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# TIME VERSUS DISTANCE FOR THE PROPAGATION OF HEAT IN BOUNDED DOMAINS

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**Introduction.** In an important paper [1] G. Fichera defended the classical Fourier theory of heat propagation against the charge that it leads to the paradox that heat propagates at infinite speed. Fichera drew attention to an observation of J. C. Maxwell's to the effect that the sensible propagation of heat, so far from being instantaneous, is an excessively slow process, and the time taken for the bulk of the heat to propagate is proportional to the square of the distance.

In [2, 3] I have proved certain theorems that are in line with Maxwell's observation and lend further support to the classical theory over and above that provided by Fichera. Both Fichera's work and my own were concerned solely with the propagation of heat in all of space ($\mathbf{R}^1$ in [1] and [3], but $\mathbf{R}^n$ in [2]).

I turn here to studying heat propagation on a bounded domain $\Omega$ in $\mathbf{R}^n$. The boundary $\partial\Omega$ is maintained at zero temperature, and there is a compact subset $A$, at positive distance from the boundary, such that the initial temperature is nonnegative on $A$ but zero outside $A$. Thus the temperature $u(x,t)$ is the solution of an initial-and-boundary value problem

$$\begin{aligned} \Delta u = \frac{\partial u}{\partial t} \quad & \text{on } \Omega \times (0,\infty), \\ u = 0 \quad & \text{on } \partial\Omega \times (0,\infty), \\ u|_{t=0} = f \quad & \text{on } \Omega, \end{aligned} \tag{1}$$

where $f$ is continuous, $f(x) \geq 0$ for $x \in A$ and $f(x) = 0$ for $x \in \Omega \backslash A$. The temperature is unique and can be constructed by using separation of variables.

The net heat flux out of $\Omega$ is

$$Q = -\int_{\partial\Omega} \partial_\nu u \, d\sigma, \tag{2}$$

the $n$-vector $\nu$ being the exterior unit normal and $d\sigma$ the element of hypersurface area. The net heat flux is positive and tends to zero at an exponential rate as $t \to \infty$. It might

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be expected that all the heat within $\Omega$ would flow out eventually, that is to say,

$$\int_0^\infty Q(t)dt = \int_\Omega f(x)\,dx, \tag{3}$$

where $dx$ is the element of hypervolume; this expectation will be confirmed at a later stage.

The purpose of this note is to prove a theorem which, in the context of bounded domains, lends precision to Maxwell's observation. The theorem exhibits explicitly a time interval $\tau < t < T$ on which all but a fraction $\varepsilon$ of the total heat is emitted. It turns out that the time $\tau$ is proportional to $r^2$, where

$$r = \inf\{\|x - y\| : x \in A, y \in \partial\Omega\}$$

is the distance from $A$ to $\partial\Omega$, while the time $T$ is proportional to $R^2$, $R$ being the radius of the smallest ball that contains $\Omega$.

THEOREM. Let $\varepsilon$ be any number lying in the interval $0 < \varepsilon < 1$. Then there exists a constant $c(n)$, depending only upon the dimension $n$, such that if

$$\tau = r^2 \Big/ \left(16 \log\left[\frac{c(n)}{\varepsilon}\right]\right), \qquad T = R^2/(n\varepsilon), \tag{4}$$

then

$$\int_\tau^T Q(t)\,dt > (1-\varepsilon)\int_0^\infty Q(t)\,dt.$$

Moreover, $c(1) = c(2) = c(3) = c(4) = 4$ but $c(n) > 4$ if $n > 4$.

*Proof of the theorem.* It will be enough to establish the inequalities

$$\int_0^\tau Q(t)dt \le \frac{\varepsilon}{2}\int_0^\infty Q(t)\,dt, \tag{5}$$

$$\int_T^\infty Q(t)dt \le \frac{\varepsilon}{2}\int_0^\infty Q(t)\,dt. \tag{6}$$

The proof of these will be made to depend upon an examination of the Laplace transform

$$\overline{u}(x,s) = \int_0^\infty e^{-st}u(x,t)\,dt,$$

but at all stages of the argument the variable $s$ is restricted to be real and nonnegative.

The Laplace transform is the solution of the boundary-value problem

$$\Delta\overline{u} = s\overline{u} - f \quad \text{on } \Omega, \qquad \overline{u} = 0 \quad \text{on } \partial\Omega.$$

If we introduce the solution, $G(x,s)$, of the boundary-value problem

$$\Delta G = sG - 1 \quad \text{on } \Omega, \qquad G = 0 \quad \text{on } \partial\Omega, \tag{7}$$

and note that

$$\operatorname{div}(G\nabla\overline{u} - \overline{u}\nabla G) = -Gf + \overline{u},$$

then, on integrating over $\Omega$ and appealing to the fact that $\overline{u}$ and $G$ both vanish on $\partial\Omega$, we arrive at the equation

$$\int_\Omega \overline{u}\, dx = \int_\Omega Gf\, dx.$$

On the other hand, it follows from Eqs. (1) and (2) that

$$Q = -\int_\Omega \frac{\partial u}{\partial t}\, dx$$

and, hence, that

$$\overline{Q} = \int_\Omega f\, dx - s\int_\Omega \overline{u}\, dx,$$

that is to say,

$$\overline{Q}(s) = \int_\Omega (1 - sG(x,s))f(x)\, dx, \tag{8}$$

a formula that makes explicit the dependence of $\overline{Q}$ upon $f$. The region of integration can, of course, be replaced by the set $A$. On setting $s = 0$ we recover Eq. (3).

Now let $x \in A$ be arbitrary, let $r(x)$ be the distance from $x$ to $\partial\Omega$ and let $B(x)$ be the open ball with centre $x$ and radius $r(x)$. Let us define $H(y,s)$ for $y \in B(x)$ by

$$H(y,s) = \frac{1}{s}\left[1 - \left(\frac{r(x)}{\|y-x\|}\right)^{(n-2)/2} \frac{I_{(n-2)/2}(\sqrt{s}\|y-x\|)}{I_{(n-2)/2}(\sqrt{s}r(x))}\right], \quad y \neq x,$$

$$H(x,s) = \frac{1}{s}\left[1 - \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{\sqrt{s}r(x)}{2}\right)^{(n-2)/2} \frac{1}{I_{(n-2)/2}(\sqrt{s}r(x))}\right],$$

where $I_k$ is the Bessel function of imaginary argument and order $k$. Then $H(y,s)$ is the solution of the boundary-value problem

$$\begin{aligned} \Delta H(y,s) &= sH(y,s) - 1 \quad &&\text{for } y \in B(x), \\ H(y,s) &= 0 &&\text{for } y \in \partial B(x). \end{aligned}$$

The maximum principle tells us that $G(y,s) \geq H(y,s)$ for $y \in B(x)$ and, since $r(x) \geq r > 0$, that, for $x \in A$,

$$G(x,s) \geq H(x,s) \geq \frac{1}{s}\left[1 - \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{\sqrt{s}r}{2}\right)^{(n-2)/2} \frac{1}{I_{(n-2)/2}(\sqrt{s}r)}\right].$$

Hence, for $s > 0$ and $x \in A$,

$$1 - sG(x,s) \leq \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{\sqrt{s}r}{2}\right)^{(n-2)/2} \frac{1}{I_{(n-2)/2}(\sqrt{s}r)}.$$

On returning to Eq. (8), replacing $\Omega$ by $A$ on the right-hand side, and appealing to the inequality just established, we arrive at the estimate

$$\overline{Q}(s) \leq \frac{1}{\Gamma(\frac{n}{2})(\frac{\sqrt{s}r}{2})^{(n-2)/2}} \frac{\int_A f(x)\, dx}{I_{(n-2)/2}(\sqrt{s}r)},$$

and, hence, at the estimate

$$\overline{Q}(s) \le \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{\sqrt{s}r}{2}\right)^{(n-2)/2} \frac{\int_0^\infty Q(t)\,dt}{I_{(n-2)/2}(\sqrt{s}r)},$$

valid for every $s > 0$.

Now let $\tau > 0$ be arbitrary. Because $Q > 0$ and $s > 0$, we can argue that

$$\overline{Q}(s) = \int_0^\infty e^{-st}Q(t)\,dt > e^{-s\tau}\int_0^\tau Q(t)\,dt.$$

Accordingly,

$$\int_0^\tau Q(t)\,dt \le \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{\sqrt{s}r}{2}\right)^{(n-2)/2} \frac{e^{s\tau}}{I_{(n-2)/2}(\sqrt{s}r)}\int_0^\infty Q(t)\,dt$$

for every $s > 0$.

At this stage we introduce the function

$$F_n(\lambda) = \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{\lambda}{2}\right)^{(n-2)/2} \frac{\cosh(\frac{\lambda}{2})}{I_{(n-2)/2}(\lambda)}, \quad \lambda > 0.$$

Then $F_n(\lambda) \to 1$ as $\lambda \to 0$ and

$$F_n(\lambda) \sim \frac{2\sqrt{\pi}}{\Gamma(\frac{n}{2})}\left(\frac{\lambda}{2}\right)^{(n-1)/2} e^{-\lambda}\cosh\left(\frac{\lambda}{2}\right) \quad \text{as } \lambda \to \infty.$$

Hence $F_n(\lambda)$ is bounded for $0 < \lambda < \infty$, and if we introduce the constants

$$c(n) = 4\sup_{\lambda>0} F_n(\lambda),$$

it follows that

$$\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{\sqrt{s}r}{2}\right)^{(n-2)/2} \frac{e^{s\tau}}{I_{(n-2)/2}(\sqrt{s}r)} \le \frac{c(n)e^{s\tau}}{4\cosh(\frac{1}{2}\sqrt{s}r)} < \frac{1}{2}c(n)e^{s\tau-\frac{1}{2}\sqrt{s}r}$$

and, hence, that

$$\int_0^\tau Q(t)\,dt \le \frac{1}{2}c(n)e^{s\tau-\frac{1}{2}\sqrt{s}r}\int_0^\infty Q(t)\,dt, \quad s > 0.$$

The choice $s = r^2/(16\tau^2)$ minimizes the right-hand side and leads to the estimate

$$\int_0^\tau Q(t)\,dt \le \frac{1}{2}c(n)e^{-r^2/(16\tau)}\int_0^\infty Q(t)\,dt$$

and, therefore, the inequality (5) holds if $\tau$ is chosen as in Eq. (4).

Since

$$F_1(\lambda) = \frac{\cosh(\frac{\lambda}{2})}{\cosh\lambda}, \qquad F_3(\lambda) = \frac{\lambda}{2\sinh(\frac{\lambda}{2})},$$

and each of these functions attains its supremum at $\lambda = 0$, we see immediately that $c(1) = c(3) = 4$. Furthermore,

$$F_2(\lambda) = \frac{\cosh(\frac{\lambda}{2})}{I_0(\lambda)}, \qquad F_4(\lambda) = \frac{\lambda\cosh(\frac{\lambda}{2})}{2I_1(\lambda)},$$

and in view of the inequalities

$$I_0(\lambda) > \cosh(\frac{\lambda}{2}), \qquad I_1(\lambda) > \frac{\lambda}{2}\cosh(\frac{\lambda}{2}),$$

which are valid for $\lambda > 0$ and which follow from the power series expansions of the Bessel functions, it must be that $c(2) = c(4) = 4$. On the other hand, we have

$$F_n(\lambda) = 1 + \left(\frac{1}{8} - \frac{1}{2n}\right)\lambda^2 + o(\lambda^2) \quad \text{as } \lambda \to 0.$$

Hence, if $n > 4$, $F_n(\lambda)$ increases initially as $\lambda$ increases from 0 and so $c(n) > 4$.

The proof of the inequality (6) will be made to depend upon the observation that if we knew the mean time

$$\mu = \int_0^\infty tQ(t)dt \Big/ \int_0^\infty Q(t)\,dt$$

we could argue that

$$\int_0^\infty tQ(t)\,dt > T\int_T^\infty Q(t)\,dt$$

for every $T > 0$, and, therefore, that

$$\int_T^\infty Q(t)\,dt < \frac{\mu}{T}\int_0^\infty Q(t)\,dt.$$

However,

$$\int_0^\infty tQ(t)\,dt = -\left.\frac{d}{ds}\overline{Q}(s)\right|_{s=0}$$

and, in view of Eq. (8), we have

$$\int_0^\infty tQ(t)\,dt = \int_\Omega \phi(x)f(x)\,dx,$$

where $\phi(x) = G(x,0)$ $(x \in \Omega)$. On setting $s = 0$ in the boundary-value problem (7), we see that $\phi$ is the solution of the boundary-value problem

$$\Delta\phi = -1 \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega,$$

and an appeal to the maximum principle now establishes that if the ball of smallest radius that contains $\Omega$ has centre $x_0$ and radius $R$ then

$$\phi(x) \le \frac{1}{2n}(R^2 - |x - x_0|^2), \quad x \in \Omega.$$

Hence,

$$\int_0^\infty tQ(t)\,dt \le \frac{R^2}{2n}\int_\Omega f(x)\,dx = \frac{R^2}{2n}\int_0^\infty Q(t)\,dt.$$

Thus the mean time $\mu \le R^2/(2n)$ and, for every $T > 0$,

$$\int_T^\infty Q(t)\,dt \le \frac{R^2}{2nT}\int_0^\infty Q(t)\,dt,$$

and so the inequality (6) holds if $T$ is defined as in Eq. (4).

## References

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