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VANISHING SPECIFIC HEAT FOR THE CLASSICAL SOLUTIONS OF A MULTIDIMENSIONAL STEFAN PROBLEM WITH KINETIC CONDITION

By

FAHUAI YI (*Department of Mathematics, Suzhou University, Suzhou, China*)

AND

JINDUO LIU (*North China Mining College, Yanjiao, Beijing, China*)

Abstract. In this paper we prove that the multidimensional Hele-Shaw problem with kinetic condition at the free boundary is the limit case of the Stefan problem with kinetic condition at the free boundary in the classical sense when the specific heat ε goes to zero. The method is the use of a fixed point theorem; the key step is to construct a suitable function space in which we can get the existence and uniform estimates with respect to $\varepsilon > 0$ at the same time as for classical solutions of the multidimensional Stefan problem with kinetic condition at the free boundary. For the sake of simplicity, we only consider one-phase problems in three space dimensions, although the method used here is also applicable for two-phase problems and any space dimensions.

1. Introduction. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded annual domain with $\partial\Omega = \Gamma_0 \cup \Gamma_1$; here Γ_0 is the outside boundary of Ω and Γ_1 is the inside one. Let $\bigcup_{0\le t\le T} \Gamma(t)$ be the (unknown) free boundary with $\Gamma(0) = \Gamma_0$ and $\Gamma_{1T} = \Gamma_1 \times [0,T]$. Denote the domain between $\Gamma(t)$ and Γ_1 by $\Omega(t)$.

The Stefan problem with kinetic condition at the free boundary is to find a temperature field $u(x,t)$, $x \in \Omega(t)$, $0 < t \le T$, and a free boundary $\bigcup_{0<t\le T} \Gamma(t)$, satisfying (see [1])

$$\varepsilon\partial_t u - \Delta u = 0, \qquad x \in \Omega(t),\ 0 < t \le T,$$

$$\begin{aligned} u &= g(x,t) && \text{on } \Gamma_{1T}, \\ u &= V_n && \text{on } \Gamma(t),\ 0 \le t \le T, \\ V_n &= -\frac{\partial u}{\partial n} && \text{on } \Gamma(t),\ 0 \le t \le T, \\ u(x,0) &= u_0(x) && \text{on } t = 0, \end{aligned} \tag{1.1}$$

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where $\varepsilon > 0$ is the specific heat, n is the unit normal on $\Gamma(t)$ towards the outside of $\Omega(t)$ and V_n is the normal velocity in the n direction. $g(x,t)$ and $u_0(x)$ are known functions.

For fixed $\varepsilon > 0$, the problem (1.1) is called the Stefan problem with kinetic condition. At the one-space-dimensional case this problem had been considered in [1] and [2], and, recently, with special two-dimensional geometries in [3] or, under strong technical conditions in [4] for any space dimensions by using Nash's implicit function theorem as in [5].

For $\varepsilon = 0$, the problem (1.1) without initial condition is called the Hele-Shaw problem. Its classical existence in the two-dimensional case was considered in [6] and [7], respectively, in Hölder space and in analytic function space.

In this work we prove that for fixed $\varepsilon > 0$ the problem has a unique classical solution. At the same time we get uniform estimates with respect to $\varepsilon > 0$ which ensure that the solutions converge to the solution with $\varepsilon = 0$ when ε goes to zero. So our result also provides a proof of the classical existence of a local solution for the Hele-Shaw problem with kinetic condition at the free boundary in any space dimensions.

The final session of the European Scientific Foundation meeting on "Problems with Regularized Free Boundaries" (Oxford, December 1993) was a general discussion of ten open questions related to the Hele-Shaw problem that the participants thought particularly interesting. Our problem arises from the ninth one (see [8]) which asks if the zero-specific-heat limit in the classical sense of the Stefan problem is the Hele-Shaw problem. In [9] we had proved that the Hele-Shaw problem with the Gibbs-Thomson relation which includes surface tension as well as a kinetic condition at the free boundary is the zero-specific-heat limit in the classical sense of the Stefan problem with the same free boundary conditions in any space dimensions.

Following the idea of [5], we introduce a (unknown) distance function $\rho(\omega,t)$, $\omega = (\omega_1,\omega_2) \in \mathbb{R}^2$, $t > 0$, to describe the free boundary. More precisely, suppose $\Gamma_1 \cap \Gamma_0 = \varnothing$. For the points on the surface Γ_0, we introduce coordinates $\omega = (\omega_1,\omega_2)$; we also denote by $x(\omega) \in \Gamma_0$ and $n(\omega)$ the unit exterior normal to Γ_0.

Let γ_0 be a given positive number such that the surface $\{x = x(\omega) \pm n(\omega)\gamma,\ 0 < \gamma < \gamma_0\}$ has no self-intersection and does not intersect Γ_1. Let $\rho(\omega,t)$ be a smooth function defined on $\Gamma_0 \times [0,T]$ such that $\rho(\omega,0) = 0$ and $\max|\rho(\omega,t)| \le \gamma_0/4$. We denote by $\Omega_{\rho T}$ the region bounded by the planes $t = 0$, T, surface Γ_{1T} and $\Gamma_{\rho T} = \{(x,t)\colon x = x(\omega) + \rho(\omega,t)n(\omega)\}$. The problem (1.1) can be written as follows:

$$\varepsilon\partial_t u - \Delta u = 0 \quad \text{in } \Omega_{\rho T}, \tag{1.2}$$

$$u = g(x,t) \quad \text{on } \Gamma_{1T}, \tag{1.3}$$

$$\frac{\partial u}{\partial n} + u = 0 \quad \text{on } \Gamma_{\rho T}, \tag{1.4}$$

$$V_n = u \quad \text{on } \Gamma_{\rho T}, \tag{1.5}$$

$$u(x,0) = u_0(x) \quad \text{on } t = 0. \tag{1.6}$$

In the next section we use a Hanzawa diffeomorphism to change the problem (1.2)–(1.6) into a cylindrical domain and straighten the free boundary. In Sec. 3, we prove the existence of a solution to problem (1.2)–(1.6) and get a uniform estimate with respect to $\varepsilon > 0$ following the ideas in [6] and [9]. In the last section, we prove the convergence theorem.

2. Straighten the free boundary and main results. To prove the solvability of the problem (1.2)–(1.6), it is convenient to reduce it to a problem in a fixed domain. To this end, we use the Hanzawa diffeomorphism presented in [5]. Suppose γ_0 introduced above is so small that the mapping $x\colon \Gamma_0 \times [-\gamma_0, \gamma_0] \to \mathbb{R}^3$ defined by the rule $x(\omega, \lambda) = x(\omega) + \lambda n(\omega)$ is regular and one-to-one. Let the range of this mapping be

$$N_0 = \{x(\omega, \lambda); (\omega, \lambda) \in \Gamma_0 \times [-\gamma_0, \gamma_0]\}.$$

The inverse mapping from N_0 to $\Gamma_0 \times [-\gamma_0, \gamma_0]$ is defined as follows: $x \to (\omega(x), \lambda(x))$. We set

$$\varphi^{(i)}(\omega, \lambda) = \nabla_x \omega_i(x)|_{x=x(\omega,\lambda)}, \quad i = 1, 2,$$
$$\varphi^{(3)}(\omega, \lambda) = \nabla_x \lambda(x)|_{x=x(\omega,\lambda)},$$

from which we can see that $\partial\Omega \in C^{3+\alpha}$ reduces to $\varphi^{(i)}(\omega, \lambda) \in C^{2+\alpha}(\Gamma_0 \times [-\gamma_0, \gamma_0])$, $i = 1, 2, 3$. We shall show below that for sufficiently small T, the free boundary surface $\Gamma_{\rho T}$ can be described by the equation

$$h_\rho(x, t) \equiv \lambda(x) - \rho(\omega(x), t) = 0.$$

This makes it possible to compute the normal velocity for $\Gamma(t)$:

$$V_n(\omega, t) = \left.\frac{\partial_t \rho}{|\nabla_x h_\rho|}\right|_{x=x(\omega,\rho(\omega,t))}.$$

We note that by construction $\varphi^{(3)}(\omega, 0) = n(\omega)$ is orthogonal to $\varphi^{(j)}(\omega, 0)$, $j = 1, 2$.

Further, let $\chi(\lambda) \in C^\infty([-\gamma_0, \gamma_0])$ be such that

$$\chi(\lambda) = 0, \ \text{in } |\lambda| > \tfrac{3}{4}\gamma_0,$$
$$\chi(\lambda) = 1, \ \text{in } |\lambda| < \tfrac{1}{4}\gamma_0,$$
$$\chi'(\lambda) \le 3\gamma_0^{-1}.$$

Then $1 + \chi'(\lambda)\mu \ge \frac{1}{4}$ if $|\mu| \le \gamma_0/4$.

For any $\rho(\omega, t) \in C^{2,1}(\Gamma_{0T})$, with $\rho|_{t=0} = 0$ and $\max_{\Gamma_{0T}} |\rho| < \gamma_0/4$, we define the Hanzawa diffeomorphism

$$\ell_{\rho T}\colon \mathbb{R}^3_y \times [0, T] \to \mathbb{R}^3_x \times [0, T]$$

in the manner $(x, t) = \ell_{\rho T}(y, t)$ with

$$\begin{cases} x = y, & \text{if } \operatorname{dist}(y, \Gamma_0) \ge \frac{3}{4}\gamma_0, \\ x = x(\omega) + [\eta + \chi(\eta)\rho(\omega, t)]n(\omega), & \text{if } \operatorname{dist}(y, \Gamma_0) \le \frac{3}{4}\gamma_0, \end{cases}$$

where $y = y(\omega, \eta)$ is in the neighborhood N_0 of Γ_0 and (ω, η) are local coordinates of y in N_0. In local coordinates of $N_0 \times [0, T]$, we have

$$\ell_{\rho T}(\omega, \eta; t) = (\omega, \eta + \chi(\eta)\rho(\omega, t); t) \equiv (\omega, \lambda; t),$$

where $\lambda = \eta + \chi(\eta)\rho(\omega, t)$.

The transformation $\ell_{\rho T}^{-1}$ takes the noncylindrical domain $\Omega_{\rho T}$ to the cylindrical domain $\Omega_T = \Omega \times [0,T]$. In the following, we make the change of variables $(x,t) = \ell_{\rho T}(y,t)$ and let

$$v(y,t) = u(\ell_{\rho T}(y,t)).$$

Notice that

$$\frac{\partial u}{\partial n} = (|\nabla_x h_\rho|^2 \partial_\eta v - \sum_{i=1}^{2} \partial_{\omega_i}\rho |\varphi^{(i)}(\omega,\rho)|^2 \partial_{\omega_i} v)/|\nabla_x h_\rho|.$$

If we define

$$S_\rho \equiv |\nabla_x h_\rho|^2 = 1 + \left| \sum_{i=1}^{2} \partial_{\omega_i}\rho \varphi^{(i)}(\omega,\rho) \right|^2,$$

then problem (1.2)–(1.6) becomes

$$\varepsilon\partial_t v - \mathcal{L}_\rho v = 0 \quad \text{in } \Omega_T, \tag{2.1}$$

$$v = g(y,t) \quad \text{on } \Gamma_{1T}, \tag{2.2}$$

$$v(y,0) = u_0(y) \quad \text{on } t=0, \tag{2.3}$$

$$S_\rho \partial_\eta v - \sum_{i=1}^{2} \partial_{\omega_i}\rho |\varphi^{(i)}(\omega,\rho)|^2 \partial_{\omega_i} v + S_\rho^{1/2} v = 0 \quad \text{on } \Gamma_{0T}, \tag{2.4}$$

$$\partial_t \rho = S_\rho^{1/2} v \quad \text{on } \Gamma_{0T}. \tag{2.5}$$

Here

$$\begin{aligned}
\mathcal{L}_\rho &= \sum_{i,j=1}^{3} a_\rho^{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i,j=1}^{3} a_\rho^{i} \frac{\partial}{\partial y_i}, \\
a_\rho^{ij} &= \nabla_x(\ell_{\rho T}^{-1})_i \cdot \nabla_x(\ell_{\rho T}^{-1})_j|_{(x,t)=\ell_{\rho T}(y,t)} \\
&= a_{ij}(\rho, \nabla_\omega \rho) \\
a_\rho^{i} &= [\varepsilon\partial_t(\ell_{\rho T}^{-1})_i - \Delta_x(\ell_{\rho T}^{-1})_i|_{(x,t)=\ell_{\rho T}(y,t)} \\
&= a_i(\rho, \nabla_\omega\rho, D_\omega^2\rho, \varepsilon\partial_t\rho),
\end{aligned}$$

and $a_\rho^{ij}|_{t=0} = \delta_{ij}$, $\delta_{ij} = 1$ if $i=j$, $\delta_{ij} = 0$ for $i \neq j$, $a_\rho^i|_{t=0} = 0$ because the transformation $\ell_{\rho T}$ is the identity at $t=0$.

For $0 < \alpha < 1$, define the function space of $\rho(\omega,t)$,

$$\begin{aligned}
D = \{\rho \in C^{2+\alpha, 1+\alpha/2}(\Gamma_{0T}) | \rho(\omega,0) = 0, |\rho|_{L^\infty(\Gamma_{0T})} \le \gamma_0/4, \\
|\rho|_{B([0,T];C^{2+\alpha}(\Gamma_0))} \le M_0, |\partial_t\rho|_{L^\infty(\Gamma_{0T})} \le 2M_0, \\
|D_\omega^2\rho|_{C^{\alpha,\alpha/2}(\Gamma_{0T})} \le N, |\varepsilon^{1+\alpha/2}\partial_t\rho|_{C^{\alpha,\alpha/2}(\Gamma_{0T})} \le N\},
\end{aligned}$$

where γ_0 is determined before, $B([0,T];C^{2+\alpha}(\Gamma_0))$ denotes the Banach space of bounded functions from $[0,T]$ with values in $C^{2+\alpha}(\Gamma_0)$, $M_0 = |g|_{L^\infty(\Gamma_{0T})} + |u_0|_{L^\infty(\Omega)}$, and T and N are to be determined later on. We will find that the choice of N depends only on known data and is independent of the choice of T. Set

$$|\rho|_D = |\rho|_{B([0,T];C^{2+\alpha}(\Gamma_0))} + |\partial_t\rho|_{L^\infty(\Gamma_{0T})} + |D_\omega^2\rho|_{C^{\alpha,\alpha/2}(\Gamma_{0T})} + |\varepsilon^{1+\alpha/2}\partial_t\rho|_{C^{\alpha,\alpha/2}(\Gamma_{0T})}.$$

It is seen that D is a closed convex set in the Banach space $C^{2+\alpha,1+\alpha/2}(\Gamma_{0T})$. Notice that for any $\xi \in \mathbb{R}^3$,

$$\sigma_1|\xi|^2 \le a_\rho^{ij}\xi_i\xi_j \le \sigma_2|\xi|^2,$$

where the positive numbers σ_1 and σ_2 do not depend on ε. Suppose

$$\Gamma_0 \in C^{3+\alpha}, \tag{2.6}$$

$$g(x,t) \in C^{2+\alpha,1+\alpha/2}(\Gamma_{1T}), \tag{2.7}$$

$$u_0(x) \in C^{2+\alpha}(\overline{\Omega}), \tag{2.8}$$

and the consistency conditions

$$\partial_n u_0 + u_0 = 0 \qquad \text{on } \Gamma_0, \tag{2.9}$$

$$g(x,0) = u_0(x) \qquad \text{on } \Gamma_1, \tag{2.10}$$

$$\partial_t g(x,0) = \Delta u_0(x) = 0 \quad \text{on } \Gamma_1. \tag{2.11}$$

THEOREM 2.1. Under the conditions (2.6)–(2.11), for fixed $\varepsilon > 0$, the problem (2.1)–(2.5) has a unique solution $\rho = \rho_\varepsilon \in D$.

Here we suppose that ρ is the unique unknown because once ρ is known, then v can be determined.

3. The proof of Theorem 2.1. The idea of the proof for the uniqueness is the same as in [6]; we omit the details.

For given $\rho \in D, \ell_{\rho T}$ is well-defined. Let $v(y,t)$ be the solution of (2.1)–(2.4) that depends on ρ and ε. It is obvious that for fixed $\varepsilon > 0$ and given ρ, problem (2.1)–(2.4) has a unique solution $v(y,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$. In order to get a uniform estimate with respect to $\varepsilon > 0$ we let, for $0 \le t \le T$,

$$t = \varepsilon\tau, \qquad 0 \le \tau \le \varepsilon^{-1}T \equiv T_\varepsilon, \tag{3.1}$$

define $h(\omega,\tau) = \rho(\omega,\varepsilon\tau)$ and $W(y,\tau) = v(y,\varepsilon\tau)$; therefore $\partial_\tau h = \varepsilon\partial_t\rho$ and $\partial_\tau W = \varepsilon\partial_t v$.

Under the change (3.1), problem (2.1)–(2.4) becomes

$$\partial_\tau W - \mathcal{L}_h W = 0 \qquad \text{in } \Omega_{T_\varepsilon}, \tag{3.2}$$

$$W(y,\tau) = g(y,\varepsilon\tau) \equiv G(y,\tau) \qquad \text{on } \Gamma_{1T_\varepsilon}, \tag{3.3}$$

$$W(y,0) = u_0(y) \qquad \text{on } t = 0, \tag{3.4}$$

$$S_h\partial_\eta W - \sum_{i=1}^{2} \partial_{\omega_i} h|\varphi^{(i)}(\omega,h)|^2\partial_{\omega_i} W + S_h^{1/2}W = 0 \quad \text{on } \Gamma_{0T_\varepsilon}, \tag{3.5}$$

where

$$\mathcal{L}_h = \sum_{i,j=1}^{3} a_h^{ij}\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{3} a_h^i \frac{\partial}{\partial y_i},$$

$$a_h^{ij} = a_{ij}(h, \nabla_\omega h),$$

$$a_h^i = a_i(h, \nabla_\omega h, D_\omega^2 h, \partial_\tau h).$$

Considering

$$\frac{|\partial_\tau h(\omega,\tau_1)-\partial_\tau h(\omega,\tau_2)|}{|\tau_1-\tau_2|^{\alpha/2}}=\frac{\varepsilon|\partial_t\rho(\omega,t_1)-\partial_t\rho(\omega,t_2)|}{\varepsilon^{-\alpha/2}|t_1-t_2|^{\alpha/2}},$$

we have $\partial_\tau h\in C^{\alpha,\alpha/2}(\Gamma_{0T_\varepsilon})$, and by the definition of the function set D we know

$$|h,\nabla_\omega h,D^2_\omega h,\partial_\tau h|_{L^\infty(\Omega_{T_\varepsilon})}\le 3M_0,$$
$$|h,\nabla_\omega h,D^2_\omega h,\partial_\tau h|_{C^{\alpha,\alpha/2}(\overline{\Omega}_{T_\varepsilon})}\le 3(M_0+N);$$

therefore,

$$|W|_{L^\infty(\Omega_{T_\varepsilon})}\le M_0\quad\text{(by the maximum principle)},\tag{3.6}$$
$$|W|_{C^{\alpha,\alpha/2}(\overline{\Omega}_{T_\varepsilon})}\le C(M_0)M_\alpha=C(M_\alpha),\tag{3.7}$$
$$|W|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_{T_\varepsilon})}\le C(M_0,N)M_{2+\alpha}=C(M_{2+\alpha},N),\tag{3.8}$$

where

$$\begin{aligned}M_0&=|G|_{L^\infty(\Gamma_{1T_\varepsilon})}+|u_0|_{L^\infty(\Omega)}\\&=|g|_{L^\infty(\Gamma_{1T})}+|u_0|_{L^\infty(\Omega)},\\M_\alpha&=|G|_{C^{\alpha,\alpha/2}(\Gamma_{1T_\varepsilon})}+|u_0|_{C^\alpha(\overline{\Omega})}\\&\le|g|_{C^{\alpha,\alpha/2}(\Gamma_{1T})}+|u_0|_{C^\alpha(\overline{\Omega})},\\M_{2+\alpha}&=|G|_{C^{2+\alpha,1+\alpha/2}(\Gamma_{1T_\varepsilon})}+|u_0|_{C^{2+\alpha}(\overline{\Omega})}\\&\le|g|_{C^{2+\alpha,1+\alpha/2}(\Gamma_{1T})}+|u_0|_{C^{2+\alpha}(\overline{\Omega})}\end{aligned}$$

if $\varepsilon\le 1$. $C(M_\alpha)$ and $C(M_{2+\alpha},N)$ do not depend on T_ε because the maximum of $|W|$ does not depend on T_ε, and other seminorms in $C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_{T_\varepsilon})$ depend on the maximum of $|W|$ but they are independent of T_ε according to the parabolic theorem. The estimates (3.6)–(3.8) indicate that

$$|v|_{L^\infty(\Omega_T)}\le M_0,\tag{3.9}$$
$$|v|_{B([0,T];C^\alpha(\overline{\Omega}))}+|\varepsilon^{\alpha/2}v|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)}\le C(M_\alpha),\tag{3.10}$$
$$|v|_{B([0,T];C^{2+\alpha}(\overline{\Omega}))}\le C(M_{2+\alpha},N).\tag{3.11}$$

The next step is to find a $\overline{\rho}(\omega,t)$ by the condition (2.5). For known $\rho\in D$ and v determined above, let $\overline{\rho}$ be the solution of the periodic Cauchy problem of the first-order partial differential equation

$$\partial_t\overline{\rho}=v(\omega,t)\sqrt{1+\sum_{i=1}^{2}|\varphi^{(i)}(\omega,\rho)|^2\overline{\rho}^2_{\omega_i}}\quad\text{in }\Gamma_{0T},\tag{3.12}$$

$$\overline{\rho}(\omega,0)=0\quad\text{on }t=0.\tag{3.13}$$

Here we used the relation $\varphi^{(1)}\cdot\varphi^{(2)}=0$ (see [10]) and denote $v(\omega,t)\equiv v(x,t)|_{x\in\Gamma_0}$ by $v(\omega,t)$.

LEMMA 3.1. There exists a $T>0$ such that for each $\rho\in D$, there exists a unique solution $\overline{\rho}\in D$ to the problem (3.12) and (3.13).

The proof of Lemma 3.1. The uniqueness can easily be seen by observing that the difference of two solutions of (3.12) and (3.13) satisfies a linear first-order equation.

Let $\varphi_\delta^{(i)}$, $i = 1,2$ and v_δ be the molifications of $\varphi^{(i)}(\omega,\rho(\omega,t))$ and $v(\omega,t)$ in the variable $\omega = (\omega_1,\omega_2)$, so that

$$|\varphi_\delta^{(i)} - \varphi^{(i)}|_D \to 0 \quad (\text{as } \delta \to 0), \quad i = 1,2,$$
$$|v_\delta - v|_{B([0,T];C^{2+\alpha}(\Gamma_0))} \to 0 \quad (\text{as } \delta \to 0),$$
$$|\varepsilon^{\alpha/2} v_\delta - \varepsilon^{\alpha/2} v|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} \to 0 \quad (\text{as } \delta \to 0).$$

Suppose that $\overline{\rho}_{\delta\lambda}$ is the unique solution of the following periodic Cauchy problem of the parabolic equation

$$\partial_t \overline{\rho}_{\delta\lambda} = v_\delta \sqrt{1 + \sum_{i=1}^{2} |\varphi_\delta^{(i)}|^2 (\partial_{\omega_i}\overline{\rho}_{\delta\lambda})^2} + \lambda\Delta_\omega \overline{\rho}_{\delta\lambda} \quad \text{in } \Gamma_{0T}, \tag{3.14}$$

$$\overline{\rho}_{\delta\lambda} = 0 \quad \text{on } t = 0, \tag{3.15}$$

where $\lambda > 0$ is a small parameter, and Δ_ω is the Laplace operator in $\mathbb{R}^2_\omega$.

Construct comparison functions $\pm M_0 t$; here M_0 is from the estimate (3.9). From the comparison principle (see [11], p. 52) we have

$$|\overline{\rho}_{\delta\lambda}|_{L^\infty(\Gamma_{0T})} \le M_0 T; \tag{3.16}$$

consequently, we choose T to be small enough such that

$$|\overline{\rho}_{\delta\lambda}|_{L^\infty(\Gamma_{0T})} \le \gamma_0/4. \tag{3.17}$$

To obtain further estimates, we differentiate Eq. (3.14) with respect to $w_j, j = 1,2$, and set $p_j = \partial_{\omega_j}\overline{\rho}_{j\lambda}, p_{ij} = \partial_{\omega_i}\partial_{\omega_i}\overline{\rho}_{\delta\lambda}, i,j = 1,2$. Then

$$\begin{aligned}\partial_t p_j = {} & (\partial_{\omega_j} v_\delta)\sqrt{1 + \sum_{i=1}^{2} |\varphi_\delta^{(t)}|^2 p_t^2} \\ & + v_\delta \sum_{i=1}^{2} (\varphi_\delta^{(i)} \cdot \partial_{\omega_i}\varphi_\delta^{(i)} p_i^2 + |\varphi_\delta^{(i)}|^2 p_i p_{ij}) \Big/ \left(1 + \sum_{i=1}^{2} |\varphi_\delta^{(i)}|^2 p_i^2\right)^{1/2} \\ & + \lambda\Delta_\omega p_j, \quad j = 1,2.\end{aligned}$$

Multiply the above equation by $2p_j$ and sum with respect to j. Notice that

$$2\sum_{j=1}^{2} p_j p_{ij} = \partial_{\omega_i}(p_1^2 + p_2^2).$$

Set $P = p_1^2 + p_2^2$ and notice that

$$2p_j(\partial_{\omega_j} v_\delta)\sqrt{1 + \sum_{i=1}^{2} |\varphi_\delta^{(i)}|^2 p_i^2} \le C(P+1), \quad j = 1,2,$$

$$2p_j v_\delta \left(\sum_{i=1}^{2} \varphi_\delta^{(i)} \partial_{\omega_i}\varphi_\delta^{(i)} p_i^2\right) \Big/ \left(1 + \sum_{i=1}^{2} |\varphi_\delta^{(i)}|^2 p_i^2\right)^{1/2} \le CP, \quad j = 1,2,$$

and

$$\sum_{i=1}^{2} 2p_j \Delta_\omega p_j = \Delta_\omega P - \sum_{i,j=1}^{2} p_{ij}^2 \le \Delta_\omega P.$$

The above calculations reduce to

$$\begin{cases} \partial_t P \le \lambda \Delta_\omega P + v_\delta \sum_{i=1}^{2} \frac{|\varphi_\delta^{(i)}|^2 p_i}{(1+\sum_{i=1}^2 |\varphi_\delta^{(i)}|^2 p_i^2)^{1/2}} \partial_{\omega_i} P + C_1 P + C_2, \\ P(\omega, 0) = 0, \end{cases} \tag{3.18}$$

where $C_i, i = 1, 2$, are constants that are independent of λ and p_i, p_{ij}, P. Notice that $v_\delta \frac{|\varphi_\delta^{(i)}|^2 p_i}{(1+\sum_{i=1}^2 |\varphi_\delta^{(i)}|^2 p_i^2)^{1/2}}$, $i = 1, 2$, are also bounded functions. So if we choose $C = C(M_{2+\alpha}, N)$ big enough, and let $C(M_{2+\alpha}, N)t$ be the comparison function, then

$$P \le C(M_{2+\alpha}, N)T. \tag{3.19}$$

Similarly, by differentiating (3.14) twice with respect to ω_i and $\omega_j, i, j = 1, 2$, and using the estimate (3.19) we can get

$$|D_\omega^2 \overline{\rho}_{\delta\lambda}| \le C(M_{2+\alpha}, N)T. \tag{3.20}$$

Since v_δ and $\varphi_\delta^{(i)}$, $i = 1, 2$, are C^∞ in the variables ω_1 and ω_2, we can use successive differentiations in their variables and the comparison function as before to get L^∞-estimates on $D_\omega^2 \overline{\rho}_{\delta\lambda}$ depending on δ but not on λ. Hence we can let $\lambda \to 0$ in (3.14) to get that there exists $\overline{\rho}_j$ such that $\partial_{\omega_i} \overline{\rho}_{\delta\lambda} \to \partial_{\omega_i} \overline{\rho}_\delta \equiv q_i, \partial_{\omega_i}\partial_{\omega_j} \overline{\rho}_{\delta\lambda} \to \partial_{\omega_i}\partial_{\omega_j} \overline{\rho}_\delta \equiv q_{ij}$ and

$$\begin{cases} \partial_t q_{ij} - \sum_{k=1}^{2} \frac{v_\delta |\varphi_\delta^{(k)}|^2 q_k}{\sqrt{1+\sum_{i=1}^2 |\varphi_\delta^{(i)}|^2 q_l^2}} \partial_{\omega_k} q_{ij} = F_{ij} \quad \text{in } \Gamma_{0T}, \\ q_{ij}|_{t=0} = 0 \quad (i, j = 1, 2), \end{cases} \tag{3.21}$$

where the functions $F_{ij} = F_{ij}(\omega, t)$ are functions given in terms of v_δ, $\nabla_\omega v_\delta$, $D_\omega^2 v_\delta$, $\varphi_\delta^{(i)}$, $\nabla_\omega \varphi_\delta^{(i)}$, $D_\omega^2 \varphi_\delta^{(i)}$, q_i, q_{ij}, $i, j = 1, 2$.

In order to get the Hölder regularity of q_{ij} with respect to ω, we introduce the characteristics of (3.21):

$$\begin{cases} \frac{d\xi_i}{dt} = -\frac{v_\delta |\varphi_\delta^{(i)}|^2 q_l}{\sqrt{1+\sum_{l=1}^2 |\varphi_\delta^{(l)}|^2 q_l^2}} (\omega, t), \\ \xi_i(0) = \omega_i, \quad i = 1, 2. \end{cases} \tag{3.22}$$

Suppose the solution of the above ordinary differential system is

$$\xi_i = \xi_i(\omega, t), \quad i = 1, 2.$$

Therefore,

$$\left(\frac{\partial \xi_i}{\partial \omega_j}\right)_{2\times 2} = \exp\left\{\int_0^t \left(\frac{\partial \beta_i}{\partial \omega_j}\right)_{2\times 2} (\omega, \tau) d\tau\right\},$$

where $\beta_i(\omega, t) = -\frac{v_\delta |\varphi_\delta^{(i)}|^2 q_i}{\sqrt{1+\sum_{l=1}^2 |\varphi_\delta^{(l)}|^2 q_i^2}} (\omega, t)$, $i = 1, 2$. So if we let T be small enough, then

$$\frac{1}{2} \le \left|\left(\frac{\partial \xi_i}{\partial \omega_j}\right)_{2\times 2}\right| \le 2.$$

Rewriting the system (3.21) in integral form along the characteristics, we have, for any $\omega, \overline{\omega} \in \mathbb{R}^2$, $0 < t < T$,

$$\begin{aligned}|q_{ij}(\xi(\omega,t),t) - q_{ij}(\xi(\overline{\omega},t),t)|\\ \le C\int_0^t \left[\sum_{i,j=1}^2 |q_{ij}(\xi(\omega,\tau),\tau) - q_{ij}(\xi(\overline{\omega},\tau),\tau)| + |\xi(\omega,\tau) - \xi(\overline{\omega},\tau)|\right] d\tau,\end{aligned}$$

where C is independent of δ.

Summing up the above inequality with respect to i and j using the Gronwall inequality, we can obtain that there is a constant C that is independent of δ, but depends on $M_{2+\alpha}$ and N, such that

$$|q_{ij}|_{C([0,T];C^\alpha(\Gamma_0))} \le C(M_{2+\alpha}, N)T, \quad i,j = 1,2. \tag{3.23}$$

By the equation

$$\partial_t \overline{\rho}_\delta = v_\delta \sqrt{1 + \sum_{i=1}^2 |\varphi_\delta^{(i)}|^2 (\partial_{\omega_i}\overline{\rho}_\delta)^2}, \tag{3.24}$$

we know that

$$\begin{aligned}|\partial_t\overline{\rho}_\delta|_{L^\infty(\Gamma_{0T})} &\le M_0[1 + C(M_{2+\alpha}, N)T],\\ |\partial_t\overline{\rho}_\delta|_{B([0,T];C^\alpha(\Gamma_0))} &\le C(M_\alpha)[1 + C(M_{2+\alpha}, N)T].\end{aligned}$$

If T is small enough such that

$$C(M_{2+\alpha}, N)T \le 1, \tag{3.25}$$

we have the following estimates:

$$|\partial_t\overline{\rho}_\delta|_{L^\infty(\Gamma_{0T})} \le 2M_0, \tag{3.26}$$

$$|\partial_t\overline{\rho}_\delta|_{B([0,T];C^\alpha(\Gamma_0))} \le 2C(M_\alpha), \tag{3.27}$$

where N is to be determined later on, which will be independent of T.

The estimates (3.16), (3.19), (3.20), and (3.23) reduce to

$$|\overline{\rho}_\delta|_{B([0,T];C^{2+\alpha}(\Gamma_0))} \le M_0 \tag{3.28}$$

if T is small enough.

Applying the following interpolation inequality, where the embedding constant C_α is independent of T (see [13], Lemma 5.1.1),

$$\begin{aligned}&|\nabla_\omega\overline{\rho}_\delta|_{C^{0,1+\alpha/2}(\Gamma_{0T})} + |D_\omega^2\overline{\rho}_\delta|_{C^{0,\alpha/2}(\Gamma_{0T})}\\ &\quad\le C_\alpha(|\overline{\rho}_\delta|_{B([0,T];C^{2+\alpha}(\Gamma_0))} + |\partial_t\overline{\rho}|_{B([0,T];C^\alpha(\Gamma_0))})\\ &\quad\le C_\alpha[M_0 + 2C(M_\alpha)].\end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29) we obtain

$$\begin{aligned}&|\nabla_\omega\overline{\rho}_\delta|_{C^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T})} + |D_\omega^2\overline{\rho}_\delta|_{C^{\alpha,\alpha/2}(\Gamma_{0T})}\\ &\quad\le M_0 + C_\alpha[M_0 + 2C(M_\alpha)] \equiv N_1.\end{aligned} \tag{3.30}$$

Multiplying (3.24) by $\varepsilon^{\alpha/2}$ to get

$$\varepsilon^{\alpha/2}\partial_t\overline{\rho}_\delta = (\varepsilon^{\alpha/2}v_\delta)\sqrt{1+\sum_{i=1}^{2}|\varphi_\delta^{(i)}|^2(\partial_{\omega_i}\overline{\rho}_\delta)^2}, \tag{3.31}$$

and recalling the estimates (3.10) and (3.30) we have

$$|\varepsilon^{\alpha/2}\partial_t\overline{\rho}_\delta|_{C^{\alpha,\alpha/2}(\Gamma_{0T})} \le C(M_\alpha)\{1+M_0+C_\alpha[M_0+2C(M_\alpha)]\} \equiv N_2.$$

So we can choose $N=\max\{N_1,N_2\}$, and then let T be small enough such that (3.25) is true.

Passing to the limit as $\delta\to 0$, through an appropriate subsequence, we obtain a function $\overline{\rho}\in D$, which turns out to be the (unique) solution to (3.12), (3.13), and still satisfies the estimates (3.17), (3.26), (3.28), and (3.30), (3.31).

Define a mapping $\mathcal{F}\colon D\to D$ by

$$\mathcal{F}(\rho)=\overline{\rho}.$$

If we consider the uniform topology in D, then D is a closed, convex, and compact subset of the Banach space $C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$. The continuity of $\mathcal{F}$ follows by compactivity of D and the uniqueness of the mapping $\mathcal{F}$. Therefore the Shauder fixed point theorem yields the existence of

$$\rho=\mathcal{F}(\rho)\in D,$$

which solves the problem (2.1)–(2.5).

This completes the proof of Lemma 3.1 as well as Theorem 2.1.

4. The convergence of the solutions. In the previous section we have proved that the problem (2.1)–(2.5) has a unique solution $\rho=\rho_\varepsilon(\omega,t)\in D$ for fixed $\varepsilon>0$. Let $v=v_\varepsilon$. Then $(v_\varepsilon,\rho_\varepsilon)$ satisfy

$$\begin{aligned}
&\varepsilon\partial_t v_\varepsilon-\sum_{i,j=1}^{3}a_{ij}(\rho_\varepsilon,\nabla_\omega\rho_\varepsilon)\frac{\partial^2 v_\varepsilon}{\partial y_i\partial y_j}\\
&\qquad+\sum_{i,j=1}^{3}a_i(\rho_\varepsilon,\nabla_\omega\rho_\varepsilon,D_\omega^2\rho_\varepsilon,\varepsilon\partial_t\rho_\varepsilon)\frac{\partial v_\varepsilon}{\partial y_i}=0 &&\text{in } \Omega_T, &(4.1)\\
&v_\varepsilon=g(y,t) &&\text{on } \Gamma_{1T}, &(4.2)\\
&v_\varepsilon(y,0)=u_0(y) &&\text{on } t=0, &(4.3)\\
&S_{\rho_t}\partial_\eta v_\varepsilon-\sum_{i=1}^{2}\partial_{\omega_i}\rho_\varepsilon|\varphi^{(i)}(\omega,\rho_\varepsilon)|^2\partial_{\omega_i}v_\varepsilon+S_{\rho_\varepsilon}^{1/2}v_\varepsilon=0 &&\text{on } \Gamma_{0T}, &(4.4)\\
&\partial_t\rho_\varepsilon=S_{\rho_\varepsilon}^{1/2}v_\varepsilon &&\text{on } \Gamma_{0T}, &(4.5)
\end{aligned}$$

and uniform estimates

$$|\rho_\varepsilon|_{B([0,T];C^{2+\alpha}(\Gamma_0))}\le C, \tag{4.6}$$

$$|\partial_t\rho_\varepsilon|_{B([0,T];C^{\alpha}(\Gamma_0))}\le C, \tag{4.7}$$

$$|v_\varepsilon|_{B([0,T];C^{2+\alpha}(\overline{\Omega}))}\le C. \tag{4.8}$$

THEOREM 4.1. Under the assumptions (2.6)–(2.11),

(1) (The convergence of free boundaries) there is a function

$$\rho(\omega,t) \in B([0,T];C^{2+\alpha}(\Gamma_0)), \qquad \partial_t\rho \in B([0,T];C^{\alpha}(\Gamma_0))$$

such that

$$\rho_\varepsilon \to \rho \quad \text{in } C([0,T];C^{2+\beta}(\Gamma_0)),\ 0<\beta<\alpha. \tag{4.9}$$

(2) (The convergence of solutions) If $\Delta u_0(x)=0$ in Ω, then

$$|\partial_t v_\varepsilon|_{L^\infty(\Omega_T)} \le C. \tag{4.10}$$

Moreover, there is a function $v(x,t) \in B([0,T];C^{2+\alpha}(\overline{\Omega}))$ such that

$$v_\varepsilon \to v \quad \text{in } C([0,T];C^{2+\beta}(\overline{\Omega})). \tag{4.11}$$

(3) (v,ρ) is the classical solution for the Hele-Shaw problem with a kinetic condition at the free boundary.

Proof. (1) (4.9) follows by the estimates (4.6) and (4.7) and Simon's compact theorem (see [12], Corollary 4, p. 85).

(2) In order to get (4.10), let us go back to the (x,t) coordinates

$$\varepsilon\partial_t u_\varepsilon - \Delta u_\varepsilon = 0 \quad \text{in } \Omega_{\rho T}, \tag{4.12}$$
$$u_\varepsilon = g(x,t) \quad \text{on } \Gamma_{1T}, \tag{4.13}$$
$$u_\varepsilon(x,0) = u_0(x) \quad \text{on } t=0, \tag{4.14}$$
$$\frac{\partial u_\varepsilon}{\partial n} + u_\varepsilon = 0 \quad \text{on } \Gamma_{\rho T}, \tag{4.15}$$
$$V_n = u_\varepsilon \quad \text{on } \Gamma_{\rho T}. \tag{4.16}$$

Here $u_\varepsilon(x,t) = v_\varepsilon(\ell_{\rho T}^{-1}(x,t))$ and by (4.8),

$$|u_\varepsilon|_{C([0,T];C^{2+\alpha}(\overline{\Omega}(t)))} \le C. \tag{4.17}$$

Differentiating with respect to t in (4.12)–(4.15), we have

$$\varepsilon\partial_t(\partial_t u_\varepsilon) - \Delta(\partial_t u_\varepsilon) = 0 \quad \text{in } \Omega_{\rho T}, \tag{4.18}$$
$$\partial_t u_\varepsilon = \partial_t g(x,t) \quad \text{on } \Gamma_{1T}, \tag{4.19}$$
$$\partial_t u_\varepsilon = 0 \quad \text{on } t=0, \tag{4.20}$$
$$\frac{\partial(\partial_t u_\varepsilon)}{\partial n} + \partial_t u_\varepsilon + V_n\left|\nabla\left(\frac{\partial u_\varepsilon}{\partial n} + u_\varepsilon\right)\right| = 0. \tag{4.21}$$

Substituting (4.16) into the last equality to get

$$\frac{\partial(\partial_t u_\varepsilon)}{\partial n} + \partial_t u_\varepsilon = -u_\varepsilon\left|\nabla\left(\frac{\partial u_\varepsilon}{\partial n} + u_\varepsilon\right)\right| \quad \text{on } \Gamma_{\rho T}, \tag{4.22}$$

and applying the maximum principle to the system (4.18)–(4.20) and (4.22), using (4.17) we get

$$|\partial_t u_\varepsilon|_{L^\infty(\Omega_{\rho T})} \le C(|\partial_t g|_{L^\infty(\Gamma_{1T})} + |u_\varepsilon|_{C([0,T];C^2(\Omega(t)))}) \le C;$$

here C is independent of $\varepsilon > 0$.

Considering

$$\partial_t u_\varepsilon = \partial_t v_\varepsilon + \nabla_y v_\varepsilon \cdot \frac{\partial \ell_{\rho T}^{-1}}{\partial t},$$

we get

$$|\partial_t v_\varepsilon|_{L^\infty(\Omega_T)} \le |\partial_t u_\varepsilon|_{L^\infty} + C|\nabla_y v|_{L^\infty}|\partial_t \rho_\varepsilon|_{L^\infty} \le C, \tag{4.23}$$

where C is independent of $\varepsilon > 0$. From the estimates (4.8) and (4.23) we get (4.11) by the use of Simon's compact theorem again.

(3) From (4.5) we know that

$$\partial_t \rho_\varepsilon \to \partial_t \rho \quad \text{in } C([0,T]; C^{1+\beta}(\Gamma_0)). \tag{4.24}$$

Letting $\varepsilon \to 0$ in (4.1), (4.2), (4.4), and (4.5) we get

$$\begin{aligned}
&-\sum_{i,j=1}^{3} a_{ij}(\rho, \nabla_\omega \rho)\frac{\partial^2 v}{\partial y_i \partial y_j} + \sum_{i,j=1}^{3} a_i(\rho, \nabla_\omega \rho, D_\omega^2 \rho)\frac{\partial v}{\partial y_i} = 0 && \text{in } \Omega_T,\\
&v = g(x,t) && \text{on } \Gamma_{1T},\\
&S_\rho \partial_\eta v - \sum_{i=1}^{2} \partial_{\omega_i}|\varphi^{(i)}(\omega,\rho)|^2 \partial_{\omega_i} v + S_\rho^{1/2} v = 0 && \text{on } \Gamma_{0T},\\
&\partial_t \rho = S_{\rho_\varepsilon}^{1/2} v && \text{on } \Gamma_{0T}.
\end{aligned}$$

This system means that (v, ρ) is the classical solution of the Hele-Shaw problem with a kinetic condition at the free boundary.

This completes the proof of Theorem 4.1.

REMARK. The $\Delta u_0(x) = 0$ in Ω is the necessary condition for the convergence of solutions because the solution of the Hele-Shaw problem must satisfy this condition.

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