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ON COMPRESSIBLE MATERIALS CAPABLE OF SUSTAINING AXISYMMETRIC SHEAR DEFORMATIONS. PART 3: HELICAL SHEAR OF ISOTROPIC HYPERELASTIC MATERIALS

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Abstract. A helical shear deformation is a composition of non-universal, axisymmetric, anti-plane shear and rotational shear deformations, shear states that are separately controllable only in special kinds of compressible and incompressible, homogeneous and isotropic hyperelastic materials. For incompressible materials, it is only necessary to identify a specific material, such as a Mooney-Rivlin material, to determine the anti-plane and rotational shear displacement functions. For compressible materials, however, these shear deformations may not be separately possible in the same specified class of hyperelastic materials unless certain auxiliary conditions on the strain energy function are satisfied. We have recently presented simple algebraic conditions necessary and sufficient in order that both anti-plane shear and rotational shear deformations may be separately possible in the same material subclass. In this paper, under the same physical condition that the shear response function be positive, we present an essentially algebraic condition necessary and sufficient to determine whether a class of compressible, homogeneous and isotropic hyperelastic materials is capable of sustaining controllable, helical shear deformations. It is then proved that helical shear deformations are possible in a specified hyperelastic material if and only if that material can separately sustain both axisymmetric, anti-plane shear and rotational shear deformations. The simplicity of the result in applications is illustrated in a few examples.

1. Introduction. The helical shear of a circular cylindrical tube consists of coupled axisymmetric, anti-plane shear and rotational shear deformations and is described by

$$r = R, \quad \theta = \Theta + \psi(R), \quad z = Z + u(R), \tag{1.1}$$

where $\psi(R)$ is the plane rotational shear angle, $u(R)$ is the axial, anti-plane shear displacement, and (r, θ, z) are the current cylindrical coordinates of a material point initially

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at (R, Θ, Z) in the same Cartesian reference system. We shall see in Section 2 that the total helical shear strain $\kappa \equiv \pm\sqrt{(du(R)/dR)^2 + (Rd\psi(R)/dR)^2}$ in a nontrivial, helical shear deformation cannot be constant. Moreover, unlike a homogeneous simple shear deformation, which is controllable in every compressible or incompressible, isotropic and homogeneous hyperelastic material, the helical shear deformation (1.1) is not a universal deformation. On the other hand, helical shear deformations may be sustained by surface tractions alone in special kinds of compressible and incompressible materials.

Spencer [1] has shown that circumferential and axisymmetric, anti-plane shear deformations are both separately and simultaneously possible in a general class of compressible materials that vary slightly from the incompressible Mooney-Rivlin model for which Rivlin first derived similar general results. Rivlin [2] has shown that the helical shear problem for an arbitrary incompressible, isotropic and homogeneous hyperelastic material leads to two coupled nonlinear ordinary differential equations for $u(R)$ and $\psi(R)$ whose solutions may be obtained only upon specification of the strain energy function. In particular, for an incompressible Mooney-Rivlin material, these equations are separable and yield the exact axial and rotational shear displacement functions given by

$$u(R) = A\ln(BR), \quad \psi(R) = C + \frac{D}{R^2}, \tag{1.2}$$

where A, B, C, and D are constants determined by assigned boundary conditions. The remaining equilibrium equation then determines the unknown pressure function, and hence the stress components and traction conditions may be found for this special class of incompressible materials. Thus, except for the details leading eventually to the determination of $u(R)$ and $\psi(R)$, the axisymmetric, helical shear problem for any specified incompressible material may be considered solved. Therefore, for an *incompressible* material, a simple shear, a pure rotational shear, and a pure axisymmetric, anti-plane shear deformation may be identified as trivial cases among the general class of axisymmetric, helical shear deformation problems. For a *compressible* material, however, the situation is less clear and hence only the simple shear case may be considered trivial. In fact, the axisymmetric, helical shear deformation (1.1) may not be possible for a specified class of compressible materials unless certain auxiliary conditions on the strain energy function are satisfied. Our purpose in this paper is to present an algebraic condition on any specified strain energy function of a compressible, isotropic and homogeneous hyperelastic material necessary and sufficient for the material to sustain axisymmetric, helical shear deformations.

A necessary and sufficient algebraic condition on the strain energy function in order that a pure azimuthal shear defined by (1.1) when $u(R) \equiv 0$ may be sustained in a compressible material was first obtained by Haughton [3]. Similar results are given by Beatty and Jiang [4] in a simpler and somewhat different form that does not require use of a certain monotonicity condition specified in [3]. More recently, Jiang and Ogden [5] presented a simpler variation of the same result deduced from equations for a general azimuthal plane strain for which $r = r(R)$ and $u(R) \equiv 0$ in (1.1). Here we are interested in the composition of pure rotational and anti-plane shear deformations in a general helical shear deformation (1.1).

In previous papers [4, 6, 7], we have presented simple algebraic conditions necessary and sufficient to determine whether or not a specified compressible, isotropic and homogeneous hyperelastic material is capable of separately sustaining axisymmetric, anti-plane shear and rotational shear deformations. Sometimes a certain subclass of a specified primary hyperelastic material is capable of supporting one or the other of these deformations, but not both; and sometimes the separate shear deformations may be possible in different subclasses of the same specified primary material, or possibly not at all. Although rotational shear and anti-plane shear deformations may be separately possible in the same material subclass, it is not evident that their superposition in helical shear deformations will be possible in the same material subclass. In Sec. 3, we present a single necessary and sufficient algebraic condition in order that nontrivial, axisymmetric, helical shear deformations may be possible in a specified primary class of compressible, isotropic and homogeneous hyperelastic materials. We then prove in Sec. 4 that a material is capable of sustaining controllable helical shear deformations when and only when it can separately sustain both axisymmetric, anti-plane shear and azimuthal shear deformations. Our single algebraic condition on helical shear deformations is thus decomposed into the two simple algebraic conditions derived in [4].

The simplicity of our results in applications is further illustrated in some examples in Sec. 5. In one example we consider a material model studied in work by Polignone and Horgan [8, 9]. They show that for their specified primary material class, axisymmetric anti-plane shear and azimuthal shear deformations are separately controllable in *different* subclasses of the material. They do not study helical shear deformations. We show in Sec. 5 that helical shear deformations are possible in a *distinct* subclass of their primary material, a class for which the shear response function is constant and for which both states of shear are separately possible.

2. Helical shear deformations of compressible isotropic materials. Let us consider a body $\mathcal{B}$ that occupies a cylindrical region D in a fixed reference configuration χ_o, typically the unstressed natural state. Let (R,Θ,Z) denote the cylindrical coordinates of a material point $\mathbf{X}$ in χ_o, with the Z-axis parallel to the generators of the cylinder. A deformation is called an *axisymmetric, helical shear* if, in a common fixed Cartesian reference frame $\varphi = \{O; \mathbf{i}_k\}$ at O, it carries the particle with coordinates (R, Θ, Z) to the place $\mathbf{x}$ with corresponding cylindrical coordinates (r, θ, z) in a deformed configuration χ so that

$$r = R, \quad \theta = \Theta + \psi(R), \quad z = Z + u(R), \qquad \forall (R, \Theta, Z) \in D. \tag{2.1}$$

The *rotational displacement angle* $\psi(\cdot)$ and the *axial displacement function* $u(\cdot)$ are twice continuously differentiable functions on the open interval $\Lambda : R \in [0, \infty)$, which is related to the open cross section S of the cylindrical region D through $S = \Lambda \times [0, 2\pi)$.

The deformation gradient tensor $\mathbf{F} = \partial \mathbf{x}(\mathbf{X})/\partial \mathbf{X}$ for an axisymmetric, helical shear deformation is given by

$$\mathbf{F} = \mathbf{1} + \kappa_r \mathbf{e}_\theta \otimes \mathbf{e}_R + \kappa_a \mathbf{e}_z \otimes \mathbf{e}_R, \quad \forall (R, \Theta, Z) \in D, \tag{2.2}$$

where $\kappa_r(R)$, called the *rotational shear strain,* and κ_a, named the *axial shear strain,* are defined by

$$\kappa_r(R) \equiv R\frac{d\psi(R)}{dR}, \quad \kappa_a(R) \equiv \frac{du(R)}{dR}. \tag{2.3}$$

Also, $(\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z)$ and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are the usual orthonormal, physical vector bases associated with (R, Θ, Z) and (r, θ, z), respectively, in frame φ; and $\mathbf{1} = \delta_{i\alpha}\mathbf{e}_{i\alpha}$ is the identity tensor in the mixed, physical tensor basis $\mathbf{e}_{i\alpha} \equiv \mathbf{e}_i \otimes \mathbf{e}_\alpha$. It may be seen that the principal invariants $I_k(\mathbf{C})$ of the left Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ for the helical shear are given by

$$I_1 = I_2 = 3 + \kappa^2, \quad I_3 = 1, \tag{2.4}$$

in which the *helical shear strain* κ is defined by

$$\kappa \equiv \sqrt{\kappa_a^2 + \kappa_r^2}. \tag{2.5}$$

The last relation in (2.4) shows that the helical shear deformation is isochoric.

We now consider a compressible, isotropic and homogeneous hyperelastic material that is characterized by a strain energy function $W(I_1, I_2, I_3)$ of the three principal invariants $I_k(\mathbf{C})$. The strain energy function relates the deformation gradient tensor $\mathbf{F}$ to the first Piola-Kirchhoff (engineering) stress tensor σ through the constitutive relation

$$\sigma = 2\frac{\partial W}{\partial I_1}\mathbf{F} + 2\frac{\partial W}{\partial I_2}\mathbf{F}(I_1\mathbf{1} - \mathbf{C}) + 2I_3\frac{\partial W}{\partial I_3}\mathbf{F}^{-\mathrm{T}}, \quad \forall\, \mathbf{F} \in \Gamma, \tag{2.6}$$

wherein $\mathbf{1} = \delta_{jk}\mathbf{e}_{jk}$ is the usual identity tensor and Γ denotes the set of all nonsingular, second-order tensors. Substitution of (2.2) into (2.6) and use of (2.4) yield the physical components of the engineering stress tensor σ for axisymmetric, helical shear deformations:

$$\begin{aligned}
\sigma_{rR} &= 2(W_1 + 2W_2 + W_3), & \sigma_{r\Theta} &= -2\kappa_r(W_2 + W_3),\\
\sigma_{\theta\Theta} &= 2[W_1 + (2 + \kappa_a^2)W_2 + W_3], & \sigma_{rZ} &= -2\kappa_a(W_2 + W_3),\\
\sigma_{zZ} &= 2[W_1 + (2 + \kappa_r^2)W_2 + W_3], & \sigma_{\theta R} &= 2\kappa_r(W_1 + W_2),\\
\sigma_{zR} &= 2\kappa_a(W_1 + W_2), & \sigma_{\theta Z} &= \sigma_{z\Theta} = -2\kappa_a\kappa_r W_2,
\end{aligned} \tag{2.7}$$

$\forall\, (R, \Theta, Z) \in D$ and where $W_k \equiv \partial W(I_1, I_2, I_3)/\partial I_k$ are evaluated for the invariants in (2.4). We note that the strain energy and the stress, by (2.7), vanish in the reference configuration χ_o provided that

$$W(3,3,1) = 0, \quad \hat{W}_1 + 2\hat{W}_2 + \hat{W}_3 = 0, \tag{2.8}$$

where $\hat{W}_k \equiv W_k(3,3,1)$ in χ_o. The universal relations

$$\begin{aligned}
&\sigma_{\theta R} - \sigma_{r\Theta} = \kappa_r\sigma_{rR}, \quad \sigma_{zR} - \sigma_{rZ} = \kappa_a\sigma_{rR},\\
&\sigma_{\theta Z} - \sigma_{z\Theta} = \kappa_r\sigma_{zR} - \kappa_a\sigma_{\theta R} = \kappa_r\sigma_{rZ} - \kappa_a\sigma_{r\Theta}
\end{aligned} \tag{2.9}$$

are evident from (2.7). The reader will find that these arise from the condition that the Cauchy stress tensor $\mathbf{T} = (\det \mathbf{F})^{-1}\ \sigma\ \mathbf{F}^{\mathrm{T}}$ be symmetric. For future convenience, we introduce

(2.10) $$\sigma(\kappa^2) \equiv \sigma_{rR} = 2(W_1 + 2W_2 + W_3).$$

It is also helpful to note that the nontrivial components of $\mathbf{T}$ and σ are related by

(2.10a) $$T_{rr} = \sigma_{rR}, \quad T_{\theta\theta} = \kappa_r \sigma_{\theta R} + \sigma_{\theta\Theta},$$
$$T_{r\theta} = \kappa_r \sigma_{rR} + \sigma_{r\Theta}, \quad T_{rz} = \kappa_a \sigma_{rR} + \sigma_{rZ}.$$

In the absence of body forces, equilibrium of $\mathcal{B}$ requires that $\operatorname{div} \mathbf{T} = \mathbf{0}$. Since the axial and rotational shear strains $\kappa_a(R)$ and $\kappa_r(R)$ depend on only the radial coordinate R, we note that the stress components in (2.7), and hence (2.10a), are functions of R alone. Therefore, the radial, axial, and circumferential equilibrium equations are provided by

(2.10b) $$\frac{\partial T_{rr}}{\partial R} + \frac{T_{rr} - T_{\theta\theta}}{R} = 0, \quad \frac{\partial (R T_{rz})}{\partial R} = 0, \quad \frac{\partial (R^2 T_{r\theta})}{\partial R} = 0.$$

Use of (2.10a) in (2.10b) yields, respectively, the following representations of the equations of equilibrium:

(2.11) $$R \frac{d}{dR} \sigma(\kappa^2) = \kappa_r \tau_r(\kappa_a, \kappa_r) + 2\kappa_a^2 W_2, \qquad \forall\, R \in \Lambda,$$

(2.12) $$\frac{d}{dR}[R \tau_a(\kappa_a, \kappa_r)] = 0, \quad \forall\, R \in \Lambda,$$

(2.13) $$\frac{d}{dR}[R^2 \tau_r(\kappa_a, \kappa_r)] = 0, \quad \forall\, R \in \Lambda,$$

where the *axial* and *rotational shear stress response functions* $\tau_a(\cdot,\cdot)$ and $\tau_r(\cdot,\cdot)$, respectively, are defined by

(2.14) $$\tau_a(\kappa_a, \kappa_r) \equiv \sigma_{zR} = \kappa_a \mu(\kappa^2), \quad \tau_r(\kappa_a, \kappa_r) \equiv \sigma_{\theta R} = \kappa_r \mu(\kappa^2)$$

for $-\infty < \kappa_a, \kappa_r < \infty$, and

(2.15) $$\mu(\kappa^2) \equiv 2(W_1 + W_2), \qquad 0 \le \kappa < \infty,$$

is called the *shear response function.* We shall assume that the material has a positive shear modulus in χ_o so that

(2.16) $$\mu_o \equiv \mu(0) = 2(\hat{W}_1 + \hat{W}_2) = -2(\hat{W}_2 + \hat{W}_3) > 0.$$

It then follows from (2.14) that the axial and the rotational shear stress response are in the direction of their corresponding shear strain if and only if the shear response function is positive-valued for all $\kappa \in [0, \infty)$:

(2.17) $$\mu(\kappa^2) > 0.$$

The *helical shear stress response function* $\tau(\cdot)$ is defined by the rule

(2.18) $$\tau(\kappa) \equiv \kappa \mu(\kappa^2).$$

It then follows from (2.14) and (2.5) that

$$\tau(\kappa) \equiv \sqrt{\tau_a^2(\kappa_a, \kappa_r) + \tau_r^2(\kappa_a, \kappa_r)}. \tag{2.19}$$

It is easy to confirm that regardless of the form of the strain energy $W(\cdot,\cdot,\cdot)$, the only constant-valued helical shear strain for which the equilibrium equations (2.11), (2.12), and (2.13) are satisfied is the *trivial* helical shear strain $\kappa = 0$; that is, in accordance with (2.5), $\kappa_a = 0$ and $\kappa_r = 0$.

Integration of (2.12) and (2.13) yields the general relations

$$R\tau_a(\kappa_r(R), \kappa_a(R)) = A, \quad R^2\tau_r(\kappa_r(R), \kappa_a(R)) = B, \quad \forall R \in \Lambda, \tag{2.20}$$

where A and B are integration constants. If Λ contains the origin $R = 0$, (2.20) shows that $A = B = 0$. Therefore, $\tau_a(R) = \tau_r(R) = 0$ for all $R \in \Lambda$; and by (2.14) and (2.17), we obtain the trivial shear strains $\kappa_a = \kappa_r = 0$. Henceforward, we shall exclude the trivial case $R = 0$ by requiring D to be a tubular cylinder denoted by D_o. It is not necessary that the tube be circular. Henceforward, we write $\Lambda_o : R \in (0, \infty)$ so that Λ_o is related to the open cross section S_o of the cylinder D_o through $S_o = \Lambda_o \times [0, 2\pi)$. Thus, independent of the radial equilibrium condition, it follows from (2.14) and (2.20) that *axisymmetric, helical shear deformations of a compressible (or an incompressible) material have the form* (1.2) *when and only when the corresponding shear response function is a constant. Moreover, the shear response function* (2.15) *for helical shear deformations is a constant if and only if* $W(\cdot,\cdot,\cdot)$ *satisfies the relation*

$$W_{11} + 2W_{12} + W_{22} = 0 \tag{2.21}$$

on the line $\mathrm{L} : I_1 = I_2 \geq 3,\ I_3 = 1$. We note that the function (2.15) has the same form for separate axisymmetric, anti-plane shear and azimuthal shear deformations, and the result (2.21) follows in the same manner shown in [6].

In general, however, (2.20) is a system of two coupled, nonlinear ordinary differential equations to determine $u(R)$ and $\psi(R)$, and only special solutions of these equations will also satisfy (2.11) subject to certain further restrictions on the form of the strain energy function. Consequently, the requirement that every solution pair (u, ψ) of (2.20) also satisfy (2.11) should restrict the form of the strain energy function $W(\cdot,\cdot,\cdot)$, and hence distinguish a subclass of compressible, isotropic and homogeneous hyperelastic materials for which nontrivial, axisymmetric, helical shear deformations may be controllable, that is, produced by application of surface tractions alone. We shall say that, by definition, *a material characterized by the strain energy function* $W(\cdot,\cdot,\cdot)$ *is capable of sustaining controllable,* ***nontrivial*** *states of axisymmetric, helical shear deformations if every solution pair* $(u(\cdot), \psi(\cdot))$ *of* (2.20) *also satisfies* (2.11). We seek necessary and sufficient conditions on $W(\cdot,\cdot,\cdot)$ to distinguish this class of materials.

3. Isotropic materials capable of sustaining axisymmetric, helical shear deformations. We shall establish in this section a necessary and sufficient condition for a compressible, isotropic, homogeneous hyperelastic material to sustain controllable, nontrivial, axisymmetric, helical shear deformations. To this end, we first note the following Lemma essential to the proof.

LEMMA. For axisymmetric, helical shear deformations of a compressible, isotropic and homogeneous hyperelastic material whose shear stress response function (2.15) satisfies the condition

$$\mu(\kappa^2) > 0 \quad \forall \kappa \in [0, \infty), \tag{3.1}$$

the helical shear strain $\kappa(R)$ vanishes identically if either the helical shear strain itself or its derivative $d\kappa/dR$ vanishes at a single location.

Proof. Our proof is similar to a parallel result in [6]. We note by (2.5) that κ_a and κ_r vanish together if and only if the helical shear strain $\kappa = 0$. Though this is the principal case of interest here, in fact, we can show that if κ_a and κ_r vanish together at the same location R_o, or if they respectively vanish at separate locations R_a and R_r in D_o, then, in accordance with (2.14) and (2.20), they must both vanish identically in Λ_o. Indeed, in this case $A = 0$ at $R = R_a$ and $B = 0$ at $R = R_r$, or both vanish at R_o. In any event, by (2.20), $\tau_a(R) = 0$ and $\tau_r(R) = 0$ everywhere in D_o; and with (3.1) it follows from (2.14) that κ_a and κ_r together vanish identically for all R in D_o. Therefore, (2.5) shows that $\kappa(R)$ vanishes identically in Λ_o. Conversely, it is now evident that if the helical shear strain vanishes at a single location, it must vanish identically in Λ_o. Notice that we actually require only that $\mu(\kappa^2) \neq 0$.

To establish the second part of the lemma, we differentiate (2.20) with respect to R to obtain

$$\frac{d\tau_a}{dR} = -\frac{\tau_a}{R}, \quad \frac{d\tau_r}{dR} = -2\frac{\tau_r}{R}. \tag{3.2}$$

Then differentiation of (2.19) and substitution of (3.2) yields

$$\tau \frac{d\kappa}{dR}\frac{d\tau}{d\kappa} = -(\frac{\tau_a^2}{R} + 2\frac{\tau_r^2}{R}). \tag{3.3}$$

Therefore, $d\kappa/dR$ vanishes at one location $R = R_o$ in D_o only when both τ_a and τ_r vanish at the same location, and hence the constants $A = B = 0$ in (2.20). Thus, both τ_a and τ_r must vanish everywhere in D_o. It follows from our previous argument that κ_a, κ_r, and κ all vanish identically in Λ_o. This concludes the proof. □

With the aid of this result, we next establish our theorem on controllable helical shear deformations. It proves convenient, however, to first recall (2.10), (2.15), and (2.18) to define

$$\dot{\sigma}(\kappa^2) \equiv \frac{d\sigma(\kappa^2)}{d\kappa^2} = 2(W_{11} + 3W_{12} + 2W_{22} + W_{13} + W_{23}), \tag{3.4}$$

$$\tau'(\kappa) \equiv \frac{d\tau(\kappa)}{d\kappa} = 2(W_1 + W_2) + 4\kappa^2(W_{11} + 2W_{12} + W_{22}), \tag{3.5}$$

$$\dot{\mu}(\kappa^2) \equiv \frac{d\mu(\kappa^2)}{d\kappa^2} = 2(W_{11} + 2W_{12} + W_{22}), \tag{3.6}$$

all on the line $\mathrm{L} : I_1 = I_2 = 3+\kappa^2,\ I_3 = 1$. The rule (3.6) was applied in (3.5). Notice also that (3.6) confirms the result stated earlier in (2.21) for materials having a constant shear response function. We are now prepared to prove the following theorem on controllable helical shear deformations.

THEOREM 1. A compressible, isotropic and homogeneous hyperelastic material whose strain energy function $W(\cdot,\cdot,\cdot)$ satisfies the condition (3.1) is capable of sustaining controllable, nontrivial, axisymmetric, helical shear deformations if and only if W also satisfies the following condition:

$$\kappa_a^2(\mu\,\dot{\sigma}+W_2\tau')+2\kappa_r^2\mu(\dot{\sigma}+\frac{\tau'}{4})=0 \tag{3.7}$$

on the line L: $I_1=I_2\geq 3,\ I_3=1$.

Proof. We recall from our previous definition at the end of Sec. 2 that the rotational displacement $\psi(\cdot)$ and the axial displacement $u(\cdot)$ of an axisymmetric, helical shear deformation must satisfy the three differential equations given in (2.11) and (2.20). To prove necessity of (3.7), we thus consider a solution pair $(u(\cdot),\psi(\cdot))$ of (2.20) that also satisfies (2.11). According to the Lemma, $u(\cdot)$ and $\psi(\cdot)$ are constants and hence satisfy (2.11) and (2.20) identically if κ vanishes at a single location $R=R_o\neq 0$, that is, if κ_a and κ_r vanish at $R=R_o$. Hence, we suppose the derivatives of $u(\cdot)$ and $\psi(\cdot)$ never vanish and consequently, by the Lemma,

$$\kappa(R)\neq 0,\quad \frac{d\kappa(R)}{dR}\neq 0,\quad \forall R\in\Lambda_o. \tag{3.8}$$

We note that the equilibrium equations (2.12) and (2.13) are recast in (3.2), which then yield the single equilibrium relation (3.3). Thus, with the aid of (2.14) and (3.1), the pair of equilibrium equations (2.11) and (3.3) may be rewritten as

$$2R\kappa\frac{d\kappa}{dR}\,\dot{\sigma}\,(\kappa^2)=\kappa_r^2\mu+2\kappa_a^2W_2\equiv S(\kappa), \tag{3.9}$$

$$R\kappa\frac{d\kappa}{dR}\tau'(\kappa)=-\mu(\kappa_a^2+2\kappa_r^2)\equiv -T(\kappa). \tag{3.10}$$

Multiply (3.9) by $T(\kappa)$ and (3.10) by $S(\kappa)$, as defined above; sum the results, recall (3.8) and thus remove the factor $2R\kappa d\kappa/dR$ to derive the following necessary condition:

$$\mu(\kappa_a^2+2\kappa_r^2)\,\dot{\sigma}+(\kappa_r^2\mu+2\kappa_a^2W_2)\frac{\tau'}{2}=0, \tag{3.11}$$

which is the same as (3.7). Hence, (3.7) is a general necessary condition for which every solution pair $(u(R),\psi(R))$ of (2.20) also satisfies (2.11).

Conversely, to prove sufficiency of (3.7), we need to show that every solution pair $(u(\cdot),\psi(\cdot))$ of the equilibrium equations (2.12) and (2.13) for which (3.11) holds also satisfies the radial equilibrium equation (2.11). Of course, we still have (3.3) cast in the form (3.10). We multiply (3.11) by $2R\kappa d\kappa/dR$ and substitute (3.10), remove the factor $\mu(\kappa_a^2+2\kappa_r^2)\neq 0$, in accordance with the Lemma, and thereby recover the radial equilibrium equation (2.11). This establishes the sufficiency of (3.7) and thus completes the proof of the theorem. □

It is useful to note other forms of (3.7). First note that use of (2.15) in (3.7) yields the equivalent necessary and sufficient condition

$$(\kappa_a^2+2\kappa_r^2)(\mu\,\dot{\sigma}+W_2\tau')+\kappa_r^2\tau'(W_1-W_2)=0. \tag{3.12}$$

Substitution of (2.15), (3.4), and (3.5) into (3.11) yields the expanded relation

(3.13)
$$2(\kappa^2+\kappa_r^2)(W_1+W_2)(W_{11}+3W_{12}+2W_{22}+W_{13}+W_{23})$$
$$+(\kappa_r^2 W_1+\kappa^2 W_2)[W_1+W_2+2\kappa^2(W_{11}+2W_{12}+\ W_{22})=0.$$

Finally, in terms related to earlier results in [4] and [6], we have

(3.14)
$$2\kappa_a^2\left[(W_1+W_2)(W_{11}+3W_{12}+2W_{22}+W_{13}+W_{23}+\tfrac{1}{2}W_2)\right.$$
$$\left.+\kappa_a^2 W_2(W_{11}+2W_{12}+W_{22})\right]$$
$$+\kappa_r^2(W_1+W_2)\left[W_1+W_2+4(W_{11}+3W_{12}+2W_{22}+W_{13}+W_{23})\right.$$
$$\left.+2\kappa_r^2(W_{11}+2W_{12}+W_{22})\right]$$
$$+2\kappa_a^2\kappa_r^2(W_1+2W_2)(W_{11}+2W_{12}+W_{22})=0.$$

Each of the relations (3.12), (3.13), and (3.14) is equivalent to (3.7) on the line L and is therefore an equivalent condition necessary and sufficient for controllable helical deformations. We see again that the proof of the theorem requires only that $\mu(\kappa^2)\neq 0$; however, the physical nature of (3.1) was noted earlier.

4. Relation to previous results. In accordance with (3.4), (3.5), and (3.6), it is seen that the terms enclosed in parentheses in (3.7) are functions of κ^2 alone:

$$f\left(\kappa^2\right)\equiv\mu\,\dot{\sigma}+W_2\tau',\quad g\left(\kappa^2\right)\equiv\dot{\sigma}+\frac{\tau'}{4},\tag{4.1}$$

functions that depend only on the form of $\hat{W}(\kappa^2)=W(3+\kappa^2,3+\kappa^2,1)$. It is evident that the restrictions on $\hat{W}(\kappa^2)$ for which

$$f\left(\kappa^2\right)=0,\quad g\left(\kappa^2\right)=0,\ \forall\,\kappa\in[0,\infty)\tag{4.2}$$

hold, suffice for (3.7). Moreover, (3.7) holds for arbitrary choices of κ_a and κ_r when and only when their respective coefficients vanish; and hence *both*

$$f\left(\kappa_a^2\right)=\mu\,\dot{\sigma}+W_2\tau'=0\text{ at }\kappa^2=\kappa_a^2,\ \kappa_r^2=0,\ \forall\ \kappa_a\in(-\infty,\infty)\,,\tag{4.3}$$

$$g(\kappa_r^2)=\dot{\sigma}+\frac{\tau'}{4}=0\text{ at }\kappa^2=\kappa_r^2,\ \kappa_a^2=0,\ \forall\ \kappa_r\in(-\infty,\infty)\,,\tag{4.4}$$

are concurrent necessary conditions on $\hat{W}(\kappa^2)$. However, these are general functions of κ alone valid for all $\kappa\in[0,\infty)$ on the line $\mathrm{L}: I_1=I_2\geq 3,\ I_3=1$. In consequence, (4.2) are necessary and sufficient conditions for which $\hat{W}(\kappa^2)$ is capable of sustaining helical shear deformations with shear strain κ^2 in (2.5).

In fact, $(4.2)_1$ and $(4.2)_2$, respectively, are necessary and sufficient conditions for which $\hat{W}(\kappa^2)$ is capable of sustaining anti-plane shear and rotational shear deformations. We thus have the following theorem on helical shear deformations, that is, simultaneous anti-plane and rotational shear deformations.

THEOREM 2. A compressible, isotropic and homogeneous hyperelastic material whose strain energy function satisfies (3.1) is capable of sustaining controllable, nontrivial, axisymmetric helical shear deformations if and only if the material can separately sustain both axisymmetric, anti-plane and circumferential shear deformations.

For elliptic materials, that is, materials for which $\tau' \neq 0$, (3.12) reveals that (4.3) and (4.4) hold for all κ on the line $\mathrm{L} : I_1 = I_2 \geq 3,\ I_3 = 1$, provided that

$$W_1 = W_2 \tag{4.5}$$

holds for all κ on L. We notice also that (4.3) and (4.4) are straight lines in $(\dot{\sigma}, \tau')$ space; these have the same slope, $-W_2/\mu = -1/4$, if and only if (4.5) holds. Thus, (4.3), (4.4), and (4.5) are precisely the conditions established in [4] in order that both axisymmetric, anti-plane shear and azimuthal shear deformations are separately possible in the *same* subclass of isotropic and compressible hyperelastic elliptic materials.

Indeed, with the aid of (3.4) and (3.5), we find that (4.3) and (4.4) may be respectively written in expanded form as

(4.6)

$$\begin{aligned} \tfrac{1}{4}(\mu\,\dot{\sigma} + W_2\tau') = (W_1 + W_2)(W_{11} + 3W_{12} + 2W_{22} + W_{13} + W_{23} + \tfrac{1}{2}W_2) \\ + \kappa^2 W_2 (W_{11} + 2W_{12} + W_{22}) = 0, \end{aligned}$$

$$\begin{aligned} 2(\dot{\sigma} + \frac{\tau'}{4}) = W_1 + W_2 + 4(W_{11} + 3W_{12} + 2W_{22} + W_{13} + W_{23}) \\ + 2\kappa^2 (W_{11} + 2W_{12} + W_{22}) = 0, \end{aligned} \tag{4.7}$$

valid for all κ on the line $\mathrm{L} : I_1 = I_2 \geq 3,\ I_3 = 1$. The condition (4.6) is the condition derived in [6, 7] necessary and sufficient for axisymmetric, anti-plane shear deformations to be controllable in a specified class of materials; and (4.7) is the corresponding necessary and sufficient condition obtained in [4] in order that axisymmetric, circumferential shear deformations are possible in a given class of materials, perhaps different from the other. It is also proved in [4] that these shear deformations may be separately possible in the *same* elliptic material subclass if and only if (4.5) holds for all κ on L. When this is so, $W_{11} = W_{22}$ holds on L; and it is thus shown that the single condition necessary and sufficient for both shears to be separately possible is given by

$$W_1 + 2(\kappa^2 + 3)(W_{11} + W_{12}) + 2(W_{13} + W_{23}) = 0 \tag{4.8}$$

for all κ on the line L. Therefore, the same condition is necessary and sufficient for helical shear deformations.

Finally, use of (2.15) and (2.21) in (3.5) shows that all materials having a constant shear response function are elliptic. We thus find that for a material having a constant shear response function (4.8) simplifies to

$$W_{13} + W_{23} = -\frac{W_1}{2} = -\frac{\mu_0}{8}. \tag{4.9}$$

It is easily verified that the same conclusions derive from (3.13) or (3.14) for materials having a constant shear response function. In consequence, no further restrictions are required for simultaneous, superimposed anti-plane and rotational shear deformations beyond those necessary and sufficient for both states of shear to be separately possible in the same material subclass. Bearing in mind the results in (4.5) and (4.8), we have the following reduced result on helical shear deformations.

COROLLARY 1. A compressible, isotropic and homogeneous hyperelastic elliptic material whose strain energy function satisfies (3.1) is capable of sustaining controllable, nontrivial, axisymmetric helical shear deformations if and only if (4.5) and (4.8) hold for all κ on the line $\mathbb{L} : I_1 = I_2 \geq 3,\ I_3 = 1$. The latter condition is reduced further to (4.9) for materials having a constant shear response function.

Although superposition of deformations is not a property typical of nonlinear materials, the result clearly supports our intuition in this case. As a consequence, all of the examples presented in [4] for which both states of shear are separately possible in the same material subclass are valid here. In particular, it is shown that axisymmetric, anti-plane shear and circumferential shear deformations are separately possible in a subclass of Hadamard materials whose shear response function is constant, but cannot be sustained in a Blatz-Ko foamed rubber material. Therefore, we know by Corollary 1 that helical shear deformations can be produced by surface tractions alone in the same Hadamard material subclass, but not in the foamed Blatz-Ko material. To conclude, we next present three new illustrations.

5. Examples. In this section, the conditions (4.5) and (4.8), or (4.9), are applied to determine subclasses of specified compressible, isotropic and homogeneous hyperelastic materials capable of sustaining nontrivial, controllable, axisymmetric, helical shear deformations. We begin with an example due to Polignone and Horgan. They show in [8, 9] that axisymmetric, anti-plane shear and azimuthal shear deformations are separately controllable in *different* material subclasses of a certain specified principal hyperelastic material model. Their results are based on two necessary conditions, one being a first-order, the other a second-order nonlinear ordinary differential equation. The same results, however, may be derived most readily from our algebraic conditions (4.6) for axisymmetric, anti-plane shear and (4.7) for azimuthal shear deformations, conditions that are simpler to apply and are both necessary and sufficient as well. Helical shear deformations, however, have not been considered previously. We now show that both states of shear are separately possible in a *common* material subclass of the Polignone-Horgan material model, a subclass that is distinct from those found in [8, 9]. Therefore, helical shear deformations are possible in this special class of materials. Two further examples for a class of quadratic materials and another general class of materials follow.

5.1. *Helical shear of a Polignone-Horgan material.* Polignone and Horgan [8, 9] introduced a general class of compressible and isotropic hyperelastic materials defined by the strain energy function

$$W = \frac{\gamma}{2}[P(I_1 - I_2, I_3)(I_1 - 3) + Q(I_1 - I_2, I_3)(I_2 - 3) + R(I_1 - I_2, I_3)] \tag{5.1}$$

in which $\gamma > 0$ is a constant and the response functions $P(\cdot,\cdot)$, $Q(\cdot,\cdot)$, and $R(\cdot,\cdot)$ are at least twice continuously differentiable functions of their arguments. In accordance with

(2.8), the strain energy and the stress vanish in the reference configuration, provided the response functions satisfy

$$R(0,1) = 0, \quad P(0,1) + 2Q(0,1) - R_1(0,1) + R_2(0,1) = 0. \tag{5.2}$$

Here and below, for any response function $H(I_1 - I_2, I_3)$, the derivatives with respect to its arguments are written as

$$H_1(\xi,\eta) \equiv \frac{\partial H(\xi,\eta)}{\partial \xi}, \quad H_2(\xi,\eta) \equiv \frac{\partial H(\xi,\eta)}{\partial \eta}. \tag{5.3}$$

It is readily verified that (2.21) is satisfied identically; so the shear response function for all materials in the class (5.1) must be constant, and hence the class of materials (5.1) is elliptic. In fact, recalling (2.15) and (3.1), we require

$$\mu(\kappa^2) = \gamma[P(0,1) + Q(0,1)] \equiv \mu_0 > 0. \tag{5.4}$$

In view of Theorem 2, helical shear deformations are possible in the material subclass of (5.1) having a constant shear modulus if and only if (4.5) and (4.9) hold for all $\kappa \in [0,\infty)$ on the line $\mathbf{L} : I_1 = I_2 = 3 + \kappa^2,\ I_3 = 1$. It is easily seen that (4.5) requires

$$P(0,1) - Q(0,1) + 2R_1(0,1) + 2\kappa^2[P_1(0,1) + Q_1(0,1)] = 0 \tag{5.5}$$

for all $\kappa \in [0,\infty)$. Hence, the response functions must satisfy

$$P(0,1) - Q(0,1) + 2R_1(0,1) = 0, \quad P_1(0,1) + Q_1(0,1) = 0. \tag{5.6}$$

The final condition (4.9) yields

$$P_2(0,1) + Q_2(0,1) = -\frac{\mu_0}{4\gamma} < 0. \tag{5.7}$$

We thus find that *nontrivial, axisymmetric, helical shear deformations can be sustained in every hyperelastic material subclass of* (5.1) *for which the foregoing algebraic conditions in* (5.2), (5.4), (5.6), *and* (5.7) *hold.*

5.1.1. *A special subclass of admissible materials.* In particular, consider the subclass of materials for which $\gamma = \mu_0$. The foregoing conditions on the response functions then reduce to the following conditions necessary and sufficient for controllable helical shear deformations to be possible for materials in the class (5.1):

$$P(0,1) + Q(0,1) = 1,\ P_1(0,1) + Q_1(0,1) = 0, \quad P_2(0,1) + Q_2(0,1) = -\tfrac{1}{4}, \tag{5.8}$$

$$Q(0,1) - R_1(0,1) = P(0,1) + R_1(0,1) = \tfrac{1}{2}, \quad R_2(0,1) = -\tfrac{3}{2}. \tag{5.9}$$

5.1.2. *A model for biological tissue.* A wide variety of admissible material models are included in the class defined by (5.1). A specific subclass of materials of potential interest in biomechanics [10, 11] that might model compressible soft tissues is described by the strain energy function

$$W = \frac{\gamma}{2}[f(I_3)e^{\alpha(I_1-I_2)}(I_1 - 3) + g(I_3)e^{-\alpha(I_1-I_2)}(I_2 - 3) + h(I_3)], \tag{5.10}$$

where $\alpha \neq 0$ is an arbitrary constant and the response functions $\gamma,\ f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ satisfy

$$\gamma = \frac{\mu_0}{2}, \quad f(1) = g(1) = 1, \quad h(1) = 0. \tag{5.11}$$

It is easily seen that $(5.2)_1$, (5.4), and (5.6) are identically satisfied and $(5.2)_2$ and (5.7) require

$$h'(1) = -3, \quad f'(1) + g'(1) = -\tfrac{1}{2}, \tag{5.12}$$

where a prime denotes d/dI_3. In (5.12), we might have $h(I_3) \equiv -3(I_3 - 1)$, for example. The results (5.12) show that controllable helical shear deformations are possible in the class of materials (5.10), a subclass that is not among those studied by Polignone and Horgan [8, 9].

5.1.3. *The role of boundary conditions.* Additional constraints may be required to satisfy specified boundary conditions. In particular, consider a circular cylindrical tube whose inside surface $R = R_i$ is fixed so that $u(R_i) = 0$ and $\psi(R_i) = 0$, and for which the outside surface $R = R_o$ is subjected to pure shear tractions so that

$$\sigma_{rR}(R_o) = 0, \quad \sigma_{\theta R}(R_o) = \sigma_r, \quad \sigma_{zR}(R_o) = \sigma_a, \tag{5.13}$$

where σ_r and σ_a are constants. Since we require $r = R$ in (2.1), no radial displacement of material points in χ_o is possible; therefore, we should expect that the null radial traction condition $(5.13)_1$ will place further restrictions on the class of materials for which helical shear deformations may be possible. From $(2.7)_1$, this requires

$$\sigma_{rR}(R_o) = 2(W_1 + 2W_2 + W_3)\,|_{R_o} = 0. \tag{5.14}$$

Thus, for the material (5.1), we find that (5.14) yields the further restriction

$$P_2(0,1) + Q_2(0,1) = 0. \tag{5.15}$$

But this contradicts (5.7) and is therefore impossible. Hence, *no Polignone-Horgan material* (5.1) *can sustain helical shear deformations without application of normal, radial tractions on its boundaries.* These tractions can be supplied by appropriate bonded rigid attachments, in which case no further restrictions need be imposed on the form of the strain energy function (5.1). It is commonly assumed in such studies that the tube is sufficiently long that end effects may be ignored, and we adopt the same position here.

5.1.4. *Helical shear displacement functions.* Since the shear response function is constant, the axial and rotational shear displacement functions have the classical form (1.2). Thus, as shown in [4, 6], these functions are given by

$$u(R) = \frac{R_o \sigma_a}{\mu_0} \log(\frac{R}{R_i}), \quad \psi(R) = \frac{R_o^2 \sigma_r}{2\mu_o} \left(\frac{1}{R_i^2} - \frac{1}{R^2} \right). \tag{5.16}$$

The helical displacement pair $(u(R_o),\, \psi(R_o))$ of the outside surface may be read from (5.16).

5.2. *A class of quadratic materials.* Consider a class of materials quadratic in I_1 and I_2 and characterized by the strain energy function

$$\begin{aligned} W(I_1, I_2, I_3) =& H_1(I_3)(I_1 + I_2 - 6) + H_2(I_3)[(I_1 - 3)^2 \\ &+ (I_2 - 3)^2] + H_3(I_3)(I_1 - 3)(I_2 - 3) + H_4(I_3), \end{aligned} \tag{5.17}$$

where $H_k(\cdot)$, $k = 1, 2, 3, 4$, are at least twice continuously differentiable functions of I_3. We write $H_k'(I_3) \equiv dH_k(I_3)/dI_3$. In accordance with (2.8), it follows from (5.17) that the strain energy and the stress vanish in χ_o provided that

$$H_4(1) = 0, \quad 3H_1(1) + H_4'(1) = 0. \tag{5.18}$$

It is apparent that on the line $\mathbf{L} : I_1 = I_2 \geq 3,\ I_3 = 1$,

$$W_1 = W_2 = H_1(1) + \kappa^2[2H_2(1) + H_3(1)]; \tag{5.19}$$

and hence the shear response function (2.15) is a quadratic function of the helical shear strain given by

$$\mu(\kappa^2) = \mu_0 + 2\mu_1\kappa^2 \quad \forall\ \kappa \in [0, \infty), \tag{5.20}$$

where μ_0 and μ_1 are constant shear moduli defined by

$$\mu_o \equiv \mu(0) = 4H_1(1) > 0, \quad \mu_1 \equiv 2[2H_2(1) + H_3(1)] \geq 0. \tag{5.21}$$

Thus, the shear response function (5.20) is quadratic in κ if and only if the inequality holds in $(5.21)_2$. The null stress condition $(5.18)_2$ is satisfied with

$$H_4'(1) = -\frac{3\mu_0}{4}. \tag{5.22}$$

Because the material, in view of (5.21), is elliptic, using (5.17) in (4.8), we find that *nontrivial, helical shear deformations are possible for the subclass of quadratic materials for which* $(5.18)_1$, (5.21), (5.22), *and the following conditions hold:*

$$H_1'(1) = -\frac{1}{16}(\mu_0 + 12\mu_1), \quad 2H_2'(1) + H_3'(1) = -\frac{3\mu_1}{8}. \tag{5.23}$$

If we now set $2H_2(I_3) = H_3(I_3)$ for all I_3, we have a quadratic material with strain energy

$$W(I_1, I_2, I_3) = H_1(I_3)(I_1 + I_2 - 6) + H_2(I_3)(I_1 + I_2 - 6)^2 + H_4(I_3), \tag{5.24}$$

in which the response functions must satisfy the following conditions:

$$\begin{gathered} H_1(1) = \frac{\mu_0}{4}, \quad H_2(1) = \frac{\mu_1}{8}, \quad H_4(1) = 0, \\ H_1'(1) = -\tfrac{1}{16}(\mu_0 + 12\mu_1), \quad H_2'(1) = -\frac{3\mu_1}{32}, \quad H_4'(1) = -\frac{3\mu_0}{4}, \end{gathered} \tag{5.25}$$

and whose shear response function is quadratic in the helical shear strain κ in (2.5). We thus recover the quadratic material model for which it is shown in Part 2 that both axisymmetric, anti-plane shear and rotational shear are separately possible. Here we have established that this material is indeed capable of sustaining more complex axisymmetric, helical shear deformations.

5.3. *A general material model.* Let us consider a general hyperelastic material with strain energy function

$$W = E(I_1 + I_2, I_3)(I_1 - 3) + F(I_1 + I_2, I_3)(I_2 - 3) + G(I_1 + I_2, I_3), \tag{5.26}$$

wherein the response functions $E(\cdot,\cdot)$, $F(\cdot,\cdot)$, $G(\cdot,\cdot)$ are at least twice continuously differentiable functions of the indicated arguments. In order that helical shear deformations may be possible in this material, we require that $W_1 = W_2$, and hence $E(I_1 + I_2, I_3) = F(I_1 + I_2, I_3)$ on the line $\mathrm{L} : I_1 = I_2 \geq 3,\ I_3 = 1$. We are thus led to investigate the elliptic material subclass of (5.26) defined by

$$\begin{aligned} W &= F(I_1 + I_2, I_3)(I_1 + I_2 - 6) + G(I_1 + I_2, I_3) \\ &\equiv H(I_1 + I_2, I_3). \end{aligned} \tag{5.27}$$

In fact, it is seen that (5.24) is a special case of (5.27). Adopting notation similar to (5.3), it is evident that $W_1 = W_2 = H_1(I_1 + I_2, I_3)$. In accordance with (3.5), the material (5.27) is elliptic if and only if

$$H_1(6 + 2\kappa^2, 1) + 4\kappa^2 H_{11}(6 + 2\kappa^2, 1) \neq 0. \tag{5.28}$$

The null referential energy and stress conditions (2.8) require

$$H(6, 1) = 0, \quad 3H_1(6, 1) + H_2(6, 1) = 0; \tag{5.29}$$

and the shear response function (2.15) is defined by

$$\mu(\kappa^2) = 4H_1(6 + \kappa^2, 1) > 0 \quad \text{with } \mu_0 = \mu(0) = 4H_1(6, 1) > 0. \tag{5.30}$$

We recall that helical shear deformations are possible for elliptic materials if and only if (4.8) holds for all $\kappa \in [0, \infty)$. For materials in the class (5.27), this requires that

$$\frac{\mu(\kappa^2)}{4} + 4(\kappa^2 + 3)H_{11}(6 + \kappa^2, 1) + 4H_{12}(6 + \kappa^2, 1) = 0 \ \forall \kappa \in [0, \infty), \tag{5.31}$$

wherein we recall (5.30); and hence for $\kappa = 0$,

$$12H_{11}(6, 1) + 4H_{12}(6, 1) = -\frac{\mu_0}{4}. \tag{5.32}$$

Thus, any elliptic material with strain energy function (5.27) *that satisfies* (5.29) *through* (5.31) *is capable of sustaining controllable, helical shear deformations.*

Consider the particular case for which

$$H(I_1 + I_2, I_3) \equiv P(I_1 + I_2, I_3) + Q(I_3), \tag{5.33}$$

where the response functions $P(\cdot,\cdot)$, $Q(\cdot)$ are appropriately smooth functions of their arguments. The null energy and stress conditions (5.29) in the reference configuration require

$$P(6, 1) + Q(1) = 0, \quad 3P_1(6, 1) + P_2(6, 1) + Q'(1) = 0; \tag{5.34}$$

and (5.30) gives the shear response function

$$\mu(\kappa^2) = 4P_1(6 + \kappa^2, 1) > 0 \quad \text{with } \mu_0 = \mu(0) = 4P_1(6, 1) > 0. \tag{5.35}$$

Helical shear deformations are thus possible if and only if (5.31) holds for all $\kappa \in [0, \infty)$. This requires that

(5.36)
$$\frac{\mu(\kappa^2)}{4} + 4(\kappa^2 + 3)P_{11}(6 + \kappa^2, 1) + 4P_{12}(6 + \kappa^2, 1) = 0 \quad \forall \kappa \in [0, \infty);$$

and (5.32) yields

$$12P_{11}(6, 1) + 4P_{12}(6, 1) = -\frac{\mu_0}{4}. \tag{5.37}$$

Notice that the function $Q(I_3)$ plays a relatively insignificant role in this example.

For a more specific illustration, suppose that

$$P(I_1 + I_2, I_3) = \frac{\mu_0}{16}[4 - f(I_3)](I_1 + I_2 - 6), \tag{5.38}$$

where $f(I_3)$ is an arbitrary, smooth function whose derivative is written as $f'(I_3)$. It is then seen that $P_{11}(I_1 + I_2, I_3) \equiv 0$, and (5.34) through (5.37) require

(5.39)
$$Q(1) = 0, \quad f(1) = 0, \quad f'(1) = 1, \quad Q'(1) = -\frac{3\mu_0}{4}, \quad \mu(\kappa^2) = \mu_0 > 0.$$

Notice in this case that the shear response function is constant and hence the condition (4.8) may be replaced by (4.9). Of course, this leads to the same results.

This concludes our study of compressible, isotropic hyperelastic materials capable of sustaining axisymmetric shear deformations. In Part 4, we present a parallel study of axisymmetric shear deformations of compressible, anisotropic hyperelastic materials.

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