QUARTERLY OF APPLIED MATHEMATICS
VOLUME LVII, NUMBER 4
DECEMBER 1999, PAGES 723–740

ON SECOND SOUND AT THE CRITICAL TEMPERATURE

By

K. SAXTON (*Dept. of Math. and Computer Science, Loyola University, New Orleans, LA*),

R. SAXTON (*Dept. of Mathematics, University of New Orleans, LA*),

AND

W. KOSINSKI (*Center of Mechanics, IPPT, Polish Academy of Sciences, Warsaw*)

Abstract. Based on low-temperature experimental data in solid dielectric crystals, we derive a model of heat conduction for rigid materials using the theory of thermodynamic internal state variables. The model is intended to admit wavelike propagation of heat below—and diffusive conduction above—a particular temperature value ϑ_λ. A rapid decay of the speed of thermal waves occurs just below this temperature, coincident with the conductivity of the material reaching a peak. An analysis of weak and strong discontinuity waves is given in order to exhibit several main features of the proposed model.

1. Introduction. In the classical theory of thermodynamics, heat conduction is viewed as a purely diffusive process, typically described using a Fourier law. This results in the associated evolution equations being of parabolic type and absent of wavelike motion. Experimentally, however, finite speed thermal waves have been observed in materials at very low temperatures. These waves, known as second sound, were first detected in ^{3}He, ([1]), and then in high purity dielectric crystals of sodium flouride, NaF, ([13]), and bismuth, Bi, ([22]). It was observed that there exists a (material dependent) temperature below which second sound begins to be observed, and that temperatures of this type are close to those at which the conductivity of the material reaches a peak. A useful discussion can be found in the review papers [10], [15], [16].

The range of temperature for which this second sound is detectable is in fact quite small, and normal diffusive propagation takes place above it. Our purpose will be to allow the two forms of propagation to exist in separate temperature ranges by taking into account the influence of the history of the absolute temperature on the heat flux

Received December 16, 1997.
1991 *Mathematics Subject Classification.* Primary 73B30, 35L67.
E-mail address: saxton@loyno.edu
E-mail address: rsaxton@math.uno.edu
E-mail address: wkos@ippt.gov.pl

vector. To introduce this memory dependence, we employ an internal state variable known as the *semi-empirical* temperature ([3], [4]), which is related to the absolute temperature through an evolution equation. Such models have previously been found to be numerically in good agreement ([2], [11], [12]) with experimental data, over the range of temperature where wavelike behaviour dominates.

We will adapt this approach to permit these distinct forms of propagation to appear in temperature regimes separated by a *critical* temperature. As our starting point we use a modified version ([19], [20]) of the model in ([3], [4]), which allows the heat flux dependence to be on both the gradients of the absolute and the semi-empirical temperatures.

In the derivation, we use qualitative experimental results from the literature to specify admissible forms of constitutive equations and material functions. In particular, our derivation is based on two experimentally-observed phenomena not included in existing thermodynamic theories of second sound. The first is related to the propagation of heat pulses in solid specimens. It has been observed, ([13]), that in some range of absolute temperature at which experiments have been performed, the time of arrival of heat pulses sent through a specimen is an approximately linear function of the reference temperature. However, near the upper limit of measured temperature values, the time measured by the leading edge of heat pulses rises rapidly with increasing temperature. The latter suggests a very fast decay to zero of the second sound speed, U_E, the same effect that occurs in superfluid helium at the well-known λ-point. We denote by ϑ_λ the *critical* temperature where $U_E = 0$. The second phenomenon concerns the heat conductivity which shows a pronounced peak, ([14]), whose height depends on the purity of the specimen. In our model, motivated by the experimental data, we make the hypothesis that the temperature of maximum heat conductivity coincides with ϑ_λ, below which second sound appears. Above this temperature, heat conduction becomes purely diffusive, obeying a general nonlinear Fourier law. In previous thermodynamically consistent theories ([17], [19], [23], [24]) second sound persists to certain degrees at all temperatures, which does not reflect the observed transition to diffusive behaviour above ϑ_λ, ([13], [21]).

Next we provide an analysis of weak and strong discontinuity waves below ϑ_λ in the inviscid limit. We obtain a further, *structural*, critical temperature, ϑ_m, separating two distinct ("hot" and "cold") classes of propagating discontinuity waves. Such a temperature was already found by Ruggeri et al. ([23], [24]) within the context of extended thermodynamics (a comparison of the extended thermodynamics model with that of [17], [3], [4] appears in [5]). However, our approach permits us to establish a direct connection between ϑ_m and ϑ_λ, which was previously not possible. That is, we are able to relate ϑ_λ, the temperature at which $U_E = 0$, to ϑ_m, the temperature at which discontinuity waves fail to propagate. In our analysis, we find that $\vartheta_m < \vartheta_\lambda$. Further, ϑ_m is the temperature corresponding to a failure of genuine nonlinearity of the equations, and ϑ_λ coincides with the temperature at which hyperbolicity itself is lost.

The organization of the paper is as follows. In Sec. 2 we list results of the gradient generalization of heat conduction developed in [19]. In Sec. 3 we introduce constitutive hypotheses based on physically-observed phenomena. In Sec. 4 weakly discontinuous

temperature-rate waves are investigated in the inviscid limit, and shocks are discussed under the same conditions in Sec. 5. Final conclusions are presented in Sec. 6.

2. General framework. We base the model in the present paper on the *generalized semi-empirical* approach developed in [19], but with modifications to the constitutive equations as a result of the existence of the critical temperature, ϑ_λ.

The principal assertion made by *semi-empirical* theories ([17], [19]) is that the absolute temperature ϑ is not by itself sufficient in describing some highly nonequilibrium phenomena, including the observed occurrence of low temperature heat pulses. Thus, besides the temperature, ϑ, a further internal variable, β, is introduced into the constitutive equations. The variable β is in a certain sense a nonequilibrium temperature, related to the absolute temperature through an initial value problem, and represents a history of the temperature field. In the original description of [17] the governing equations are of hyperbolic type. These equations account for various phenomena observed within the range of temperatures where second sound is present. The generalization proposed in [19] introduces viscosity into the equations by allowing dependence of heat conductivity on the spatial gradient of ϑ, as well as on β, ϑ and the gradient of β. This makes it possible to consider such diffusive effects as the observed broadening of travelling pulses of second sound with increasing temperature, as well as providing a physical selection mechanism analogous to that of the viscosity admissibility criterion found in gas dynamics.

In our application, we further develop the principles layed down above by employing the fact that second sound only appears in low temperature regions. The presence of an upper bound, ϑ_λ, for such a region is suggested by experimental measurements of NaF and is consistent with observations of other materials such as superfluid helium. This introduces a phase transition about ϑ_λ separating modes of heat propagation via travelling waves from those via pure diffusion.

A rather general dependence of the free energy ψ was allowed in [19] on the various variables. However, to avoid constraints between ϑ and β, this framework reduces to the following set of constitutive relations:

$$\psi = \psi(\vartheta, \beta, \nabla\beta), \qquad \eta = -\partial_\vartheta \psi(\vartheta, \beta, \nabla\beta), \tag{2.1}$$

$$\boldsymbol{q} = \boldsymbol{q}(\vartheta, \nabla\vartheta, \beta, \nabla\beta), \qquad \dot{\beta} = f(\vartheta, \beta), \tag{2.2}$$

in which the symbol ∇ denotes the gradient operator. Here $\boldsymbol{q}$ is the heat flux vector, η the entropy density, ϑ the absolute temperature, and ψ the free energy per unit volume related to ε, the internal energy per unit volume, by

$$\psi = \varepsilon - \eta\vartheta. \tag{2.3}$$

Balance of energy and the second law of thermodynamics imply

$$\varepsilon_t + \operatorname{div} \boldsymbol{q} = r, \tag{2.4}$$

$$\eta_t + \operatorname{div}(\boldsymbol{q}/\vartheta) \geq r/\vartheta, \tag{2.5}$$

where r is the body heat supply per unit volume. In this case the second law will take the form of the residual inequality

$$-\partial_{\nabla\beta}\psi \cdot \partial_\beta f \nabla\beta - \partial_\beta \psi f - (\partial_{\nabla\beta}\psi \partial_\vartheta f + \vartheta^{-1}\boldsymbol{q}) \cdot \nabla\vartheta \geq 0. \tag{2.6}$$

In the isotropic case, the dependence of $\boldsymbol{q}$ on the gradients $\nabla\vartheta$ and $\nabla\beta$ can take the form

$$\boldsymbol{q} = k\nabla\vartheta + \alpha\nabla\beta, \tag{2.7}$$

where the coefficients k and α may depend on the scalar quantities $\vartheta, \beta, |\nabla\vartheta|, |\nabla\beta|$, and $\nabla\vartheta \cdot \nabla\beta$.

However, as discussed in [5], it becomes reasonable to make the following assumptions while remaining consistent with classical thermostatics, at the same time making it straightforward to use experimental results to identify material functions needed:

- the free energy is independent of β and quadratic in $|\nabla\beta|$,
- the coefficients k and α depend only on ϑ.

Then we have the following representation for the free energy (cp. [5]):

$$\psi = \psi_1(\vartheta) + \tfrac{1}{2}\psi_2(\vartheta)|\nabla\beta|^2 \tag{2.8}$$

and the residual inequality simplifies to the form

$$-\psi_2 \partial_\beta f |\nabla\beta|^2 - (\vartheta^{-1}\alpha + \psi_2 \partial_\vartheta f)\nabla\vartheta \cdot \nabla\beta - \vartheta^{-1}k|\nabla\vartheta|^2 \geq 0. \tag{2.9}$$

We note that the form of (2.8) is one of the consequences of the second law of thermodynamics in the original semi-empirical theory (i.e., when $k = 0$) under the hypothesis that α depends only on ϑ, as we have assumed above.

It is not hard to show that the last inequality will be satisfied for any choice of $\nabla\vartheta$ and $\nabla\beta$ if and only if

$$\partial_\beta f(\vartheta, \beta)\psi_2(\vartheta) \leq 0, \quad k(\vartheta) \leq 0, \tag{2.10}$$

and

$$(\partial_\vartheta f(\vartheta, \beta)\psi_2(\vartheta) + \vartheta^{-1}\alpha(\vartheta))^2 \leq 4\partial_\beta f(\vartheta, \beta)\psi_2(\vartheta)\vartheta^{-1}k(\vartheta). \tag{2.11}$$

The latter inequality should hold for any choice of $k(\vartheta) \leq 0$, in particular for $k(\vartheta) = 0$. This gives the compatibility condition

$$\alpha(\vartheta) = -\vartheta\psi_2(\vartheta)\partial_\vartheta f(\vartheta, \beta) \tag{2.12}$$

(cp. [5]). From (2.12), we obtain the consequence $\partial_\beta \partial_\vartheta f(\vartheta, \beta) = 0$, which leads to the existence of two single-variable functions f_1, f_2, and to the splitting

$$f(\vartheta, \beta) = f_1(\vartheta) + f_2(\beta). \tag{2.13}$$

In this way we have the same set of compatibility conditions as in the previous setup; however, now the heat flux vector can satisfy the more general constitutive equation (2.7).

3. Derivation of the equations. We now specialize to one space dimension and make some refinements in the behaviour of constitutive terms, particularly in the light of experimental evidence concerning NaF, ([14]).

In the absence of a body heat supply, balance of energy, Eq. (2.4) reduces to

$$\varepsilon_t + q_x = 0, \tag{3.1}$$

and using (2.2) and (2.13), the evolution of β is described by

$$\beta_t = f_1(\vartheta) + f_2(\beta). \tag{3.2}$$

The heat flux, (2.7), is given by

$$q = k(\vartheta)\vartheta_x + \alpha(\vartheta)\beta_x, \tag{3.3}$$

while the second law implies

$$\alpha(\vartheta) = -\psi_{20}\vartheta^2 f_1'(\vartheta) \tag{3.4}$$

using (2.12) and the following simple choice:

$$\psi = \psi_1(\vartheta) + \tfrac{1}{2}\psi_{20}\vartheta\beta_x^2 \tag{3.5}$$

for ψ, where $\psi_2(\vartheta) = \psi_{20}\vartheta$, and ψ_{20} is a constant (see (2.8)). In this case ε reduces to a function of ϑ alone, by (2.1) and (2.3).

Finally, we define the specific heat c_v by

$$c_v(\vartheta) = \varepsilon'(\vartheta). \tag{3.6}$$

Combining Eqs. (3.1), (3.3), and (3.6) provides an equation describing the evolution of ϑ, which can be used in conjunction with (3.2) to give a (first-order) system in the pair (ϑ, β),

$$c_v(\vartheta)\vartheta_t + (k(\vartheta)\vartheta_x + \alpha(\vartheta)\beta_x)_x = 0. \tag{3.7}$$

Several important conditions will be made concerning the constitutive terms.

(1) The equations are intended here to adopt parabolic form outside a region $\vartheta \in (0, \vartheta_\lambda), \vartheta_\lambda > 0$, of observed second sound. This is accomplished by allowing the evolution equation (3.2) governing β to decouple from ϑ above ϑ_λ. So, for convenience, we first incorporate ϑ_λ in f_1 by redefining, henceforth,

$$f_1(\cdot - \vartheta_\lambda) \equiv f_1(\cdot), \tag{3.8}$$

and we will assume

$$f_1(z) = 0, \quad z \geq 0. \tag{3.9}$$

Consequently, the governing equations (3.7), (3.2), take on two forms according to whether $\vartheta < \vartheta_\lambda$ or $\vartheta \geq \vartheta_\lambda$. If $\vartheta < \vartheta_\lambda$, then

$$c_v(\vartheta)\vartheta_t + (k(\vartheta)\vartheta_x + \alpha(\vartheta)\beta_x)_x = 0, \tag{3.10}$$

where

$$\vartheta = \vartheta(\xi) = f_1^{-1}(\xi) + \vartheta_\lambda, \qquad \xi = \beta_t - f_2(\beta). \tag{3.11}$$

This system will be considered in the form of a second-order equation β below (see (3.19)).

If $\vartheta \geq \vartheta_\lambda$, then by (3.4), $\alpha(\vartheta) = 0$,

$$c_v(\vartheta)\vartheta_t + (k(\vartheta)\vartheta_x)_x = 0, \tag{3.12}$$

and

$$\beta_t = f_2(\beta). \tag{3.13}$$

Here, Eq. (3.12) is the classical nonlinear diffusion equation one obtains using Fourier's law. Equation (3.13) can now be regarded as describing the evolution of β in a regime in which it has indirect impact (and is here allowed to decay in time) since observed phenomena are covered by the temperature variable there. A continued evolution of β is required to predict the behaviour of materials in which differing, connected spatial regions are at temperatures on both sides of ϑ_λ.

(2) We suppose there exists an invertible *quasistatic* relation between $z = \vartheta - \vartheta_\lambda$ and β, i.e., a homeomorphism $B(z)$ such that

$$f_1(z) + f_2(B(z)) = 0, \quad z \in (-\vartheta_\lambda, 0). \tag{3.14}$$

The function $B(z)$ preserves the order of the variables ϑ and β; hence its derivative, wherever defined, should be positive. Since the derivative of this function is given by

$$B'(z) = \frac{-f_1'(z)}{f_2'(f_2^{-1}(-f_1(z)))} \tag{3.15}$$

and $f_2' < 0$ (see below), we have the condition

$$f_1'(z) > 0, \quad z \in (-\vartheta_\lambda, 0). \tag{3.16}$$

(3) As observed experimentally and explained in the Introduction, the *quasistatic* conductivity, K, becomes large, or possibly unbounded, in absolute value as $\vartheta \to \vartheta_\lambda^-$. In the quasistatic case, β is given by $B(\vartheta - \vartheta_\lambda)$, and $q = K(\vartheta)\vartheta_x$ where $K(\vartheta)$ is given by $k(\vartheta) + \alpha(\vartheta)B'(\vartheta - \vartheta_\lambda) \leq 0$. By (3.4) and (3.15), then, in this case

$$q = (k(\vartheta) + \psi_{20}\vartheta^2\mathcal{F}(\vartheta - \vartheta_\lambda))\vartheta_x \tag{3.17}$$

where

$$\mathcal{F}(z) = \frac{f_1'^2(z)}{f_2'(f_2^{-1}(-f_1(z)))}, \tag{3.18}$$

and we will require either that $\mathcal{F}(z) \to -\infty$ or $|\mathcal{F}(z)|$ becomes large ($\mathcal{F}(z) < 0$) as $z \to 0^-$.

(4) In the inviscid limit ($k(\vartheta) = 0$) the speed of second sound, U_E, vanishes (see Introduction) as $\vartheta \to \vartheta_\lambda^-$. Combining (3.7) and (3.11), one finds that if $k(\vartheta) = 0$,

$$c_v(\vartheta)\vartheta'(\xi)\beta_{tt} + \alpha(\vartheta)\beta_{xx} = c_v(\vartheta)\vartheta'(\xi)f_2'(\beta)\beta_t - \alpha'(\vartheta)\vartheta'(\xi)(\beta_t - f_2(\beta))_x\beta_x. \tag{3.19}$$

Given bounded, smooth initial data for

$$\beta(x, 0) = \beta_0(x),$$

and for

$$\beta_t(x, 0) = f_1(\vartheta(x, 0) - \vartheta_\lambda) + f_2(\beta(x, 0)) = \beta_1(x),$$

this equation leads to a locally well-posed Cauchy problem of quasilinear hyperbolic type, ([6]). The connection with physical data involves using either the dependence of β on

the history of ϑ in (3.2), or the spatial dependence of β on q and ϑ in (3.3) (with $k = 0$ here). In this first approach, one integrates (3.2) from a state of quasistatic equilibrium at $t = -\infty$ to $t = 0$ for $\beta_0(x)$. In the second, initial data for temperature and heat flux are used at $t = 0$ in integrating (3.3) from quasistatic equilibrium at $x = \pm\infty$ to x in order to obtain $\beta_0(x)$.

The expression for the equilibrium second sound speed U_E is the characteristic speed of this damped wave equation linearized by neglecting the higher-order β terms on the right, ([17]). Using the fact that $\vartheta'(\xi) = 1/f_1'(\vartheta - \vartheta_\lambda)$ gives

$$U_E^2(\vartheta) = -\frac{\alpha(\vartheta)}{c_v(\vartheta)\vartheta'(\xi)} = -\frac{\alpha(\vartheta)f_1'(\vartheta - \vartheta_\lambda)}{c_v(\vartheta)} = \frac{\psi_{20}}{c_v(\vartheta)}\vartheta^2 f_1'(\vartheta - \vartheta_\lambda)^2, \tag{3.20}$$

leading to the condition

$$f_1'(0) = 0. \tag{3.21}$$

We remark that the expression for U_E coincides with that of s in the next section, which gives the speed of weakly discontinuous waves propagating into an equilibrium state ($q = 0$).

(5) The specific heat is given by

$$c_v(\vartheta) = c_0\vartheta^r \tag{3.22}$$

where c_0 and r are positive constants. If $r = 3$, this becomes Debye's law.

(6) The possible existence and stability of an equilibrium solution for β. In consideration of Eq. (3.2), we will assume

$$f_2'(z) < 0, \qquad f_2(0) = 0. \tag{3.23}$$

As concrete examples, let us define two C^1-homeomorphisms $f_1 \colon \mathbb{R} \to (-\infty, 0]$, and $f_2 \colon \mathbb{R} \to \mathbb{R}$. For the first, we set

$$f_1(z) = a(|z|^{p-1}z)_-, \quad p > 1, \tag{3.24}$$

where a is a positive constant, and the subscript "$-$" means that when $z \geq 0$, f_1 is taken to be zero. The restriction on p allows f_1 to satisfy (3.21). For the second, put

$$f_2(z) = -b|z|^{q-1}z, \quad q > 0, \tag{3.25}$$

where b is another positive constant, and $q > 0$ in order to satisfy (3.23).

The inverse of f_2 takes the form

$$f_2^{-1}(y) = -\frac{1}{b^{1/q}}y|y|^{(1/q)-1}, \tag{3.26}$$

which leads to $B(z), B'(z)$ reading

$$B(z) = (a/b)^{1/q}(z|z|^{(p/q)-1})_-, \qquad B'(z) = (a/b)^{1/q}p/q|z_-|^{(p/q)-1}. \tag{3.27}$$

For the function $\mathcal{F}(z)$ (see (3.18)) we have the following expression:

$$\mathcal{F}(z) = -c|z_-|^{p(1+1/q)-2} \tag{3.28}$$

where c is a positive constant depending on a, b, p, and q. Since $q > 0$, for heat conductivity to be significant as ϑ approaches ϑ_λ, that is, for i) $\mathcal{F}(z) \not\to 0$ as $z \to 0^-$, or ii) $|\mathcal{F}(z)| \to \infty$ as $z \to 0^-$, we require $p < 2$ and $q \geq p/(2-p)$.

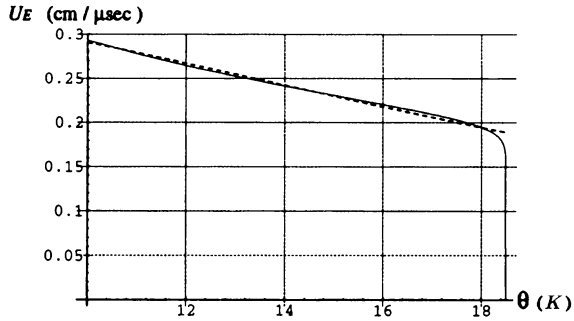

FIG. 1. Characteristic velocity (solid curve), $U_E = 0.85(18.5 - \vartheta)^{0.04}/\vartheta^{0.5}$, ahead of wave for $p = 1.04$, $\vartheta_\lambda = 18.5$, together with empirical data (dotted curve), $U_E = (9.09 + 0.00222\vartheta^{3.1})^{-0.5}$ (Coleman and Newman, [7]).

In summary, to satisfy the conditions given for f_1, f_2, and $\mathcal{F}$, the particular choices made in (3.24) and (3.25) lead to the restrictions on p and q that

$$1 < p < 2, \qquad p/(2-p) \leq q. \tag{3.29}$$

Raw data for $U_E(\vartheta)$ is given for crystals of NaF of varying purity in [13], with an empirical relation, $U_E = (9.09 + 0.00222\vartheta^{3.1})^{-0.5}\text{cm}/\mu\text{sec}$ provided in [7]. The measure for ϑ_λ of $18.5K$ is taken at the upper limit for which second sound in NaF has been found to exist (Table II of [13]). The dependence of conductivity on temperature and purity is also described in [14], temperature of peak conductivity increasing with purity. The purest sample had a peak in conductivity at close to $18.5K$ which we let here coincide with ϑ_λ, below which second sound waves begin to appear. In Figure 1, we have the same behaviour below ϑ_λ as in the ad hoc form of $U_E(\vartheta)$, ([7], [12]), used to interpolate the available experimental data. We have obtained this behaviour using the example for f_1 above when $p = 26/25$. The rapid drop at $18.5K$ reflects our assumption that U_E vanishes at ϑ_λ. On reaching this temperature the pulse disappears into the diffusive signal.

The choices we have made for f_1 and f_2 in this special case lead to a finite conductivity peak for $K(\vartheta)$ as $\vartheta \to 18.5K$ if $q = 13/12$, and to infinite conductivity in the same limit if $q > 13/12$. The definition of $f_1(\vartheta - \vartheta_\lambda)$, (3.24), then makes the value of $K(\vartheta)$ drop to $k(\vartheta)$ for $\vartheta > \vartheta_\lambda$. Precise details of the form of $k(\vartheta)$ may be found in the future through the analysis of observed waveform broadening. In the following sections, we investigate the inviscid limit $k(\vartheta) = 0$ further.

4. Weakly discontinuous temperature-rate waves. In this section, we will consider solutions $\beta, \vartheta \in C^0, \vartheta < \vartheta_\lambda$, to Eq. (3.19). We note that the evolution equation for β, (3.2), immediately implies that $\beta_t \in C^0$. We define a one-dimensional temperature-rate wave as a smooth curve $x = \varphi(t)$, with speed of propagation

$$s(t) = \dot{\varphi}(t) \tag{4.1}$$

across which a function ϕ may have a jump

$$[\![\phi]\!](x) = \phi(x-) - \phi(x+). \tag{4.2}$$

The directional derivative of $[\![\phi]\!]$ is given by

$$\frac{d}{dt}[\![\phi]\!] = [\![\phi_t]\!] + s[\![\phi_x]\!]. \tag{4.3}$$

By (4.3) and our previous remarks, we observe that $\beta_x \in C^0$. In general, however, the following terms are discontinuous on $\varphi(t)$,

$$[\![\beta_{tt}]\!] = f_1'(\vartheta^+ - \vartheta_\lambda)[\![\vartheta_t]\!] \neq 0. \tag{4.4}$$

We now rewrite Eq. (3.19) in the form

$$c_v(\vartheta)\beta_{tt} + \alpha'(\vartheta)\beta_x\beta_{xt} + \alpha(\vartheta)f_1'(\vartheta - \vartheta_\lambda)\beta_{xx} - c_v(\vartheta)f_2'(\beta)\beta_t - \alpha'(\vartheta)f_2'(\beta)\beta_x^2 = 0, \tag{4.5}$$

where ϑ is considered a function of β and β_t using Eq. (3.11). Evaluating Eq. (4.5) across $\varphi(t)$ gives an equation for the speed of the temperature-rate waves,

$$s^2 c_v(\vartheta^+) - s\alpha'(\vartheta^+)\beta_x^+ + \alpha(\vartheta^+)f_1'(\vartheta^+ - \vartheta_\lambda) = 0. \tag{4.6}$$

Here we used (4.3) to obtain the identities

$$[\![\beta_{xt}]\!] = -1/s[\![\beta_{tt}]\!] \tag{4.7}$$

and

$$[\![\beta_{xx}]\!] = 1/s^2[\![\beta_{tt}]\!] \tag{4.8}$$

for the continuous functions β_t, β_x. Next, we assume that the wave propagates into a region of quasistatic equilibrium at constant temperature, i.e., for $x \geq \varphi(t)$,

$$\beta_t(x,t) \equiv 0, \qquad \vartheta(x,t) \equiv \vartheta^+, \tag{4.9}$$

where $\vartheta^+ < \vartheta_\lambda$ is a positive constant. It follows from (3.2) that $f_2'(\beta)\beta_x \equiv 0$ in this region; hence $\beta_x(x,t) \equiv 0$ and so the wave moves into a region at constant β, i.e., if $x \geq \varphi(t)$,

$$\beta(x,t) \equiv \beta^+, \tag{4.10}$$

where, by (3.14), $\beta^+ = B(\vartheta^+ - \vartheta_\lambda)$.

In this case, we find

$$s^2 = -\frac{\alpha(\vartheta^+)f_1'(\vartheta^+ - \vartheta_\lambda)}{c_v(\vartheta^+)} = U_E^2(\vartheta^+) = \text{constant}. \tag{4.11}$$

Let us define the amplitude of the temperature-rate wave, m, by

$$m(t) = [\![\beta_{tt}]\!](t). \tag{4.12}$$

In order to derive an equation describing the amplitude evolution in time along $x = \varphi(t)$, we differentiate Eq. (4.5) with respect to time and evaluate it across the wave. Using the relations

$$[\![\beta_{ttt}]\!] = \frac{d}{dt}m - s[\![\beta_{xtt}]\!] \tag{4.13}$$

and

$$[\![\beta_{xxt}]\!] = -1/s^2 \frac{d}{dt} m - 1/s[\![\beta_{xtt}]\!], \tag{4.14}$$

which came from (4.3), (4.7), and (4.8), one obtains a differential equation for m with the help of (4.11),

$$\frac{d}{dt} m + \frac{1}{2\vartheta^+ f_1'^2} \left\{ \frac{c_v'}{c_v} \vartheta^+ f_1' - (4f_1' + 3\vartheta^+ f_1'') \right\} m^2 - \tfrac{1}{2} f_2' m = 0. \tag{4.15}$$

All the coefficients in this equation are constants, evaluated at $\varphi(t)^+$ in the undisturbed region in front of the wave, as in (4.9), (4.10), and so the f_1 terms are functions of $\vartheta^+ - \vartheta_\lambda$, and f_2 is a function of β^+. We will use the quasistatic relation, (3.14), to write β^+ in terms of $\vartheta^+ - \vartheta_\lambda$ when necessary.

On differentiating Eq. (3.2) with respect to time, it follows that

$$m = f_1'(\vartheta^+ - \vartheta_\lambda)[\![\vartheta_t]\!]. \tag{4.16}$$

In order to simplify the discussion, we normalize the amplitude and work with $\overline{m}$ defined by

$$\overline{m} = [\![\vartheta_t]\!] = \frac{m}{f_1'(\vartheta^+ - \vartheta_\lambda)}. \tag{4.17}$$

For the new amplitude we obtain

$$\frac{d\overline{m}}{dt} + \frac{1}{2\vartheta^+ f_1'} \left\{ \frac{c_v'}{c_v} \vartheta^+ f_1' - (4f_1' + 3\vartheta^+ f_1'') \right\} \overline{m}^2 - \tfrac{1}{2} f_2' \overline{m} = 0. \tag{4.18}$$

Next we discuss the case when the specific heat is of the form (3.22), f_1 is given by (3.24), and f_2 by (3.25). In this case (4.18) reduces to the form

$$\frac{d}{dt}\overline{m} + \mathcal{A}\overline{m}^2 + \mathcal{B}\overline{m} = 0 \tag{4.19}$$

where

$$\begin{aligned} \mathcal{A} &= \frac{(r-3p-1)\vartheta^+ - (r-4)\vartheta_\lambda}{2\vartheta^+(\vartheta^+ - \vartheta_\lambda)}, \\ \mathcal{B} &= -\tfrac{1}{2} f_2'(\beta^+) = \frac{a^{1-1/q}}{2b^{1/q}} q|\vartheta^+ - \vartheta_\lambda|^{p-p/q} > 0. \end{aligned} \tag{4.20}$$

We note that the $\overline{m}^2$ term in (4.18) has a coefficient that may change sign at a particular value of ϑ, which we denote by ϑ_m. This is in contrast with previous results concerning temperature-rate waves ([18], [8]).

For any p satisfying (3.29) we obtain the following expression for ϑ_m from $\mathcal{A}$,

$$\vartheta_m = \frac{4-r}{3p+1-r} \vartheta_\lambda, \tag{4.21}$$

and we can see that this value is always positive and less than ϑ_λ, provided that $r < 4$. In the case when $r = 4$ the coefficient standing in front of $\overline{m}^2$ has constant sign in the range of temperature in which we are interested, i.e., when $\vartheta \in (0, \vartheta_\lambda)$. If the exponent r exceeds 4 then the coefficient also does not change sign in $(0, \vartheta_\lambda)$; however, we do not know of physical reasons to consider such values of r, and so we confine our attention to the range $r \in [1, 4]$ which includes the important case of Debye's law, $r = 3$.

Now, since $\mathcal{B} > 0$, it is straightforward to find that solutions $\overline{m}(t)$ of (4.19) tend to infinity in finite time if and only if $\mathcal{A}\overline{m}_0 + \mathcal{B} < 0$, where $\overline{m}_0 = \overline{m}(0)$. This immediately leads to the following results on the finite-time blowup of temperature-rate waves, where we define a *hot* temperature-rate wave as one for which $\overline{m}_0 = \vartheta_t^-(0) > 0$, and a *cold* temperature-rate wave as one for which $\overline{m}_0 < 0$.

THEOREM 1. Let $\vartheta^+ < \vartheta_\lambda$ be the temperature in front of the temperature-rate wave, and let $\overline{m}_0 = \vartheta_t^-(0)$. Then,

1. for $\vartheta_m < \vartheta^+ < \vartheta_\lambda$ (i.e., $\mathcal{A} > 0$), the amplitude $\overline{m}(t)$ blows up in finite time if $\overline{m}_0 < -\frac{B}{A} < 0$ (hot temperature-rate wave);
2. for $0 < \vartheta^+ < \vartheta_m$ (i.e., $\mathcal{A} < 0$), the amplitude $\overline{m}(t)$ blows up in finite time if $\overline{m}_0 > -\frac{B}{A} > 0$ (cold temperature-rate wave);
3. for $\vartheta^+ = \vartheta_m$ (i.e., $\mathcal{A} = 0$), the amplitude $\overline{m}(t)$ is a decreasing function of time, and blowup does not occur.

REMARK. It is interesting to notice that by taking $\beta_x \neq 0$ in front of the wave, an apparent structural instability in the equations may be revealed. In this case, under the condition $1 < p < 2$, Eq. (4.6) shows the existence of two waves with different speeds as $\vartheta \to \vartheta_\lambda^-$. One of these has unbounded speed in the limit, while the other tends to zero as before. Because both wave speeds previously tended to zero one observes that, in the limit, s cannot depend continuously on the state in front of the wave.

5. Shocks. In this section, we consider solutions $\beta \in C^0$, for which $\vartheta \notin C^0$, $\vartheta < \vartheta_\lambda$. β_t, β_x, and ϑ will admit jumps, and a one-dimensional shock is defined, in an analogous fashion to Eqs. (4.1)–(4.3), as a smooth curve $x = \varphi(t)$, with speed of propagation now denoted by

$$\sigma(t) = \dot{\varphi}(t). \tag{5.1}$$

We will examine the case of shocks propagating into a region, $x \geq \sigma(t)$, of quasistatic equilibrium at constant temperature, $\beta_t(x,t) \equiv 0$, $\vartheta(x,t) \equiv \vartheta^+$, in which $\beta^+ = B(\vartheta^+ - \vartheta_\lambda)$ is constant (see Eqs. (4.9), (4.10)).

As in the previous section, we are assuming $k(\vartheta) = 0$. Because the case $k(\vartheta) < 0$ induces viscosity in the equations, this can be regarded as a mechanism to avoid nonuniqueness of weak solutions. However, in this section we will pursue instead the consequences of the Lax shock criterion on admissibility, [9], [23].

The Rankine-Hugoniot relations obtained from the weak formulations of (3.1) and (3.2) read

$$\sigma[\![\varepsilon]\!] - [\![q]\!] = 0, \tag{5.2}$$

$$\sigma[\![\beta]\!] = 0. \tag{5.3}$$

Equations (3.2) and (3.8) also imply

$$[\![\beta_t]\!] = [\![f_1(\vartheta - \vartheta_\lambda)]\!], \tag{5.4}$$

and as long as $\sigma \neq 0$, the continuity of β (Eq. (5.3)) implies

$$[\![\beta_t]\!] + \sigma[\![\beta_x]\!] = 0 \tag{5.5}$$

(see (4.3)). Finally $q(x,t) = 0$ ahead of the wave as a consequence of (3.3), since the wave propagates into a constant temperature, quasistatic region.

Combining (5.2), (5.4), and (5.5) leads to the following expression for the speed of the shock wave:

$$\sigma^2 = -\alpha(\vartheta^-)\frac{[\![f_1(\vartheta - \vartheta_\lambda)]\!]}{[\![\varepsilon(\vartheta)]\!]}. \tag{5.6}$$

Since σ must satisfy Lax's admissibility condition, then

$$s_r \leq \sigma \leq s_l, \tag{5.7}$$

where s_r and s_l denote the characteristic speeds, respectively, in front of and behind the shock. Here s_r is the same as the speed of temperature-rate waves, (4.11),

$$s_r^2(\vartheta^+) = -\frac{\alpha(\vartheta^+)f_1'(\vartheta^+ - \vartheta_\lambda)}{c_v(\vartheta^+)}, \tag{5.8}$$

and $s_l(\vartheta^+, \vartheta^-)$ satisfies Eq. (4.6) with ϑ^+, β_x^+ replaced by ϑ^- and β_x^-.

Using (5.3), (5.4), and (5.5), the latter equation can be written more conveniently as

$$s_l^2(\vartheta^+, \vartheta^-) - s_l(\vartheta^+, \vartheta^-)\mathcal{K}(\vartheta^+, \vartheta^-)\sigma(\vartheta^+, \vartheta^-) - s_r^2(\vartheta^-) = 0, \tag{5.9}$$

with

$$\mathcal{K}(\vartheta^+, \vartheta^-) = \frac{\alpha'(\vartheta^-)[\![\varepsilon]\!]}{\alpha(\vartheta^-)c_v(\vartheta^-)}. \tag{5.10}$$

In the remainder of this section we will take $\varepsilon = \frac{1}{4}c_0\vartheta^4$, $c_v(\vartheta) = c_0\vartheta^3$ (Debye's law), $\alpha(\vartheta) = -\psi_{20}\vartheta^2 f_1'(\vartheta - \vartheta_\lambda)$ and $f_1(z) = a(|z|^{p-1}z)_-$, $1 < p < 2$, (see (3.4), (3.6), (3.24), and (3.29)). This reduces $\mathcal{K}$ to the form

$$\frac{(p+1)(\vartheta_p - \vartheta^-)}{4(\vartheta_\lambda - \vartheta^-)}\left(1 - \frac{\vartheta^{+^4}}{\vartheta^{-^4}}\right) \tag{5.11}$$

where $\vartheta_p = (2/(p+1))\vartheta_\lambda$.

For our analysis, we will suppose that $s_r(\vartheta^+) > 0$ (so $\sigma > 0$ and $s_l > 0$ by (5.7)) and we begin by examining the lower extreme of the inequalities (5.7), $\sigma = s_r$.

Fixing $\vartheta^+ < \vartheta_\lambda$ and taking the difference between Eqs. (5.7) and (5.5) gives

$$s_r^2(\vartheta^+) - \sigma^2(\vartheta^+, \vartheta^-) = \frac{Q(\vartheta^+, \vartheta^-)}{\vartheta^+(\vartheta^{-^4} - \vartheta^{+^4})} \tag{5.12}$$

where

$$Q(\vartheta^+, \vartheta^-) \tag{5.13}$$

$$= p(\vartheta_\lambda - \vartheta^+)^{2(p-1)}(\vartheta^{-^4} - \vartheta^{+^4}) - 4\vartheta^+\vartheta^{-^2}(\vartheta_\lambda - \vartheta^-)^{p-1}((\vartheta_\lambda - \vartheta^+)^p - (\vartheta_\lambda - \vartheta^-)^p). \tag{5.14}$$

It is easily seen that Q has the following properties:

$$Q(\vartheta^+,0)<0, \qquad Q(\vartheta^+,\vartheta^+)=0, \qquad Q(\vartheta^+,\vartheta_\lambda)>0, \tag{5.15}$$

$$\frac{\partial Q}{\partial \vartheta^-}(\vartheta^+,\vartheta^+)=0, \tag{5.16}$$

$$\frac{\partial^2 Q}{\partial \vartheta^{-2}}(\vartheta^+,\vartheta^+)<0 \quad \text{if } \vartheta^+<\vartheta_m, \tag{5.17}$$

$$\frac{\partial^2 Q}{\partial \vartheta^{-2}}(\vartheta^+,\vartheta^+)>0 \quad \text{if } \vartheta^+>\vartheta_m, \tag{5.18}$$

$$\frac{\partial^2 Q}{\partial \vartheta^{-2}}(\vartheta_m,\vartheta_m)=0, \tag{5.19}$$

where ϑ_m is given by (4.21) with $r=3$. Thus, in particular, from (5.15) and (5.16), $\sigma\to s_r$ as $\vartheta^-\to\vartheta^+$. These properties also imply that there exists a further temperature, $\vartheta_*\neq\vartheta^+, \vartheta_*<\vartheta_\lambda$, where $Q(\vartheta^+,\vartheta_*)=0$ and $\sigma(\vartheta^+,\vartheta_*)=s_r(\vartheta^+)$. An example of this can be seen in Figs. 2(i)–(iii) below. The following results can also be established straightforwardly.

LEMMA 1. For a given temperature ϑ^+ in the quasistatic region, there exists $\vartheta_*\neq\vartheta^+$ such that

$$\vartheta_*>\vartheta^+, \quad \text{if } \vartheta^+<\vartheta_m, \tag{5.20}$$

$$\vartheta_*<\vartheta^+, \quad \text{if } \vartheta^+>\vartheta_m, \tag{5.21}$$

where

$$\sigma(\vartheta^+,\vartheta^-)>s_r(\vartheta^+), \quad \text{if } \vartheta^-\in(\vartheta^+,\vartheta_*), \tag{5.22}$$

$$\sigma(\vartheta^+,\vartheta^-)>s_r(\vartheta^+), \quad \text{if } \vartheta^-\in(\vartheta_*,\vartheta^+). \tag{5.23}$$

The relations (5.22) and (5.23) imply that the left side of (5.7) will be satisfied for ϑ^- lying in the interval between ϑ^+ and ϑ_*.

Next we examine the upper extreme of (5.7), $\sigma=s_l$.

Suppose there exists some $\vartheta_{**}\neq\vartheta^+$ at which $\sigma=s_l$. In order to find ϑ_{**}, we consider the substitution $\sigma=s_l$ in (5.9), which leads to consideration of the term

$$\chi(\vartheta^+,\vartheta^-)=\sigma^2(\vartheta^+,\vartheta^-)(1-\mathcal{K}(\vartheta^+,\vartheta^-))-s_r^2(\vartheta^-). \tag{5.24}$$

Using (5.11), we define

$$\begin{aligned} P(\vartheta^+,\vartheta^-) &= \frac{c_0}{\psi_{20}a^2p}(\vartheta_\lambda-\vartheta^-)^{p-2}(\vartheta^{-^4}-\vartheta^{+^4})\vartheta^{-^2}\chi(\vartheta^+,\vartheta^-) \\ &= 4\vartheta^{-^4}(\vartheta_\lambda-\vartheta^-)((\vartheta_\lambda-\vartheta^+)^p-(\vartheta_\lambda-\vartheta^-)^p) \\ &\quad -(p+1)(\vartheta_p-\vartheta^-)((\vartheta_\lambda-\vartheta^+)^p-(\vartheta_\lambda-\vartheta^-)^p)(\vartheta^{-^4}-\vartheta^{+^4}) \\ &\quad -p\vartheta^-(\vartheta_\lambda-\vartheta^-)^p(\vartheta^{-^4}-\vartheta^{+^4}). \end{aligned} \tag{5.25}$$

P has similar properties to Q, namely

$$P(\vartheta^+, 0) < 0, \qquad P(\vartheta^+, \vartheta^+) = 0, \qquad P(\vartheta^+, \vartheta_\lambda) > 0, \tag{5.26}$$

$$\frac{\partial P}{\partial \vartheta^-}(\vartheta^+, \vartheta^+) = 0, \tag{5.27}$$

$$\frac{\partial^2 P}{\partial \vartheta^{-2}}(\vartheta^+, \vartheta^+) < 0 \quad \text{if } \vartheta^+ < \vartheta_m, \tag{5.28}$$

$$\frac{\partial^2 P}{\partial \vartheta^{-2}}(\vartheta^+, \vartheta^+) > 0 \quad \text{if } \vartheta^+ > \vartheta_m, \tag{5.29}$$

$$\frac{\partial^2 P}{\partial \vartheta^{-2}}(\vartheta_m, \vartheta_m) = 0 \tag{5.30}$$

Using the definition of $\mathcal{K}$ in (5.11), it follows that $\chi(\vartheta^+, \vartheta^+) = 0$, and the above properties for P lead to the result $\sigma \to s_l$ as $\vartheta^- \to \vartheta^+$. Further,

$$\vartheta_{**} > \vartheta^+, \quad \text{if } \vartheta^+ < \vartheta_m, \tag{5.31}$$

$$\vartheta_{**} < \vartheta^+, \quad \text{if } \vartheta^+ > \vartheta_m, \tag{5.32}$$

as with ϑ_* (see Figs. 2(i)–(iii)). To show that $\sigma < s_l$ for values of ϑ^- lying between ϑ^+ and ϑ_{**}, we use the following simple comparison result.

For fixed ϑ^+, write Eq. (5.9) in the form

$$F(x(t), t) = x^2(t) - x(t)a(t)b(t) - c^2(t) = 0, \tag{5.33}$$

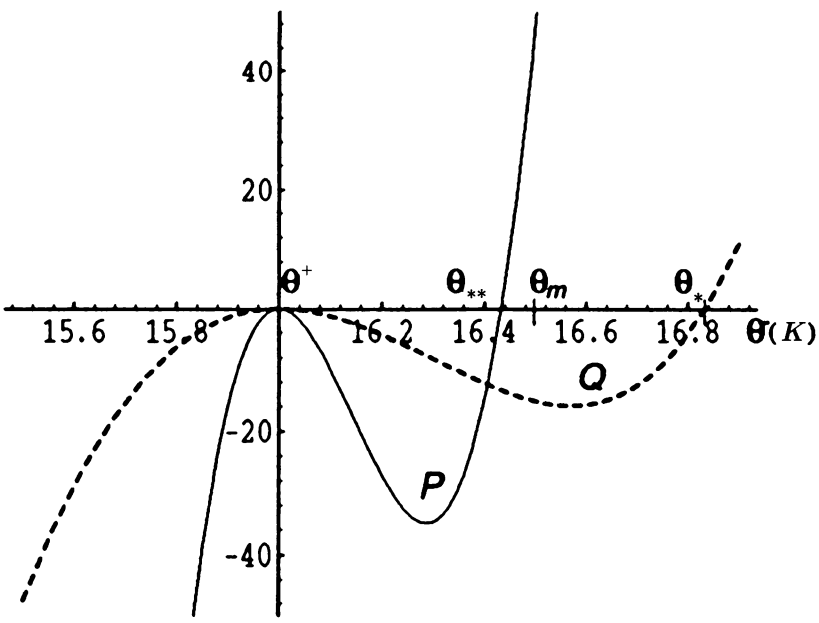

FIG. 2(i). Plots of P and Q with $p = 1.04$, $\vartheta_m = 16.518$, and $\vartheta_\lambda = 18.5$. $\vartheta^+ = 16K < \vartheta_m$ ("hot" shock).

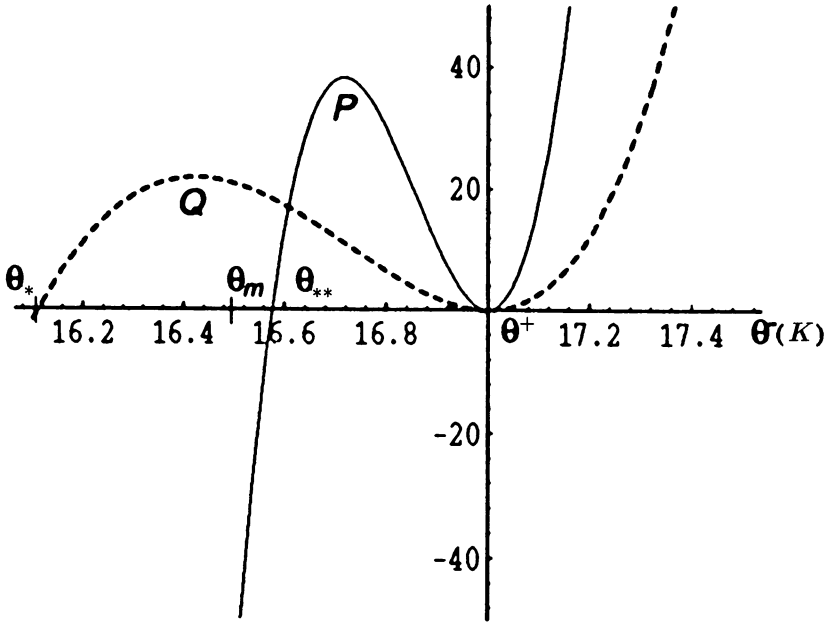

FIG. 2(ii). Plots of P and Q with $p = 1.04$, $\vartheta_m = 16.518$, and $\vartheta_\lambda = 18.5$. $\vartheta^+ = 17K < \vartheta_m$ ("cold" shock).

where t represents the independent variable ϑ^-, $x > 0$, a, b, and c represent $s_l, \mathcal{K}, \sigma$, and s_r, respectively. Consequently (cp. (5.24)),

$$F(b(t), t) = b^2(t) - a(t)b^2(t) - c^2(t). \tag{5.34}$$

Since χ has two distinct roots $\vartheta^- = \vartheta^+, \vartheta_{**}$ for $\vartheta^+ \neq \vartheta_m$, F has two corresponding roots, t_1, t_2, i.e.,

$$F(b(t_1), t_1) = F(b(t_2), t_2) = 0. \tag{5.35}$$

Next, assuming that $F(b(t), t) < 0$ for t between t_1 and t_2 (cp. (5.25)–(5.29)), here

$$b^2(t) - a(t)b^2(t) - c^2(t) < 0 \tag{5.36}$$

and

$$x^2(t) - x(t)a(t)b(t) - c^2(t) = 0. \tag{5.37}$$

Combining these gives

$$(b(t) - x(t))(b(t) + x(t) - a(t)b(t)) < 0 \tag{5.38}$$

for t in this region. However, since $x(t) > 0$ is a solution to (5.37), (5.38) can only be satisfied provided $b(t) < x(t)$ and $b(t) + x(t) > a(t)b(t)$, and so in the original variables, $\sigma < s_l$ for ϑ^- lying between ϑ^+ and ϑ_{**}. Based on the above, an analogue to Lemma 1 can be obtained.

LEMMA 2. For a given temperature ϑ^+ in the quasistatic region, there exists $\vartheta_{**} \neq \vartheta^+$ such that

$$\vartheta_{**} > \vartheta^+, \quad \text{if } \vartheta^+ < \vartheta_m, \tag{5.39}$$

$$\vartheta_{**} < \vartheta^+, \quad \text{if } \vartheta^+ > \vartheta_m, \tag{5.40}$$

where

$$\sigma(\vartheta^+, \vartheta^-) < s_l(\vartheta^+, \vartheta^-), \quad \text{if } \vartheta^- \in (\vartheta^+, \vartheta_{**}), \tag{5.41}$$

$$\sigma(\vartheta^+, \vartheta^-) < s_l(\vartheta^+, \vartheta^-), \quad \text{if } \vartheta^- \in (\vartheta_{**}, \vartheta^+). \tag{5.42}$$

We therefore have that Q and P define two intervals $\vartheta \in (\vartheta^+ - \vartheta_*, \vartheta^+ + \vartheta_*)$ and $\vartheta \in (\vartheta^+ - \vartheta_{**}, \vartheta^+ + \vartheta_{**})$. The admissibility condition (5.7) holds throughout whichever is the smaller of these. Moreover, numerical and analytic results (the latter valid for small values of $|\vartheta^+ - \vartheta_m|$) show that generally $(\vartheta^+ - \vartheta_{**}, \vartheta^+ + \vartheta_{**}) \subseteq (\vartheta^+ - \vartheta_*, \vartheta^+ + \vartheta_*)$. The value ϑ_{**} can also be compared with ϑ_m in the sense that for $p > \sim 1.04$ one finds

$$\vartheta^+ < \vartheta_m \Rightarrow \vartheta^+ < \vartheta_{**} < \vartheta_m \tag{5.43}$$

and

$$\vartheta^+ > \vartheta_m \Rightarrow \vartheta^+ > \vartheta_{**} > \vartheta_m. \tag{5.44}$$

These results are analogous to those of Ruggeri et al, ([23], [24]), in the sense that there exists a temperature, ϑ_m, below which (admissible) "hot" shocks propagate and above which "cold" shocks propagate. In other words, if ϑ^+ is the temperature in front of the wave, then for $\vartheta^+ < \vartheta_m$, the temperature behind the wave lies between ϑ^+ and

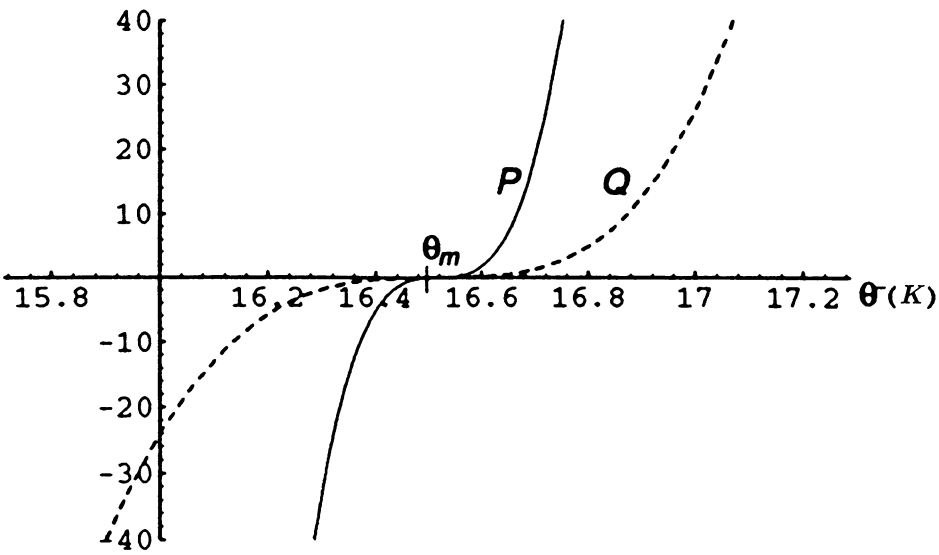

FIG. 2(iii). Plots of P and Q with $p = 1.04$, $\vartheta_m = 16.518$, and and $\vartheta_\lambda = 18.5$. $\vartheta^+ = \vartheta_m$ (no admissible region).

ϑ_{**} and is greater than ϑ^+. If $\vartheta^+ > \vartheta_m$, the temperature behind the wave is between ϑ^+ and ϑ_{**}, which is less than ϑ^+. In the case $\vartheta^+ = \vartheta_m$, no shock exists. The results are also consistent with those concerning the blowup of temperature-rate waves in the previous section.

Our interpretation of ϑ_m and ϑ_λ is however different to that in [23] and [24]. By introducing ϑ_λ into our model, we were able to respond to the speculation made by the authors of [23], [24] that the critical temperature, ϑ_m, could play the role of a λ-point, as in superfluid helium. Our analysis indicates that this is not the case, $\vartheta_m \neq \vartheta_\lambda$; however, these temperatures are related by (4.21). From Eq. (4.21), the value for ϑ_m predicted by the choice of $\vartheta_\lambda = 18.5K$ from Table II in [13], is $16.5K$. In comparison, ϑ_m is found in [23] as $15.36K$. In both approaches, given Debye's law (3.22), the only measurement required in computing ϑ_m is $U_E(\vartheta)$. Although our values for $U_E(\vartheta)$ intersect those of [23] over a range where experiments have been carried out, the computed value of ϑ_m depends in part on the extrapolation of U_E outside this range. The interpolating function for U_E from [7], used in [23], is designed to fit experimental data, but does not have a physical basis at higher temperatures (see [7], p. 1495). Instead, our use of the cut-off at ϑ_λ changes the global definition of $U_E(\vartheta)$ and this contributes to a different value for θ_m.

6. Conclusions. The analysis performed shows

- in the present framework one can specify two material functions, f_1 and f_2, and use them to model two observed phenomena:

  i) the nearly linear temperature dependence (Fig. 1) of the speed of heat pulses in a range of temperatures, together with a rapid decay of this speed at a critical temperature, ϑ_λ, and

  ii) a peak in the heat conductivity at the same critical temperature;
- the connection of ϑ_m, which separates two distinct families of shock waves, with ϑ_λ.

We have employed an approach that has some very useful flexibility. The introduction of separate temperature regimes for heat conduction provides the possibility of describing further phenomena related to ballistic phonons and second sound as discussed in [10], including broadening of smooth heat pulses, ([13], [14]), and phase transition to diffusive behaviour at higher temperature.

Acknowledgment. Part of this research has been conducted during a one month stay of one of the authors (W. K.) at the Department of Mathematics and Computer Science of Loyola University, New Orleans, in the summer of 1995. His stay and research were supported by NSF Grant GER-9450202 at Loyola University.

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