsilon\frac{\pi}{2}Y_1(X) - 2\varepsilon\ln\varepsilon J_1(X) \qquad (43)$$

and for given θ we can solve $dY_{\mathrm{lb}}/dX = \tan\theta$ for the critical value of X_c, which is related to R_b in Fig. 1 via the relationship $X_\mathrm{c} = R_\mathrm{b}/a$. In the case where θ is close to π, i.e., the slope is small, we can use the bottom solution (39) rather that the composite solution and (43) is simplified to

$$\frac{dY}{dX} = -2\varepsilon\ln\varepsilon J_1(X) - \varepsilon J_1(X)(2\gamma - 4\ln 2) + \varepsilon\frac{\pi}{2}Y_1(X) = \tan\theta \qquad (44)$$

as an equation for the critical value of X. In the case where $\theta = \pi$, i.e., at the point where the profile becomes horizontal, this reduces to the following equation to be solved for X:

$$(-4\ln 2 + 2\ln\varepsilon + 2\gamma)J_1(X) = \frac{\pi}{2}Y_1(X). \qquad (45)$$

To illustrate the point, we consider the case where $\varepsilon = 0.0739397$ and the contact angle $\theta = \pi$. Solving (45) numerically using Newton's method we find the value of X at the base to be $X = 0.6039$ using the bottom solutions and $X = 0.60606$ using the composite solution. The "exact" numerical answer from [2] is $X = 0.608160$.

As an illustration of how the solutions may be used we consider the case where the contact angle is less than $\pi/2$. Then using (18) (with $\xi = 1$), we can estimate the critical pinhole radius given the film thickness or vice versa. If the dimensional film thickness is H, the capillary length a is assumed known and so the dimensionless film thickness $h = H/a$. In this case R is the radius of the dry spot and so the dimensionless radius is $\varepsilon = R/a$ and we need to solve the following nonlinear algebraic equation for ε:

$$-H = \varepsilon\sin\theta(\ln(1+\cos\theta) - 2\ln 2 + \gamma + \ln\varepsilon). \qquad (46)$$

For very thin films, we clearly have $H \sim -\sin\theta\varepsilon\ln\varepsilon$. Note that the critical pinhole size is smaller than the thickness of the layer by a factor of $\log\varepsilon$ and in practical applications, ε is about 0.01. In Fig. 3 we show some sample graphs illustrating the composite solutions for the case $\varepsilon = 0.0739397$ and comparing them to numerical solutions of [2]. Note that the part of the asymptotic solutions where the lower/bottom curve passes through the horizontal and beyond is only valid for as long as the inclination is "small". To correctly continue these solutions any further a new rescaling of the governing equations is required as in [10] though it is not clear what the physical significance of these solutions is and whether or not they can physically be obtained. In Fig. 4 we consider the same situation and graph the corresponding lower and bottom solutions, the matching region being clearly visible.

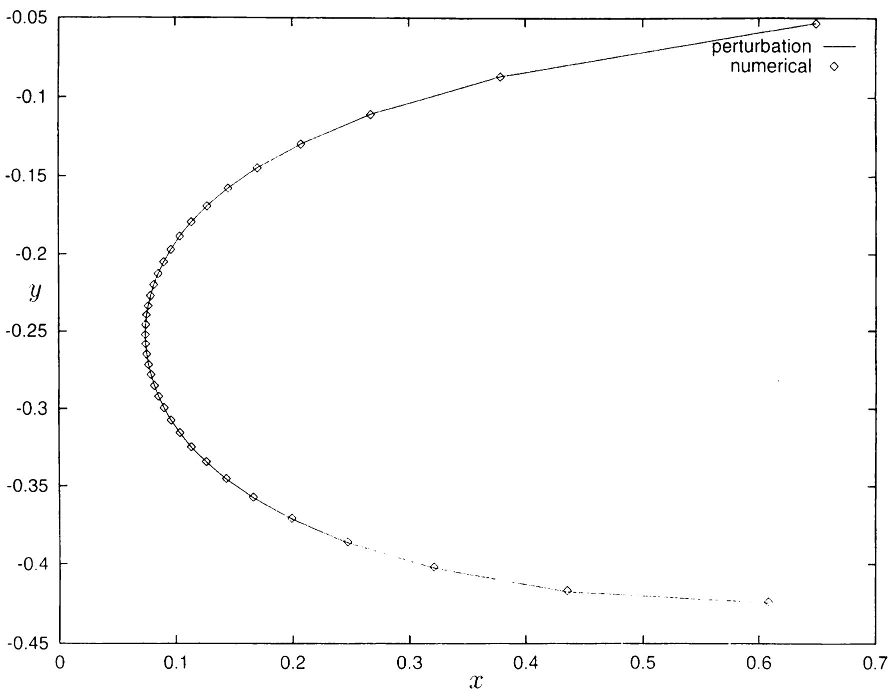

FIG. 3. Comparison of perturbation and numerical solutions with $\varepsilon = 0.0739397$

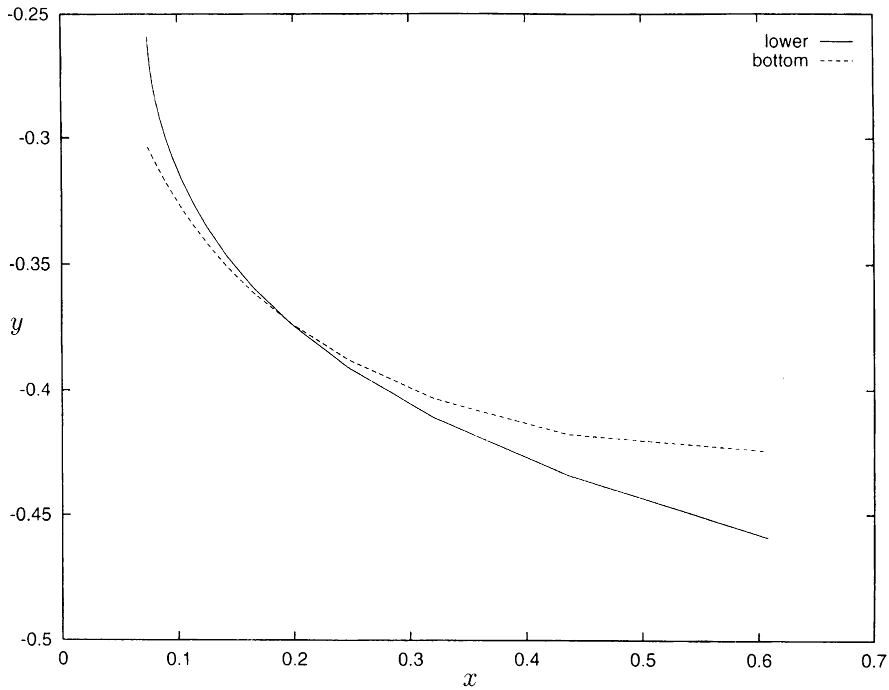

FIG. 4. Illustration of lower and bottom solutions, $\varepsilon = 0.0739397$

In Table 1 we tabulate numerical and asymptotic solutions for the composite lower/bottom solutions for the case when $\varepsilon = 0.0739397$.

TABLE 1. Comparison of numerical and asymptotic solutions, $\varepsilon = 0.0739397$

X	Y (asymptotic)	Y (numerical)
0.0742168	-0.2584821	-0.2587040
0.0750583	-0.2649197	-0.2650870
0.0764952	-0.2714440	-0.2715610
0.0785833	-0.2781075	-0.2781740
0.0814078	-0.2849658	-0.2849840
0.0850922	-0.2920817	-0.2920500
0.0898122	-0.2995281	-0.2994470
0.0958177	-0.3073916	-0.3072580
0.1034680	-0.3157767	-0.3155890
0.1132910	-0.3248131	-0.3245670
0.1260820	-0.3346610	-0.3343510
0.1430790	-0.3455175	-0.3451380
0.1662910	-0.3576187	-0.3571610
0.1990930	-0.3712013	-0.3706570
0.2473330	-0.3863593	-0.3857220
0.3210380	-0.4025499	-0.4018270
0.4355830	-0.4173049	-0.4165480
0.6081600	-0.4242719	-0.4236120

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